MAL'CEV CONDITIONS REVISITED

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ABSTRACT. We characterize cosieves in locally presentable categories which are generated by a set of objects or are even principal. We apply our results to the category of algebraic theories where they are related to Mal'cev conditions dealt with in universal algebra.

1. Introduction

In 1954, Mal'cev [19] proved a remarkable characterization of varieties $\mathcal{V}$ such that the composition of two congruences is always their join. It says that the algebraic theory of $\mathcal{V}$ contains a ternary operation $t(x, y, z)$ such that $t(x, x, y) = y$ and $t(x, y, y) = x$ for each x, y. Later on, further similar Mal'cev type characterizations were found. For instance, Jónsson [14] characterized congruence distributive varieties in 1967 and A. Day [10] described congruence modular ones in 1969. More recently, protomodular varieties $\mathcal{V}$ were characterized in [9] as those varieties whose equational theory contains, for some integer n, 0-ary terms $e_1, ..., e_n$, binary terms $t_1, ..., t_n$ and $(n+1)$-ary terms t satisfying the identities $t(x, t_1(x, y), ..., t_n(x, y)) = y$ and $t_i(x, x) = e_i$ for each $i = 1, ..., n$.

G. Grätzer [11] asked how to recognize when a given class $\mathcal{X}$ of varieties has a Mal'cev type characterization. An answer was obtained by W. Taylor [24] who characterized classes $\mathcal{X}(M)$ of varieties given by a finite set M of terms satisfying a finite number of equations. He speaks about a *strong Mal'cev condition* in this case. In the original situation of Mal'cev, M is given by a ternary operation satisfying the two equations above. A simpler proof of his result was found in [20]. There is also a model theoretic proof given in [4] (see also the survey [15]). It is natural to view Grätzer's question from the category theoretic point

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of view because a variety V belongs to the class $\mathcal{X}(M)$ if and only if there is a morphism from M to the theory of V. Thus the theories of varieties from $\mathcal{X}(M)$ form the principal cosieve in the category **Th** of algebraic theories. Recall that a *cosieve* in a category $\mathcal{K}$ is a full subcategory $\mathcal{L}$ of $\mathcal{K}$ such that whenever $L \to K$ is a morphism in $\mathcal{K}$ with L in $\mathcal{L}$ then K belongs to $\mathcal{L}$ too. A cosieve $\mathcal{L}$ is *principal* if there exists an object M of $\mathcal{K}$ such that K belongs to $\mathcal{L}$ if and only if there is a morphism $M \to K$.

Since the category **Th** is locally finitely presentable, we are lead to consider cosieves in locally presentable categories. Grätzer's question asks for a characterization of principal cosieves in **Th** generated by finitely presentable objects. Taylor [24] introduced a whole hierarchy of Mal'cev type conditions which ends with cosieves generated by countable algebraic theories (see [20] and [4] as well). The latter theories are precisely ω_1-presentable objects in **Th**. Our first observation is that a cosieve $\mathcal{L}$ in a locally presentable category $\mathcal{K}$ is principal if and only if it is closed under products and λ-pure subobjects for some regular cardinal λ. Assuming the set-theoretic semiweak Vopěnka's principle, the second condition is automatic and thus any cosieve closed under products is principal. Conversely, semiweak Vopěnka's priciple follows from the fact that any cosieve in **Th** closed under products is principal. A much more delicate question is to estimate the presentation rank of an object M making a given cosieve principal. Our optimal result is that, given a locally finitely presentable category $\mathcal{K}$ having (up to isomorphism) countably many finitely presentable objects, then a cosieve $\mathcal{L}$ is a principal cosieve determined by an ω_1-presentable object M if and only if $\mathcal{L}$ is closed under products and ω-pure subobjects. Since the category **Th** of algebraic theories satisfies this assumption, we get a characterization of principal cosieves in **Th** determined by a countable theory M. We also get the Taylor's characterization of classes of varieties given by a strong Mal'cev condition (as a consequence of 3.4). It says that the corresponding algebraic theories form a cosieve $\mathcal{L}$ in **Th** closed under products and such that for each $T \in \mathcal{L}$ there is a finitely presented $T' \in \mathcal{L}$ with a morphism $T' \to T$. Taylor mentions in his review of [20] in Mathematical Reviews that there is an unpublished proof of his result using algebraic theories due to P. D. Bacsich.

