MORE ON INJECTIVITY IN LOCALLY PRESENTABLE CATEGORIES

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Dedicated to Horst Herrlich on the occasion of his 60th birthday.

Abstract. Injectivity w.r.t. morphisms having λ-presentable domains and codomains is characterized: such injectivity classes are precisely those closed under products, λ-directed colimits, and λ-pure subobjects. This sharpens the result of the first two authors (Trans. Amer. Math. soc. 336 (1993), 785-804). In contrast, for geometric logic an example is found of a class closed under directed colimits and pure subobjects, but not axiomatizable by a geometric theory. A more technical characterization of axiomatizable classes in geometric logic is presented.

I. Introduction

In [AR_1], classes of objects injective w.r.t. a set $\mathcal{M}$ of morphisms of a locally presentable category $\mathcal{K}$ were characterized: they are precisely the classes closed under products, λ-directed colimits and λ-pure subobjects for some cardinal λ (see Part II below for the concept of λ-pure subobject). In fact, the formulation in [AR_1] did not use λ-pure subobjects, but accessibility of the class in question. However, a full subcategory of $\mathcal{K}$, closed under λ-directed colimits, is accessible iff it is closed under λ'-pure subobjects for some λ' (see [AR_2], Corollary 2.36). The main result of our paper is a "sharpening" of the previous result to a given regular cardinal λ: by a λ-*injectivity class* we call a class of all objects injective w.r.t. $\mathcal{M}$, where $\mathcal{M}$ is a set of morphisms with λ-presentable domains and codomains. If $\mathcal{K}$ is a locally λ-presentable category, we prove that λ-injectivity classes in $\mathcal{K}$ are precisely the full subcategories closed under products, λ-directed colimits, and λ-pure subobjects.

In contrast, the generalization from injectivity w.r.t. morphisms to injectivity w.r.t. cones, which was presented by H. Hu and M. Makkai [HM], does not allow a corresponding "sharpening" – which is unfortunate, since this is closely connected to geometric logic. Hu and Makkai proved that classes given by injectivity w.r.t. a set of cones are precisely those closed under λ-directed colimits and λ-pure subobjects for some λ (again, we use Corollary 2.36 of [AR_2] to put the original result into our terminology). We present an example of a class of Σ-structures, for a finitary signature Σ, which is closed under directed colimits and pure subobjects, but which is *not* a cone-injectivity class for any set of cones with finitely presentable domains and codomains. In particular, the class is not axiomatizable by geometric logic in $\mathbf{Str}\Sigma$. We introduce a rather technical concept of being strongly closed under

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pure subobjects. This, then, is a full characterization of classes axiomatizable by geometric logic.

Let us recall that, given an arrow $h : A \to A'$ in a category $\mathcal{K}$, an object X is called *h-injective*, provided that every morphism from A to X factors through h, i.e. the map

$$\hom(h, X) : \hom(A', X) \to \hom(A, X), \quad f \mapsto fh$$

is surjective. This is an important concept in algebra, where h is usually required to be a monomorphism, and in model theory (where general h's are considered). A class $\mathcal{A}$ of objects is called an *injectivity class* provided that there is a collection $\mathcal{M}$ of morphisms such that $\mathcal{A}$ consists of precisely all objects that are h-injective for all $h \in \mathcal{M}$ (we write $\mathcal{A} = \mathcal{M}$-Inj); if $\mathcal{M}$ is small, we speak about a *small-injectivity class*, and if all domains and all codomains of $\mathcal{M}$-maps are λ-presentable objects of $\mathcal{K}$, we speak about a λ-*injectivity class*. In case that the base category $\mathcal{K}$ is locally presentable (in the sense of Gabriel and Ulmer [GU]), then every small-injectivity class is a λ-injectivity class for some λ, and conversely, every λ-injectivity class is a small-injectivity class. But small-injectivity classes are interesting also in such categories as **Top** - see e.g. [H], where they are called implicational subcategories.

A full characterization of small-injectivity classes in a locally presentable category $\mathcal{K}$ has been presented in [AR_1]: they are precisely the classes $\mathcal{A}$ of objects, for which there exists a regular cardinal λ such that $\mathcal{A}$ is closed under

(i) products,
(ii) λ-directed colimits,

and

(iii) λ-pure subobjects.

(Let us recall that λ-*pure monomorphisms* are precisely the λ-directed colimits of retractions (as objects of $\mathcal{K}^{\to}$), and if $\mathcal{K}$ is the category of all Σ-structures for some λ-ary signature Σ, then this categorical concept coincides with that of a λ-pure submodel, used in model theory.) In the present paper, we prove that for each locally λ-presentable category $\mathcal{K}$, the conditions (i) – (iii) above precisely characterize λ-injectivity classes in $\mathcal{K}$. The proof is substantially more difficult then that in [AR_1].

