Forking in accessible categories

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Accessible categories whose morphisms are monomorphisms.

They are the same as μ-abstract elementary classes.

Any μ-accessible category whose morphisms are monomorphisms is a μ-AEC.

The forgetful functor $\mathcal{K} \to \mathbf{Set}$ is given by the canonical embedding $E : \mathcal{K} \to \mathbf{Set}^{\mathcal{K}_\mu^{\mathrm{op}}}$ where $\mathcal{K}_\mu$ is the (representative) full subcategory of $\mathcal{K}$ consisting of μ-presentable objects: EK is the restriction of $\mathcal{K}(-, K)$ on $\mathcal{K}_\mu$.

Conversely, any μ-AEC is λ^+-accessible where λ is its LST number (LRV + R. Grossberg and W. Boney 2016). Moreover it has μ-directed colimits.

But the forgetful functor cannot be detected from $\mathcal{K}$.

Problem 1. Is every accessible category with μ-directed colimits whose morphisms are monomorphisms equivalent to a μ-AEC?

This is open even for $\mu = \aleph_0$. Is the category of complete metric spaces and isometries a counterexample?

Problem 2. Is there an AEC which is not ∞, ω-elementary, i.e., not given by an $L_{\infty\omega}$-theory?

Is the category of uncountable sets and monomorphisms such an example?

Accessible categories whose morphisms are monomorphisms cannot be locally presentable.

But they can be locally multipresentable ($=$ accessible with connected limits $=$ accessible with multicolimits).

A typical example of a locally multipresentable category is the category of fields – the multiinitial object is formed by $\mathbb{Q}$ and by prime fields $\mathbb{Z}_p$.

Theorem 1. Locally multipresentable categories whose morphisms are monomorphisms coincide with universal μ-AECs.

A central concept of modern model theory is the notion of independence (introduced by Shelah) generalizing linear independence in vector spaces and algebraic independence in fields. Our aim is to extend this notion to accessible categories whose morphisms are monomorphisms.

In particular, we will consider locally multipresentable categories whose morphisms are monomorphisms.

But we could also consider locally polypresentable categories (= accessible categories with wide pullbacks = accessible categories with polycolimits) whose morphisms are monomorphisms. They coincide with μ-AECs with intersections and they include algebraically closed fields.

A *polyinitial* object is a set $\mathcal{I}$ of objects of a category $\mathcal{K}$ such that for every object M in $\mathcal{K}$:

1. There is a unique $i \in \mathcal{I}$ having a morphism $i \to M$.
2. For each $i \in \mathcal{I}$, given $f, g : i \to M$, there is a unique (isomorphism) $h : i \to i$ with $fh = g$.

We get groups of automorphisms of members of a polyinitial object. In the case of a multinitial objects, they are singletons. An example of a locally polypresentable category whose morphisms are monomorphisms are algebraically closed fields. The polyinitial object is formed by algebraic closures of the multiinitial object in fields.

Lemma 1. Let $\mathcal{K}$ be a coregular locally μ-presentable category and $\mathcal{K}_{reg}$ be the category having the same objects as $\mathcal{K}$ and regular monomorphisms as morphisms. Then $\mathcal{K}_{reg}$ is locally μ-multipresentable.

Examples of coregular locally presentable categories: Grothendieck toposes, Grothendieck abelian categories, **Gra** graphs, **Gr** groups, **Bool** Boolean algebras, **Ban** Banach spaces with linear contractions, **Hilb** Hilbert spaces with linear isometries, **CAlg** commutative unital C^*-algebras, etc.

Let $\mathcal{K}$ be a coregular locally presentable category and

$$\begin{tikzcd} M_1 \\ M_0 \arrow[u] \arrow[r] & M_2 \end{tikzcd}$$

a span in $\mathcal{K}_{reg}$. Let

$$\begin{tikzcd} M_1 \arrow[r] & P \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

a pushout in $\mathcal{K}$. Then a multipushout in $\mathcal{K}_{reg}$ is formed by all squares

$$\begin{tikzcd} M_1 \arrow[r] & Q \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

where the induced morphism $P \to Q$ is an epimorphism.

The *notion of independence* $\perp\!\!\!\perp$ in $\mathcal{K}$ consists in the choice of squares

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

which are declared to be independent. We say that M_1 and M_2 are independent over M_0 in M_3.

The following properties should be satisfied

(i) invariance under isomorphisms of squares

(ii) independence on M_3,

(iii) existence,

(iv) uniqueness,

(v) symmetry,

(vi) closedness under compositions of squares, and

(vii) accessibility.

(ii) means to be closed under the equivalence generated by

$$\begin{tikzcd}
 & M_3' \arrow[r, dotted] & M_3'' \\
M_1 \arrow[ur] \arrow[rr] & & M_3 \arrow[u, dotted] \\
M_0 \arrow[u] \arrow[r] & M_2 \arrow[uu] \arrow[ur] &
\end{tikzcd}$$

(iii) any span can be completed to an independent square,

(iv) any two independent squares of the same span are equivalent,

(v) and (vi) are clear,

(vii) the category whose objects are morphisms in $\mathcal{K}$ and whose morphisms are independent squares is accessible.

Theorem 2. Let $\mathcal{K}$ be an accessible category with chain bounds whose morphisms are monomorphisms. Then $\mathcal{K}$ has at most one notion of independence.

More generally, if $\mathcal{K}$ has the notion of independence $\mathop{\smile\!\!\!\!\!\!\mid}$ then any $\overset{1}{\mathop{\smile\!\!\!\!\!\!\mid}}$ with (i-vi) equals to $\mathop{\smile\!\!\!\!\!\!\mid}$.