For a general cosieve $\mathcal{L}$, a basic question is whether it is generated by a set of objects. In a locally presentable category, it is equivalent to $\mathcal{L}$ being closed under λ-pure subobjects for some λ. Now, this condition is automatic under Vopěnka's principle and, conversely, this principle follows from the fact that each cosieve in **Th** is generated by a set of theories. Moreover, a cosieve in **Th** is generated by countable theories if and only if it is closed under ω-pure subobjects. It gives a new characterization of the bottom member of Taylor's hierarchy and shows that this class is closed under intersections.

Cosieves in locally presentable categories appear in more contexts than just in **Th**. In graph theory, the are called homomorphisms closed classes. Their complements, which are precisely sieves, apper in coloring problems because the principal sieve generated by a graph H precisely consists of H-colorable graphs (see [21]). Grothendieck [12] considered cosieves in locally presentable (and even in accessible) categories and called them closed subcategories. He proved that Vopěnka's principle is equivalent with the fact that each such a cosieve is generated by a set. We have learnt about [12] only when writing the revised version of our paper and we are grateful to G. Maltsiniotis for sending us the preliminary redaction of the Grothendieck's manuscript.

All needed facts about locally presentable and accessible categories can be found in [2]. We only recall that a morphism $h : A \to B$ is called *λ-pure* (where λ is a regular cardinal) whenever in each commutative square

$$\begin{array}{ccc} X & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[h]{} & B \end{array}$$

with X and Y λ-presentable u factorizes through f, i.e., $u = tf$ for some $t : Y \to A$. Every λ-pure morphism is a monomorphism and we say that $\mathcal{L} \subset \mathcal{K}$ is *closed under λ-pure subobjects* if for each λ-pure morphism $K \to L$ with L in $\mathcal{L}$ we have K in $\mathcal{L}$.

2. Cosieves in locally presentable categories

Definition 2.1. A full subcategory $\mathcal{L}$ of a category $\mathcal{K}$ is called a *cosieve* if K is in $\mathcal{L}$ for each morphism $L \to K$ with L in $\mathcal{L}$. A cosieve is called a *filter* provided it is closed under finite products. A cosieve $\mathcal{L}$ is called *set generated* provided that there is a set $\mathcal{X}$ of objects of $\mathcal{L}$ such that K belongs to $\mathcal{L}$ if and only if there is a morphism $X \to K$ with X in $\mathcal{X}$. Finally, a cosieve is called *principal* if it is generated by a single object.

Theorem 2.2. *Let $\mathcal{L}$ be a cosieve in a locally presentable category $\mathcal{K}$. Then the following conditions are equivalent:*

(i) $\mathcal{L}$ is set generated,

(ii) $\mathcal{L}$ is closed under λ-pure subobjects for some regular cardinal λ,

(iii) $\mathcal{L}$ is an accessible category.

Proof. (i) $\to$ (ii). Let $\mathcal{L}$ be a cosieve in $\mathcal{K}$ generated by a set $\mathcal{X}$ of objects. There is a regular cardinal λ such that each X from $\mathcal{X}$ is λ-presentable (this follows from [2] 1.16). Consider a λ-pure morphism $h : K \to L$ with L in $\mathcal{L}$. There is a morphism $v : Y \to L$ with Y in $\mathcal{X}$. Consider the commutative square

$$\begin{array}{ccc} O & \xrightarrow{f} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ K & \xrightarrow[h]{} & L \end{array}$$

where O is an initial object in $\mathcal{K}$. Since O is λ-presentable, we get a morphism $Y \to K$. Thus K is in $\mathcal{L}$.

(ii) $\to$ (iii). Conversely, let $\mathcal{L}$ be a cosieve in $\mathcal{K}$ closed under λ-pure subobjects. Since $\mathcal{L}$ is closed in $\mathcal{K}$ under all non-empty colimits, it is closed under filtered colimits. By [2] 2.36, $\mathcal{L}$ is an accessible category.