More generally, given a cone $H = (h_i : A \to A'_i)_{i \in I}$, an object X is H-injective iff every maps from A to X factors through h_i for some $i \in I$. A class of objects is called a *(small-)cone-injectivity class* if it can be axiomatized by injectivity w.r.t. a class (or set) of cones. If all domains and codomains of all those cones are λ-presentable, we speak about λ-*cone-injectivity* classes.

Injectivity in locally presentable categories is treated in [AR_2] where the reader also finds standard categorical concepts used in our paper. We use the standard notation $\mathbf{Str}\,\Sigma$ for the category of Σ-structures and homomorphisms, where Σ is a finitary many-sorted signature. This is a locally finitely presentable category.

II. Characterization of Injectivity Classes

Remark. Recall that a morphism $m : A \to B$ of a locally λ-presentable category $\mathcal{K}$ is called λ-*pure* provided that in each commutative square

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\ A \arrow[r, "m"'] & B \end{tikzcd}$$

with X and Y λ-presentable the morphism f factors through h. As an object of $A \downarrow \mathcal{K}$, m is λ-pure iff m is a λ-directed colimit of retractions, see [AR$_2$]. Every λ-pure morphism is a regular monomorphism.

Theorem 1. *Let $\mathcal{K}$ be a locally λ-presentable category. A full subcategory of $\mathcal{K}$ is a λ-injectivity class iff it is closed under*

(i) *products*
(ii) *λ-directed colimits*

and

(iii) *λ-pure subobjects.*

Proof. The necessity is easy: closedness under products and λ-directed colimits is trivial. To prove closedness under λ-pure subobjects $m : P \to Q$, we are to show that if Q is h-injective then P is h-injective for all maps $h : A \to B$ with A, B both λ-presentable. Let $u : A \to P$ be a morphism. There exists $v : B \to Q$ making the following square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ P \arrow[r, "m"'] & Q \end{tikzcd}$$

commutative (h-injectivity of Q) and since u factors through h (λ-purity of m), the h-injectivity of P follows.

For the sufficiency, we can assume that $\mathcal{K}$ is the category $\mathbf{Str}\Sigma$ of all Σ-structures for some finitary relational signature Σ. (The general case follows from the fact that $\mathcal{K}$ can be considered as a full reflective subcategory of $\mathbf{Str}\Sigma$ closed under λ-directed colimits – see 1.47 of [AR$_2$]. This implies that $\mathcal{K}$ is closed under (i) – (iii), see Remark 2.31 in [AR$_2$]. Consequently, given a full subcategory $\mathcal{L}$ of $\mathcal{K}$ closed under (i) – (iii) in $\mathcal{K}$, then $\mathcal{L} = \mathcal{M}$-Inj for some set $\mathcal{M}$ of morphisms of $\mathbf{Str}\Sigma$ having λ-presentable domains and codomains. The reflector $F : \mathbf{Str}\Sigma \to \mathcal{K}$ obviously preserves λ-presentability of objects and satisfies $\mathcal{L} = F(\mathcal{M})$-Inj.)

Thus, let $\mathcal{L}$ be a full subcategory of $\mathbf{Str}\Sigma$, where Σ is a finitary relational signature, closed under (i) – (iii). Put

$$\mathcal{M} = \{f : A \to B \text{ in } \mathbf{Str}\Sigma;\ A \text{ and } B \text{ are } \lambda\text{-presentable} \\ \text{and every object of } \mathcal{L} \text{ is injective w.r.t. } f\}.$$

We will prove that $\mathcal{L} = \mathcal{M}$-Inj. We first observe that $\mathcal{L}$ is *weakly reflective*, i.e., every object K of $\mathbf{Str}\Sigma$ has a morphism $r_K : K \to K^*$ (weak reflection) with K^*

in $\mathcal{L}$ such that every object of $\mathcal{L}$ is injective w.r.t. r_K. This follows from 2.36 and 4.8 in [AR$_2$]. Now $\mathcal{L} \subseteq \mathcal{M}$-Inj by the choice of $\mathcal{M}$, and to prove $\mathcal{M}$-Inj $\subseteq \mathcal{L}$, we verify the following implication:

$K \in \mathcal{M}$-Inj $\Rightarrow$ any weak reflection of K in $\mathcal{L}$ is λ-pure.

It then follows from (iii) that $K \in \mathcal{L}$, and this will prove

$$\mathcal{M}\text{-Inj} = \mathcal{L}\,.$$

Thus, given $K \in \mathcal{M}$-Inj and a weak reflection $r : K \to K^*$ in $\mathcal{L}$, we are to prove that in any commutative square

$$\begin{tikzcd} A \arrow[r, "h"] \arrow[d, "u"'] & B \arrow[d, "v"] \\ K \arrow[r, "r"'] & K^* \end{tikzcd}$$

with A and B λ-presentable the map u factors through h. We will work with the arrow-category $(\mathbf{Str}\Sigma)^{\to}$ and consider the morphism $(u, v) : h \to r$.