Theorem 3. Let $\mathcal{K}$ be an accessible category whose morphisms are monomorphism having a notion of independence. Then $\mathcal{K}$ is tame, stable and does not have the order property.

Theorem 4. Let $\mathcal{K}$ be a coregular locally presentable category with effective unions. Then $\mathcal{K}_{reg}$ has an independence notion (consisting of pullback squares).

$\mathcal{K}$ has effective unions if whenever we have a pullback

$$\begin{array}{ccc} M_1 & \rightarrowtail & M_3 \\ \uparrow & & \uparrow \\ M_0 & \rightarrowtail & M_2 \end{array}$$

and a pushout

$$\begin{array}{ccc} M_1 & \longrightarrow & P \\ \uparrow & & \uparrow \\ M_0 & \rightarrowtail & M_2 \end{array}$$

the induced morphism $P \to M_3$ is a regular monomorphism.

Any Grothendieck topos and any Grothendieck abelian category has effective unions. **Hilb** has effective unions. The facts that **R-Mod** and **Hilb** have an independence notion were known.

Gra, **Gr**, **Ban**, **Bool** or **CAlg** do not have effective unions. They do not have a notion of independence because they have the order property.

Let $\mathcal{K}$ be a coregular locally presentable category and take *effective pullback squares* in $\mathcal{K}_{reg}$ as independent squares.

This satisfies all axioms of the independence up to the smallness condition in (vii).

Corollary 1. Let $\mathcal{K}$ be a coregular locally presentable category with an independence notion $\downarrow$ in $\mathcal{K}_{reg}$. Then $\downarrow$ is given by effective pullback squares.

In **Gra**, in an effective pullback square

$$\begin{tikzcd} M_1 \arrow[r] & M_3 \\ M_0 \arrow[u] \arrow[r] & M_2 \arrow[u] \end{tikzcd}$$

M_3 does not contain any cross-edge between M_1 and M_2. Thus we do not have enough independent squares with M_0 and M_2 small. Thus **Gra** does not have an independence notion.

There is another attempt of independence where we include all cross-edges between M_1 and M_2. This yields $\downarrow$ satisfying (i-vi). By Theorem 2, **Gra** does not have an independence notion.

The category $\mathcal{K}$ of graphs whose vertices have order $\leq n$ form a locally finitely presentable category which does not have effective unions. But effective pullback squares provide an independence in $\mathcal{K}_{reg}$. The reason is that we can add all cross-edges to M_0 and M_2 and keeping them finite.

Theorem 2 implies that this notion of independence is unique.

Let $\mathcal{K}$ be the category of locally finite graphs, i.e., graphs such any vertex has a finite degree. This category is coregular and locally $\aleph_1$-multipresentable. Again, effective pullback squares form a notion of independence in $\mathcal{K}_{reg}$.

$\mathcal{K}$ does not have chain bounds but the proof of Theorem 2 still goes through – thus this independence is unique.

Based on Malliaris and Shelah 2011, we say that a graph is stable if it does not contain a copy of the half graph (the bipartite graph on $\mathbb{N} \times \mathbb{N}$ such that $E(i,j)$ iff $i < j$). The category $\mathcal{K}$ of stable graphs is locally $\aleph_1$-multipresentable and effective pullback squares do not form an independence notion in $\mathcal{K}_{reg}$. But we do not know whether $\mathcal{K}_{reg}$ has an independence notion. $\mathcal{K}$ does not have chain bounds and we could not adapt the proof of Theorem 2 to this case.

Let $\mathcal{K}$ be a locally polypresentable category whose morphisms are monomorphisms having a stable independence notion. Then, for each span, exactly one instance of a polypushout is independent. Moreover, a morphism of spans

$$(\mathrm{id}_{M_0}, h_1, h_2) : (M_0, M_1, M_2) \to (M_0, M_1', M_2')$$

induces a morphism of independent instances of polypushouts. Thus the independence yields a coherent choice of polypushouts.

The same holds for weak polypushouts (where 2. is omitted).

Moreover, if $\mathcal{K}$ has the amalgamation property then a coherent choice of weak polypushouts yields $\perp$ satisfying (i-vi).

Corollary 2. Let $\mathcal{K}$ be a locally polypresentable category with the amalgamation property, chain bounds and whose morphisms are monomorphisms. If $\mathcal{K}$ has two distinct coherent choices of polypushouts then it does not have a notion of independence.

Theorem 5. Let $\mathcal{K}$ be a coregular locally presentable category where regular monomorphisms are closed under directed colimits. Then $\mathcal{K}_{reg}$ has a stable independence notion iff regular monomorphisms are cofibrantly generated.

Consequently, **Gr**, **Ban**, **Bool** and **CAlg** do not have a notion of independence. **Gr** do not have enough regular injectives and thus regular monomorphisms cannot be cofibrantly generated. In all other cases, regular injectives do not form an accessible category and thus regular monomorphisms cannot be cofibrantly generated again.

Another consequence is that regular monomorphisms in **Gra** are not cofibrantly generated. Equivalently, **Gra** does not have enough regular injectives or those are not accessible.

Theorem 6. Let $\mathcal{K}$ be an accessible category whose morphisms are monomorphisms having the amalgamation property and chain bounds. Let κ be a strongly compact cardinal. If $\mathcal{K}$ does not have the order property then the full subcategory of $\mathcal{K}$ consisting of κ-saturated objects has an independence notion.

Theorem 7. Let $\mathcal{K}$ be an accessible category whose morphisms are monomorphisms having an independence notion. Then there exists a regular cardinal κ such that any independent square with M_0 κ-saturated is a pullback square.