(iii) $\to$ (i). Let $\mathcal{L}$ be accessible. Since it is accessibly embedded subcategory in $\mathcal{K}$, $\mathcal{L}$ is cone-reflective in $\mathcal{K}$ (by [2] 2.53). This means that for every object K in $\mathcal{K}$ there is a set of morphisms $g_i : K \to L_i$ with $L_i \in \mathcal{L}$ such that each morphism $K \to L$ with $L \in \mathcal{L}$ factorizes through some g_i. Then the cone-reflection $O \to L_i$ yields a generating set of $\mathcal{L}$. □

The just proved theorem is true in any accessible category having an initial object. Similarly, we could state many further results for acccessible categories having suitable limits or colimits. But we prefer to stay in the context of locally presentable categories which are precisely these accessible categories having all limits and colimits. Our motivating example **Th** has this properties.

Corollary 2.3. *A cosieve $\mathcal{L}$ in a locally presentable category $\mathcal{K}$ is principal if and only if it is closed under products and λ-pure subobjects for some regular cardinal λ.*

Proof. Let $\mathcal{L}$ be a principal cosieve generated by an object M and consider a set of objects L_i, $i \in I$ from $\mathcal{L}$. Since there are morphisms $M \to L_i$ for each $i \in I$, there is a morphism from M to the product of L_i. Thus $\mathcal{L}$ is closed under products.

Conversely, let $\mathcal{L}$ be a cosieve generated by a set of objects M_i, $i \in I$, which is closed under products. Then the product of M_i clearly generates $\mathcal{L}$. □

Let us recall that Vopěnka's principle says that the large discrete category (i.e., the category having a proper class of objects and no other morphisms than the identities) cannot be fully embedded into any locally presentable category.

Theorem 2.4. *The following statements are equivalent:*
(i) Vopěnka's principle;
(ii) every cosieve in a locally presentable category is set generated.

Proof. (i) $\to$ (ii) follows from 2.2 and [2] 6.17.
(ii)$\to$ (i). Assuming the negation of Vopěnka's principle, there is a locally presentable category $\mathcal{K}$ having a proper class of objects M_i, $i \in I$ such that the only morphisms $M_i \to M_j$ are the identities. Then the cosieve generated by M_i, $i \in I$ is not set generated. □

Remark 2.5. The just proved result can be found in [12] (see 4.16.7.10). More precisely, Grothendieck shows that every cosieve in an accessible category is accessible if and only if, given a proper class of objects A_i indexed by ordinals in an accessible category, there exists a morphism $A_i \to A_j$ for some $i < j$. But the last statement is known to be equivalent to Vopěnka's principle (see [2], 6.3).

Let us recall the statement $(*)$ from [2] 6.27 which is called *semiweak Vopěnka's principle* in [3]: a locally presentable category cannot contain objects A_i indexed by all ordinals i such that $\hom(A_i, A_j) \neq \emptyset$ if and only if $i \geq j$. Vopěnka's principle implies semiweak Vopěnka's principle but the converse implication is an open problem.

Theorem 2.6. *The following statements are equivalent:*
(i) semiweak Vopěnka's principle;
(ii) every cosieve closed under products in a locally presentable category is principal.

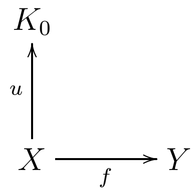
Proof. (i)$\to$ (ii) follows from [2] 6.26 and 6.27 (2) because a generator of $\mathcal{L}$ is given by a weak reflection of an initial object O.
(ii) $\to$ (i). Assuming the negation of semiweak Vopěnka's principle, there is a locally presentable category $\mathcal{K}$ and objects A_i indexed by all ordinals i such that $\hom(A_i, A_j) \neq \emptyset$ if and only if $i \geq j$. Let $\mathcal{L}$ consist of all objects K admitting a morphism $A_i \to K$ for some ordinal i. Then $\mathcal{L}$ is a cosieve closed under products. In fact, having $K_j \in \mathcal{L}$, $j \in J$ where J is a set, then there is an ordinal i with a morphism $A_i \to K_j$ for each $j \in J$; it suffices to take the upper bound of ordinals i_j with $A_{i_j} \to K_j$, $j \in J$. Assume that $\mathcal{L}$ is generated by M. Then there is $A_k \to M$ and thus $A_k \to A_i$ for each ordinal i, which is a contradiction. □

Notation 2.7. Given a locally presentable category $\mathcal{K}$ and a regular cardinal λ, there is only a set (up to isomorphism) of λ-presentable objects. We will denote such a set by $\mathrm{pres}_\lambda \mathcal{K}$.