Claim: The morphism $(u, v) : h \to r$ factors through some object $\bar{r}$ of $(\mathbf{Str}\Sigma)^{\to}$, where $\bar{r} : \bar{K} \to \bar{K}^*$ is a weak reflection of $\bar{K}$ in $\mathcal{L}$ and $\bar{K}$ is λ-presentable in $\mathbf{Str}\Sigma$.

Proof of claim. Consider all factorizations of (u, v) through weak reflections $\bar{r} : \bar{K} \to \bar{K}^*$ of Σ-structures in $\mathcal{L}$ (e.g. the trivial one through r) and take the smallest α such that $\bar{K}$ is α-presentable in $\mathbf{Str}\Sigma$. We are to prove $\alpha \leq \lambda$. Assuming $\alpha > \lambda$ we derive a contraction. Let us first remark that an object A of $\mathbf{Str}\Sigma$ is α-presentable iff it has

(a) less than α elements (i.e., the union of all underlying sets of all sorts has cardinality $< \alpha$)

and

(b) less than α relational symbols $\sigma \in \Sigma$ with σ_A nonempty.

(See 1.14 (2) in [AR$_2$].) Consequently, for every regular cardinal $\alpha \geq \aleph_0$, any α-presentable object in $\mathbf{Str}\Sigma$ is a colimit of a *smooth α-chain* (i.e., a chain whose limit steps form a colimit of the preceding part) of objects of presentability smaller than α.

Let us express the above object $\bar{K}$ as a colimit of a smooth α-chain $k_{ij} : K_i \to K_j$ $(i \leq j < \alpha)$ of objects K_i of presentability less than α, let $k_{i\alpha} : K_i \to K_\alpha$ $(i < \alpha)$ denote the colimit cocone. We define an α-chain $k^*_{ij} : K^*_i \to K^*_j$ $(i \leq j < \alpha)$ in $\mathcal{L}$ and a natural transformation

$$r_i : K_i \to K^*_i \qquad (i < \alpha)$$

by transfinite induction. (This idea has been used in [AR$_1$] already, see the proof of IV. 3.)

(i) $r_0 : K_0 \to K^*_0$ is a weak reflection of K_0 in $\mathcal{L}$.

(ii) $i \mapsto i + 1$: Form a pushout of r_i and $k_{i,i+1}$

$$\begin{tikzcd} K_i \arrow[r, "r_i"] \arrow[d, "k_{i,i+1}"'] & K_i^* \arrow[d, "\hat{k}_{i,i+1}"] \\ K_{i+1} \arrow[r, "\hat{r}_i"] & \hat{K}_{i+1} \end{tikzcd}$$

Choose a weak reflection $r^*_{i+1} : \hat{K}_{i+1} \to K^*_{i+1}$ in $\mathcal{L}$ and put $r_{i+1} = r^*_{i+1}\hat{r}_i$ and $k^*_{i,i+1} = r^*_{i+1} \cdot \hat{k}_{i,i+1}$.

(iii) Given a limit ordinal j, form a colimit

$$\hat{K}_j = \operatorname*{colim}_{i<j} K_i^*$$

with colimit maps $\hat{k}_{ij} : K_i^* \to \hat{K}_j \quad (i < j)$ and choose a weak reflection $r_j^* : \hat{K}_j \to K_j^*$ in $\mathcal{L}$, then $r_j = r_j^* \cdot (\operatorname*{colim}_{i<j} r_i)$ and $k^*_{ij} = r_j^* \hat{k}_{ij}$.

Observe that every object L of $\mathcal{L}$ is r_i-injective for all $r_i < \alpha$.

Proof. This is obvious for $i = 0$. If this holds for all smaller ordinals than $i \ (> 0)$, then it holds for i as well: given $f : K_i \to L$, we can define a compatible cocone $f^*_{ij} : K^*_j \to L \quad (j \le i)$ with $f^*_j r_j = f\, k_{ji}$ by transfinite induction as follows:

(i) f^*_0 is any map with $f^*_0 r_0 = f\, k_{0i}$ (this exists by the choice of r_0);

(ii) given f^*_j, the pushout property yields a unique $\hat{f}_{j+1} : \hat{K}_{j+1} \to L$ and we choose any map $f^*_{j+1} : K^*_{j+1} \to L$ with $\hat{f}_{j+1} = f^*_{j+1} r_{j+1}$

(iii) given a limit ordinal j and f^*_k for $k < j$, the j-chain-colimit property yields a unique $\hat{f}_j : \hat{K}_j \to L$ and we choose any map $f^*_j : K^*_j \to L$ with $\hat{f}_j = f^*_j \cdot r_j$.

The object $K^*_\alpha = \operatorname*{colim}_{i<\alpha} K^*_i$ lies in $\mathcal{L}$ because α-chains are λ-directed (recall that $\alpha > \lambda$ is a regular cardinal). Thus, for the maps

$$r_\alpha = \operatorname*{colim}_{i<\alpha} r_i : K \to K^*_\alpha$$

there exists $s : K^* \to K^*_\alpha$ with $r_\alpha = sr$. (Recall that r is a weak reflection of K.)

In $(\mathbf{Str}\Sigma)^{\to}$ we thus obtain a morphism

$$(u, sv) : h \to r_\alpha \, .$$

Now h is, obviously, λ-presentable in $(\mathbf{Str}\Sigma)^{\to}$, thus that last map factors through some of the objects $r_i,\ i < \alpha$. This is the desired contradiction with the minimality of α: each $r_i : K_i \to K^*_i$ is a weak reflection of K_i, and K_i has smaller presentation rank then α. This proves the claim.

We are ready to prove that u factors through h. Let us consider a factorization

$$\begin{tikzcd}
A \arrow[r, "h"] \arrow[d, "u_1"'] & B \arrow[d, "v_1"] \\
\bar{K} \arrow[r, "\bar{r}"] \arrow[d, "u_2"'] & \bar{K}^* \arrow[d, "v_2"] \\
K \arrow[r, "r"'] & K^*
\end{tikzcd}$$

of (u, v) through an object $\bar{r}$ which is a weak reflections of a λ-presentable Σ-structure $\bar{K}$ in $\mathcal{L}$. Let us express $\bar{K}^*$ as a λ-directed colimit of λ-presentable objects Q_t, $t \in T$, in $\mathbf{Str}\Sigma$ with a colimit cocone $q_t : Q_t \to \bar{K}^*$. Since both $\bar{K}$ and B are λ-presentable, the maps $\bar{r}$ and v_1 both factor through q_{t_0} for some $t_0 \in T$ and then there exists (since A is λ-presentable) $t_1 \geqslant t_0$ in T with a commutative diagram as follows:

$$\begin{tikzcd}
A \arrow[rr, "h"] \arrow[dd, "u_1"'] & & B \arrow[dl, "\tilde{v}_1"'] \arrow[dd, "v_1"] \\
& Q_{t_1} \arrow[dr, "q_{t_1}"] & \\
\bar{K} \arrow[ur, "\tilde{r}"] \arrow[rr, "\bar{r}"'] & & \bar{K}^*
\end{tikzcd}$$

Since all objects of $\mathcal{L}$ are $\bar{r}$-injective, they are also $\tilde{r}$-injective; moreover, $\bar{K}$ and Q_{t_1} are both λ-presentable, thus,

$$\tilde{r} \in \mathcal{M} .$$

This implies that K is $\tilde{r}$-injective. Choose $d : Q_{t_1} \to K$ with $u_2 = d\tilde{r}$ to obtain

$$u = u_2 u_1 = d\tilde{r}u_1 = d\tilde{v}_1 h ,$$

the desired factorization. Thus, r in λ-pure, and $K \in \mathcal{L}$. □

Remark. The reader may well wonder whether the condition of closedness under λ-pure subobjects is not superfluous: after all

(i) closedness under λ-directed colimits implies closedness under split subobjects

and

(ii) every λ-pure subobject is a λ-directed colimit of split subobjects, see [AR$_2$] 2.30.

However, (ii) holds in the arrow-category of $\mathcal{K}$, not is $\mathcal{K}$ itself. The following Example demonstrates that, indeed, λ-pure subobjects are essential:

Example. of a class of graphs which is closed under products and directed colimits, but is not an ω-injectivity class. We denote by **Gra** the category of graphs, i.e., pairs (X, R) of sets with $R \subseteq X \times X$, and graph homomorphisms. Let $\mathcal{A}$ be the class of all graphs containing nodes $x_n \in X \quad (n \in N)$ such that $x_n R x_{n+1}$ for each $n \in N$. It is clear that $\mathcal{A}$ is closed under products and nonempty colimits in **Gra**. However, $\mathcal{A}$ is not closed under ω-pure subobjects. In fact, let A denote the graph obtained from an infinite path $x_0, x_1, x_2, \dots$ (starting in the node x_0) by gluing a path of length n to x_0 for each $n \in N$. Then

$$A \in \mathcal{A} \text{ and } B \notin \mathcal{A}$$

where B is the strong subgraph of A over all nodes distinct from x_k for $k \geq 1$. However, B is an ω-pure subgraph of A because given any finite (= finitely presentable) subgraph V of A there exists a morphisms $h : V \to B$ with $h(x) = x$ for all nodes x of B. In fact: let n be the largest index with $x_n \in V$ and for each $k \leq n$ denote by x'_k the k-th node on the path of length n in A, then put $h(x) = x$ if $x = B$ and $h(x_k) = x'_k$.