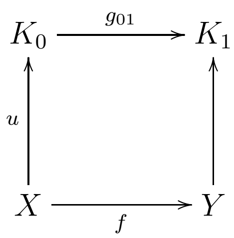
For regular cardinals $\lambda \leq \mu$, the symbol $\lambda \trianglelefteq \mu$ means that for every set X of cardinality less than μ, the set of subsets of X of cardinality less λ has a cofinal subset of cardinality less than μ (see [18], 2.3.1). The successor of a cardinal λ is denoted by λ^+.

Proposition 2.8. *Let $\mathcal{K}$ be a locally λ-presentable category and μ a regular cardinal such that $\lambda \trianglelefteq \mu$ and* $\mathrm{card}\,\mathrm{pres}_\lambda \mathcal{K} < \mu$. *Let $h : K \to L$ be a morphism in $\mathcal{K}$ having K μ-presentable. Then there is a factorization $h = h'g$ where $h' : K' \to L$ is λ-pure and K' μ-presentable.*

Proof. Put $K_0 = K$, $h_0 = h$ and consider the set $\mathcal{X}$ of all spans



such that X, Y are λ-presentable and f factorizes through $h_0 u$. By [18], 2.3.11, we have $\mathrm{card}\,\mathcal{X} < \mu$. Take a multiple pushout



over all such spans. Then there is a morphism $h_1 : K_1 \to L$ such that $h_1 g_{01} = h_0$ and $h_1 u$ factorizes through f for each our span. By [2], 1.16, K_1 is μ-presentable. We proceed with this construction over all ordinals $i \leq \lambda$. It means that we construct K_{i+1} by a multiple pushout above with K_0 replaced by K_i and get $g_{ii+1} : K_i \to K_{i+1}$ and $h_{i+1} : K_{i+1} \to L$. For a limit ordinal i, K_i is a colimit of K_j, $j < i$. In this way, we get μ-presentable objects K_i and morphisms $g_{ij} : K_i \to K_j$ and $h_i : K_i \to L$ for $i < j < \lambda$ such that $g_{jk} g_{ij} = g_{ik}$ and $h_j g_{ij} = h_i$ for $i < j < k < \lambda$. Then $K' = K_\lambda$, $h' = h_{0\lambda}$ and $g = g_\lambda$ have the desired properties. □

Theorem 2.9. *Let $\mathcal{K}$ be a locally λ-presentable category and $\lambda \trianglelefteq \mu$ regular cardinals such that* $\mathrm{card}\,\mathrm{pres}_\lambda \mathcal{K} < \mu$. *Then every cosieve $\mathcal{L}$ in $\mathcal{K}$ closed under λ-pure subobjects is generated by μ-presentable objects.*

Proof. Since $\mathcal{K}$ is locally μ-presentable (see [2], Remark 1.20), every object of K is a μ-directed colimit of μ-presentable objects. The result follows by 2.8. □

Theorem 2.10. *Let $\mathcal{K}$ be a locally λ-presentable category such that* $\operatorname{card}\operatorname{pres}_\lambda \mathcal{K} = \lambda$. *Then a cosieve $\mathcal{L}$ in $\mathcal{K}$ is closed under λ-pure subobjects if and only if it is generated by λ^+-presentable objects.*