III. Characterization of Classes Axiomatizable by Geometric Logic

Recall that *geometric logic* is the first-order logic using formulas of the following form

$$(\forall x)\big(\varphi(x) \to \bigvee_{i \in I} (\exists y_i)\ \psi(x, y_i)\big) \tag{1}$$

where x and each y_i is a finite string of variables, and φ and each ψ_i is a finite conjunction of atomic formulas. For example, given a cone H with finitely presentable domain and codomains in $\mathbf{Str}\Sigma$, injectivity w.r.t. H can be axiomatized by a formula (1), where $\varphi(x)$ expresses the presentation of the domain of H, and $\psi(x, y_i)$ expresses the presentation of the i-th codomain, and the connecting morphism. Consequently, every ω-cone-injectivity class in $\mathbf{Str}\Sigma$ is axiomatizable in geometric logic. Conversely, every formula (1) can be translated to ω-cone-injectivity in $\mathbf{Str}\Sigma$, see [AR$_2$].

Observation. Every class of structures axiomatizable by geometric logic is closed in $\mathbf{Str}\Sigma$ under directed colimits and pure subobjects.

The converse does not hold:

Example. of a class of Σ-structures (Σ finitary) closed under directed colimits and strong subobjects, which is not an ω-cone-injectivity class.

Let Σ be the (one-sorted) signature of countably many unary relation symbols $\sigma_n \quad (n \in \omega)$. For every set $A \subseteq \omega$ denote by $\bar{A}$ the Σ-structure over the set $\{0\}$ with $\sigma_n \neq \emptyset$ iff $n \in A$. Further denote by I the initial (empty) object of $\mathbf{Str}\Sigma$.

Choose a disjoint decomposition

$$\omega = \bigcup_{n \in \omega} B_n, \quad B_n \text{ infinite}$$

and denote by $\mathcal{B}$ the full subcategory of $\mathbf{Str}\Sigma$ consisting precisely of

(i) I

and

(ii) all $\bar{B}$ where $B \subseteq \omega$ is a set such that there exists $n \in \omega$ and a set $M \subseteq \omega$ of cardinality $\leq n$ with $B = B_n \cup M$.

$\mathcal{B}$ is obviously closed under substructures (strong subobjects), since $\mathcal{B}$-objects have no non-trivial substructures. And it is closed under directed colimits: if $\bar{B}_t$ $(t \in T)$ is a directed collection in $\mathcal{B}$, then there exists $n \in \omega$ such that each B_t has the form $B_t = B_n \cup M_t$ for M_t of at most n elements. Since T is directed, the set $M = \bigcup\limits_{t \in T} M_t$ also has at most n elements, and $\operatorname{colim} \bar{B}_t = B_n \cup M \in \mathcal{B}$.

However, $\mathcal{B}$ is not an ω-cone-injectivity class. In fact, $\mathcal{B}$ does not contain the terminal Σ-structure $\bar{\omega}$, although any ω-cone-injectivity class containing $\mathcal{B}$ contains $\bar{\omega}$ too. Indeed: $\bar{\omega}$ is injective w.r.t. all nonempty cones, and we observe that there exists no empty ω-cone H to which all $\mathcal{B}$-objects are injective. (In fact, let M be the domain of H. Since M is finitely presentable, there exists $n \in \omega$ such that $(\sigma_k)_M = \emptyset$ for all $k \geq n$. Put $B = B_n \cup \{0, 1, \ldots, n-1\}$, then the constant map is a Σ-homomorphism from M to $\bar{B}$, consequently, $\bar{B}$ is not injective w.r.t. the empty cone with domain H.)

Definition. A span $A \xleftarrow{n} C \xrightarrow{m} B$ is called λ-*pure* provided that for each commutative square

$$\begin{tikzcd}
 & X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, "g"] \\
A & C \arrow[l, dashed, "n"] \arrow[r, "m"'] & B
\end{tikzcd}$$

with X and Y λ-presentable the morphism nf factors through h (i.e., there is $t : Y \to A$ with $nf = th$). If $\lambda = \omega$ we say just *pure*.

Examples. (1) For each morphism $r : B \to A$ the inverse relation of the graph of r, i.e., $A \xleftarrow{r} B \xrightarrow{1} B$, is pure. Also any subspan of the inverse of a graph is pure. These are precisely the relations $A \xleftarrow{n} C \xrightarrow{m} B$ where n factors through m; we call them *split spans*.

(2) A λ-directed colimit of split spans is λ-pure. More precisely, let $\mathbf{Sp}\,(\mathcal{K})$ denote the category whose objects are spans $A \xleftarrow{n} C \xrightarrow{m} B$ in $\mathcal{K}$ and whose morphisms are commutative diagrams

$$\begin{tikzcd}
A \arrow[d, "u"'] & C \arrow[l, "n"'] \arrow[r, "m"] \arrow[d, "v"] & B \arrow[d, "w"] \\
A' \arrow[r, "n'"'] & C' \arrow[r, "m'"'] & B
\end{tikzcd}$$

If $\mathcal{K}$ has λ-directed colimits, then so does $\mathbf{Sp}(\mathcal{K})$, and a λ-directed colimit of split relations is λ-pure. This follows easily from the fact that (λ-directed) colimits are formed object-wise in $\mathbf{Sp}(\mathcal{K})$.

Remark. In a locally λ-presentable category the latter example of λ-pure spans is canonical: every λ-pure span is a λ-directed colimit of split spans. The proof is analogous to that of 2.30 in [AR$_2$].

Definition. A subcategory $\mathcal{L}$ of a category $\mathcal{K}$ is *strongly closed under λ-pure subobjects* provided that $\mathcal{L}$ contains every object A of $\mathcal{K}$ with the following property:

> given a morphism $n : C \to A$ in $\mathcal{K}$ with C λ-presentable, there exists a λ-pure span $A \xleftarrow{n} C \xrightarrow{m} B$ with $B \in \mathcal{L}$.

Remark. (1) Given a morphism $m : C \to D$, for each cardinal λ let us denote by $H_\lambda(m)$ the set of all spans (h, f) where $h : X \to Y$ is a morphism with X and Y λ-presentable, and $f : X \to C$ is a morphism with a commutative square

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "f"'] & Y \arrow[d, dashed] \\ C \arrow[r, "m"'] & D \end{tikzcd}$$

A span $A \xleftarrow{n} C \xrightarrow{m} B$ is λ-pure iff $H_\lambda(m) \subseteq H_\lambda(n)$. Thus, $\mathcal{L}$ is strongly closed under λ-pure subobjects iff $\mathcal{L}$ contains every object $A \in \mathcal{K}$ with the following property: given $n : C \to A$, C λ-presentable, there exists $m : B \to A$ with $B \in \mathcal{L}$ such that $H_\lambda(m) \subseteq H_\lambda(n)$.

(2) This is, indeed, stronger than closedness under λ-pure subobjects. For, given $L \in \mathcal{L}$ and a λ-pure subobject $i : A \to L$, we prove that $A \in \mathcal{L}$ as follows: Let $n : C \to A$ be a morphism with C λ-presentable, then $A \xleftarrow{n} C \xrightarrow{in} L$ is a λ-pure span.

(3) Strong closedness under λ-pure subobjects implies closedness under λ-directed colimits (and thus is, indeed, much stronger than closedness under λ-pure subobjects). In fact, let A be a λ-directed colimit of $A_i \in \mathcal{L}$ with a colimit cocone $\left(A_i \xrightarrow{h_i} A\right)_{i \in I}$. For each morphism $n : C \to A$ with C λ-presentable there exists $i \in I$ such that n factors as $n = a_i n'$ for some $n' : C \to A_i$, and we have a pure span $A \xleftarrow{n} C \xrightarrow{n'} A_i$, thus, $A \in \mathcal{L}$.

Theorem 2. *Let $\mathcal{K}$ be a locally λ-presentable category. Then λ-cone-injectivity classes of $\mathcal{K}$ are precisely the full subcategories strongly closed under λ-pure subobjects.*

Proof. I. Suppose that $\mathcal{L}$ is strongly closed in $\mathcal{K}$ under λ-pure subobjects. For every object A in $\mathcal{K} - \mathcal{L}$ we construct a sink γ_A as follows: choose a morphism $n : C \to A$, C λ-presentable, such that no λ-pure span $A \xleftarrow{n} C \xrightarrow{m} B$ with $B \in \mathcal{L}$ exists. Given any morphism $m : C \to B$ with $B \in \mathcal{L}$ we thus have a commutative square

$$\begin{tikzcd} & X_m \arrow[r, "h_m"] \arrow[d, "f_m"'] & Y_m \arrow[d, "g_m"] \\ A & C \arrow[l, dashed, "n"] \arrow[r, "m"'] & B \end{tikzcd}$$

on $\mathcal{K}$ with X_m and Y_m λ-presentable such that $n \cdot f_m$ does not factor through h_m. Form a pushout of f_m and h_m:

$$\begin{tikzcd}
X_m \arrow[r, "h_m"] \arrow[d, "f_m"'] & Y_m \arrow[d, "f_m^*"] \arrow[ddr, dashed, "g_m"] & \\
C \arrow[r, "m^*"'] \arrow[drr, dashed, "m"'] & B^* \arrow[dr, dashed] & \\
& & B
\end{tikzcd}$$