Proof. Sufficiency follows from 2.9. Let $\mathcal{L}$ be a cosieve generated by λ^+-presentable objects. Consider L in $\mathcal{L}$ and a λ-pure morphism $h : K \to L$. There is a λ^+-presentable object M in $\mathcal{L}$ with a morphism $v : M \to L$. By [22], Lemma 1, M is a colimit $m_i : M_i \to M$ of a smooth chain $(m_{ij} : M_i \to M_j)_{i<j<\lambda}$ consisting of λ-presentable objects. Recall that the chain (m_{ij}) is smooth if $(m_{ij} : M_i \to M_j)_{i<j}$ is a colimit for a limit ordinal j. We can assume that $M_0 = O$ and denote by $u_0 : O \to K$ a unique morphism from the initial object O. Since $vm_1m_{01} = hu_0$ and h is λ-pure, there is a morphism $u_1 : M_1 \to K$ with $u_1m_{01} = u_0$. We proceed by a transfinite induction and get morphisms $u_i : M_i \to K$ with $u_jm_{ij} = u_i$ for $i < j < \lambda$. Here, u_{i+1} is obtained from u_i in the same way as in the first step. Since the chain (m_{ij}) is smooth, $(m_{ij} : M_i \to M_j)_{i<j}$ is a colimit for a limit ordinal j, which yields u_j in this case. At the end, we get a morphism $u : M \to K$, which implies that $K \in \mathcal{L}$. Hence $\mathcal{L}$ is closed under λ-pure subobjects. □

Corollary 2.11. *Let $\mathcal{K}$ be a locally finitely presentable category such that* $\operatorname{card}\operatorname{pres}_\omega \mathcal{K} = \omega$. *Then a cosieve $\mathcal{L}$ in $\mathcal{K}$ is closed under ω-pure subobjects if and only if it is generated by ω_1-presentable objects.*

Corollary 2.12. *Let $\lambda < \mu$ regular cardinals and $\mathcal{K}$ be a locally λ-presentable category such that* $\operatorname{card}\operatorname{pres}_\lambda \mathcal{K} < \mu$. *Then every cosieve $\mathcal{L}$ in $\mathcal{K}$ closed under products and λ-pure subobjects is a principal cosieve generated by a μ-presentable object.*

Proof. Let $\mathcal{L}$ be a cosieve closed under products and λ-pure subobjects. By 2.9, $\mathcal{L}$ is generated by μ-presentable objects. Moreover, $\mathcal{L}$ is principal (by 2.3). Since its generator M admits a morphism $M' \to M$ from a μ-presentable object $M' \in \mathcal{L}$, $\mathcal{L}$ is generated by M'. □

Remark 2.13. (1) A cosieve $\mathcal{L}$ in a locally λ-presentable category $\mathcal{K}$ generated by λ-presentable objects is a λ-accessible category. This follows from the fact that any morphism $h : K \to L$ with K λ-presentable and L in $\mathcal{L}$ factorizes through an object L' in $\mathcal{L}$ which is λ-presentable in $\mathcal{K}$. It suffices to take morphism $L_1 \to L$ with L_1 in $\mathcal{L}$ and λ-presentable in $\mathcal{K}$ and consider a coproduct L' of K and L_1. Then L' is λ-presentable, belongs to $\mathcal{L}$ and h factorizes through it.

(2) Full subcategories of locally λ-presentable categories closed under products, λ-filtered colimits and λ-pure subobjects are precisely classes determined by injectivity with respect to morphisms $N \to M$ where both N and M are λ-presentable (see [23]). These classes are called λ-*injectivity classes*. In the case when $N = O$, injectivity with respect to $N \to M$ precisely means the existence of a morphism from M.

3. Mal'cev conditions

Recall that a (single-sorted) *algebraic theory* is a category T whose objects are integers $0, 1, 2, \dots$ and such that n is a product of n copies of 1 for each $n = 0, 1, \dots$ (see [17]). This means that T has finite products and, in particular, a terminal object 0. An algebraic theory T determines a *variety* $Alg(T)$ which is the full subcategory of $\mathbf{Set}^T$ consisting of all functors $T \to \mathbf{Set}$ preserving finite products. **Set** denotes the category of sets. Each variety $Alg(T)$ is a concrete category with the underlying functor $U_T : Alg(T) \to \mathbf{Set}$ given by the evaluation at 1, i.e., $U_T(A) = A(1)$. Morphisms of algebraic theories are functors $F : T \to T'$ preserving finite products and such that $F(1) = 1$. This implies that $F(n) = n$ for each $n = 0, 1, \dots$. Let **Th** denote the resulting category of algebraic theories. Each morphism $F : T \to T'$ determines a concrete functor $Alg(F) : Alg(T') \to Alg(T)$ given by precompositions with F, i.e., $Alg(F)(A) = AF$ for each A in $Alg(T')$. Conversely, every concrete functor $H : Alg(T') \to Alg(T)$, i.e., every functor H with $U_T H = U_{T'}$, is determined by a unique morphism of theories (see [5]). Hence the dual of **Th** is the category **Var** whose objects are varieties and morphisms are concrete functors.