Since C, X_m and Y_m are λ-presentable, so is B^*. We denote by γ_A the cone of all $m^* : C \to B^*$ (indexed by all $m \in \bigcup_{B \in \mathcal{L}} \hom(C, B)$), and we first observe that since γ_A has a λ-presentable domain and all codomains, it is essentially small. We claim that

$$\mathcal{L} = \{\gamma_A\}_{A \in \mathcal{K} - \mathcal{L}}\text{-Inj}\,,$$

i.e., that an object of $\mathcal{K}$ lies in $\mathcal{L}$ iff it is cone-injective w.r.t. each γ_A. In fact, given $B \in \mathcal{L}$ then for every morphism $m : C \to B$ we have a factorization of m through m^*, a member of γ_A, thus, B is $\mathcal{M}$-injective.

Conversely, let A be an object of $\mathcal{K}$ injective to all of the above cones, then we prove that $A \in \mathcal{L}$. In fact, assuming $A \in \mathcal{K} - \mathcal{L}$ we have the above cone γ_A and since A is γ_A-injective, the morphism $n : C \to A$ factors through m^* for some $m : C \to B$ with $B \in \mathcal{L}$. But then $n \cdot f_m$ factors through h_m (since if $t \cdot m^* = n$ then $t \cdot f_m^* \cdot h_m = n \cdot f_m$), in contradiction to the above choice of h_m, f_m, and g_m.

II. Suppose that $\mathcal{L}$ is a λ-cone-injectivity class in $\mathcal{K}$. Let $\mathcal{M}$ be a set of cones with λ-presentable domains and codomains such that $\mathcal{L} = \mathcal{M}$-Inj. We will prove that $\mathcal{L}$ contains any object A satisfying the condition of the above definition of strong closedness under λ-pure subobjects. Thus, given a cone $\gamma = (C \xrightarrow{c_i} C_i)_{i \in I}$ in $\mathcal{M}$, we are to show that A is γ-injective. In fact, let $n : C \to A$ be a morphism. There exists a λ-pure span $A \xleftarrow{n} C \xrightarrow{m} B$ with $B \in \mathcal{L}$. Since $B \in \mathcal{M}$-Inj, the morphism m factors through some c_i, say, $m = m' \cdot c_i$.

$$\begin{tikzcd}
& C \arrow[r, "c_i"] \arrow[d, "1"] & C_i \arrow[d, "m'"] \\
A & C \arrow[l, dashed, "n"] \arrow[r, "m"'] & B
\end{tikzcd}$$

From the λ-purity we conclude that $n \cdot 1$ factors through c_i, thus, A is γ-injective.

Corollary 1. *A class of Σ-structures can be axiomatized by geometric logic iff it is strongly closed under pure subobjects in* $\mathbf{Str}\Sigma$.

Remark. (1) We can speak, more generally, about infinitary geometric logic: given a regular cardinal λ, a formula (1) is λ-geometric provided that x and y_i are strings of less than λ variables and φ and ψ_i are conjunctions of less than λ atomic formulas. The above corollary generalizes as expected: classes axiomatized by λ-geometric theories are precisely those strongly closed under λ-pure subobjects.

(2) λ-geometric logic precisely describes λ-cone-injectivity classes in $\mathbf{Str}\Sigma$. Analogously, λ-injectivity classes are described by λ-*regular logic*, i.e., logic using formulas

$$(\forall x)\big(\varphi(x) \to \exists y\ \psi(x,y)\big) \tag{2}$$

where, again, x and y are strings of less than λ variables and φ and ψ are conjunctions of less than λ atomic formulas. Thus, Theorem 1 characterizes classes of structures axiomatizable by λ-regular logic: they are the classes closed under products, λ-directed colimits and λ-pure subobjects.

This seems to be a new result for $\lambda > \aleph_0$; let us remark that, however, for $\lambda = \aleph_0$ a more elementary proof follows from Compactness Theorem. A class closed under products and directed colimits is namely closed under ultraproducts; being closed under pure (in particular, under elementary) subobjects, the class is axiomatizable in the first-order logic, and our result then easily follows.

(3) We now turn to a special case of cone-injectivity classes for which a good characterization can be formulated. If $\mathcal{M}$ is a set of cones and each cone consists of strong epimorphisms only, we call $\mathcal{M}$-Inj a *strong cone-injectivity class*.