Proposition 3.1. **Th** *is a locally finitely presentable category.*

Proof. The category **Cat** of small categories is locally finitely presentable (see, e.g., [5], 5.2.2.f). Let $\mathbf{Cat}_{\mathrm{fp}}$ be its (non-full) subcategory consisting of categories with finite products and finite product preserving functors. $\mathbf{Cat}_{\mathrm{fp}}$ is a reflective subcategory of **Cat** which is closed under filtered colimits. A reflection of a small category $\mathcal{C}$ in $\mathbf{Cat}_{\mathrm{fp}}$ is given by a free completion of $\mathcal{C}$ under finite products. For instance, this completion is given as the full subcategory of $(\mathbf{Set}^{\mathcal{C}})^{\mathrm{op}}$ consisting of finite products of hom-functors. Since the embedding of $\mathbf{Cat}_{\mathrm{fp}} \to \mathbf{Cat}$ preserves filtered colimits, the reflector $\mathbf{Cat} \to \mathbf{Cat}_{\mathrm{fp}}$ preserves finitely presentable objects. Consequently, every object in $\mathbf{Cat}_{\mathrm{fp}}$ is a filtered colimit of finitely presentable objects, which means that $\mathbf{Cat}_{\mathrm{fp}}$ is locally finitely presentable. Since **Th** is the comma category

$$\mathbf{Fin}^{\mathrm{op}} \downarrow \mathbf{Cat}_{\mathrm{fp}}$$

where **Fin** is the category of finite sets, it is locally finitely presentable (see [2], 1.57). □

Remark 3.2. (1) **Th** is closed under connected limits in **Cat** but products in **Th** are different – morphisms $t : n \to m$ in the product of T_i, $i \in I$, are I-tuples $(t_i) : n \to m$ of morphisms t_i in T_i (cf. [5]).

(2) An algebraic theory is finitely presentable in **Th** if and only if it is generated by finitely many morphisms and finitely many equations between their compositions. As a consequence we get that

$$\operatorname{card} \operatorname{pres}_\omega \mathbf{Th} = \omega.$$

More generally, an algebraic theory $\mathcal{T}$ is λ-presentable in **Th** if and only if it is generated by less than λ morphisms and less than λ equations. For an uncountable regular cardinal λ, it is the same as having less than λ morphisms. In particular, $\mathcal{T}$ is ω_1-presentable if and only if it is countable.

We can thus apply to **Th** all results about cosieves from the previous section.

Theorem 3.3. *A cosieve in* **Th** *is generated by countable algebraic theories if and only if it is closed under ω-pure subobjects.*

Theorem 3.4. *Let $\mathcal{L}$ be a cosieve in* **Th**. *Then $\mathcal{L}$ is a principal cosieve generated by a countable theory if and only if it is closed under products and ω-pure subobjects.*

Proof. If $\mathcal{L}$ is closed under products and ω-pure subobjects then it is principal (by 2.3) and generated by a countable theory (by 3.3). Conversely, if $\mathcal{L}$ is a principal cosieve generated by a countable theory then it is closed under products and ω-pure subobjects (see 2.3 and 3.3). □

Remark 3.5. (1) Analogously, we get that a cosieve $\mathcal{L}$ in **Th** is a filter generated by a countable theories if and only if it is closed under finite products and ω-pure subobjects.

(2) Algebraic theories correspond to usual equational theories considered in universal algebra. Such an equational theory is given by a set Σ of (finitary) operation and by a set E of equations. The corresponding algebraic theory T has as morphisms $1 \to n$ n-ary algebraic operations which are obtained from Σ and E by means of substitutions. Morphisms $m \to n$ are then m-tuples of n-ary algebraic operations. In other words, T is the dual of the category of finitely generated free (Σ, E)-algebras. Any algebraic theory can be obtained in this way and varieties in our sense are categories equivalent to those resulting from varieties in universal algebra.

(3) We have mentioned in the introduction how our results are related to the top and the bottom members of Taylor's hierarchy. Let us add that, in between them, he considers filters generated by finitely presented algebraic theories. Since there is countably many finitely presentable algebraic theories, finitely presentable generators can be organized into a chain

$$\cdots \to M_n \to \cdots \to M_1 \to M_0$$

Taylor speaks about a *Mal'cev condition* in this case. We can deduce Taylor's result from our 3.3 like in the strong Mal'cev case. Add that ω-injectivity classes in **Th** were considered in [27].

(4) An algebraic theory T is called *pointed* if it has a unique morphism $0 \to 1$. Let $\mathbf{Th}_*$ denote the full subcategory of **Th** consisting of pointed theories. Since $\mathbf{Th}_*$ is evidently closed under limits and filtered colimits in **Th** (in fact, $\mathbf{Th}_*$ is a principal cosieve in **Th**, cf. [6]), $\mathbf{Th}_*$ is a locally finitely presentable category (see [2], Corollary 2.48). Thus we can apply our results to pointed algebraic theories as well. An example of a filter in $\mathbf{Th}_*$ generated by a finitely presentable algebraic theories is the class consisting of algebraic theories T having $Alg(T)$ semi-abelian (see [9]). Semi-abelian varieties are precisely pointed protomodular ones (see [13] or [6]). Let us mention that strongly protomodular varieties (see [8]) do not yield a cosieve because the variety of groups is strongly protomodular while the variety of digroups is not (see [8]). Recall that a digroup is a set equipped with two group structures sharing the same unit, which means that the theory of groups has a morphism into that of digroups.

(5) There is a characterization of congruence modular varieties in [7] which uses the following concepts. A punctual span in a pointed (i.e. when the final object is also initial) category $\mathcal{C}$ is a diagram:

$$X \underset{f}{\overset{s}{\rightleftarrows}} Z \underset{g}{\overset{t}{\leftrightarrows}} Y$$

such that $fs = 1_X$, $gt = 1_Y$, $gs = 0$ and $ft = 0$ (where 0 is the zero arrow). Now a pointed category (resp. pointed variety) $\mathcal{C}$ is *punctually congruence hypoextensible* when, given any punctual span and any equivalence relation E on Z such that $R[f] \wedge R[g] \leq E \leq R[f]$, we necessarily have $E \leq g^{-1}(t^{-1}(E))$. Here, $R[f]$ denotes the kernel of f. The pointed subtractive varieties (in the sense of [26]) are punctually congruence hypoextensible. A pointed category (resp. pointed variety) is *punctually congruence hyperextensible* when, given any punctual span and any equivalence relation E on Z such that $R[f] \wedge R[g] \leq E$, we necessarily have $R[f] \wedge g^{-1}(t^{-1}(E)) \leq E$. The pointed Jónsson-Tarski varieties ([16]) are punctually congruence hyperextensible. In connection with Day's result, it is natural to ask whether punctually congruence hypoextensible (resp. punctually congruence hyperextensible) varieties have a Mal'cev type characterization. By [7], algebraic theories T having $Alg(T)$ punctually congruence hypoextensible (resp. punctually congruence hyperextensible) form a cosieve in $\mathbf{Th}_*$. By 2.4, these cosieves are set generated under Vopěnka's principle. We do not know whether they are closed under (finite) products or under λ-pure subobjects for some λ.

Let us end with the following observations.

Theorem 3.6. *The following statements are equivalent:*
(i) Vopěnka's principle;
(ii) every cosieve in $\mathbf{Th}$ *is set generated.*

Proof. It follows from 2.4 and the fact that, assuming the negation of Vopěnka's principle, the large discrete category can be fully embedded into $\mathbf{Th}$ (see [25], (3) after Theorem 1.1). □

Theorem 3.7. *The following statements are equivalent:*
(i) semiweak Vopěnka's principle;
(ii) every cosieve in $\mathbf{Th}$ *closed under products is set generated.*

Proof. Like in 3.6, it follows from 2.4 and [25]. □

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