If the domains and codomains in all those cones are λ-generated objects (i.e., objects A such that $\hom(A,-)$ preserves λ-directed unions), then we speak about *strong λ-cone-injectivity* classes, and we prove that those are precisely the classes closed under λ-directed unions and subobjects. This depends on (strong epi, mono)-factorizations of morphisms. We work, more generally, with an abstract factorization system. Following [AHS] we say that $\mathcal{K}$ is an $(\mathcal{E},\mathcal{M})$-*structured category* provided that $\mathcal{E}$ is a class of epimorphisms, $\mathcal{M}$ is a class of monomorphisms, both dosed under composition, with $\mathcal{E}\cap\mathcal{M} = \mathrm{Iso}$ and every morphism has an essentially unique $(\mathcal{E},\mathcal{M})$-factorization.

Definition. Let $\mathcal{K}$ be an $(\mathcal{E},\mathcal{M})$-structured category. A λ-directed colimit whose colimit cocone is formed by $\mathcal{M}$-morphisms is called a λ-*directed union*. An object K is called λ-*generated* provided that $\hom(K,-)$ preserves λ-directed unions.

Lemma. *In a locally λ-presentable $(\mathcal{E},\mathcal{M})$-structured category every object is a λ-directed union of λ-generated objects.*

Proof. Every object K is a λ-directed colimit of λ-presentable objects $K_i \quad (i \in I)$. If $k_i : K_i \to K \quad (i \in I)$ denotes the colimit cocone and $k_i = e_i m_i$ is an $(\mathcal{E},\mathcal{M})$-factorization of k_i, with $m_i : K_i' \to K$, then the objects K_i' form a λ-directed diagram with a colimit cocone $m_i : K_i' \to K \quad (i \in I)$, due to the diagonal fill-in. Since K_i is λ-presentable and $e_i : K_i \to K_i'$ lies in $\mathcal{E}$, it follows easily from the diagonal fill-in that K_i' is λ-generated.

Remark. In a locally λ-presentable $(\mathcal{E},\mathcal{M})$-structured category, λ-generated objects are precisely the $\mathcal{E}$-quotients of all λ-presentable objects – this has been proved for $\mathcal{E}$ = strong epis in [AR$_2$], 1.61, and the general case is analogous.

Definition. A class of objects is called a *strong λ-cone-injectivity* class if it has the form $\mathcal{M}$-Inj for a set of cones formed by $\mathcal{E}$-morphisms whose domains and codomains are λ-generated.

Theorem 3. *Let $\mathcal{K}$ be a locally λ-presentable $(\mathcal{E},\mathcal{M})$-structured category. A full subcategory of $\mathcal{K}$ is a strong λ-cone-injectivity class iff it is closed in $\mathcal{K}$ under λ-directed unions and $\mathcal{M}$-subobjects.*

Proof. Necessity is clear. For the sufficiency, let $\mathcal{L}$ be a full subcategory of $\mathcal{K}$ closed under λ-directed unions and $\mathcal{M}$-subobjects.

(a) $\mathcal{L}$ is an accessible category. In fact, $\mathcal{L}$ is closed under λ-pure subobjects in $\mathcal{K}$, since by 2.31 in [AR$_2$],

$$\lambda\text{-pure} \Rightarrow \text{regular mono} \Rightarrow \text{member of } \mathcal{M}\,.$$

By 2.36 of [AR$_2$] it is sufficient to prove that $\mathcal{L}$ is closed under μ-directed colimits for some regular cardinal μ. By the above remark, $\mathcal{K}$ has only a set of λ-generated objects, thus, there exists a regular cardinal μ such that

$$\lambda\text{-generated} \Rightarrow \mu\text{-presentable}\,.$$

Let $L_i \ \ (I \in I)$ form a μ-directed diagram in $\mathcal{L}$ with a colimit $l_i : L_i \to K \ \ (i \in I)$ in $\mathcal{K}$. We are to prove that $K \in \mathcal{L}$. By the above Lemma, K is a λ-directed union of λ-generated objects, say, $k_j : K_j \to K \ \ (j \in J)$. Since K_j is μ-presentable, k_j factors through some l_i, i.e., there exists $i \in I$ and $d : K_j \to L_i$ with $k_j = l_i \cdot d$. Now $k_j \in \mathcal{M}$ implies $d \in \mathcal{M}$, thus, $K_j \in \mathcal{L}$ (since $\mathcal{L}$ is closed under $\mathcal{M}$-subobjects). Thus, K is a λ-directed union of $\mathcal{L}$-objects, which proves $K \in \mathcal{L}$.

(b) $\mathcal{L}$ is a cone-reflective subcategory of $\mathcal{K}$, i.e., for every object $K \in \mathcal{K}$ there exists a cone with domain K to which all $\mathcal{L}$-objects are injective. This is proved in 2.53 of [AR$_2$]. Moreover, every object $K \in \mathcal{K}$ has a cone-reflection in $\mathcal{L}$ all members of which are $\mathcal{E}$-epis: this follows immediately from the closedness of $\mathcal{K}$ under $\mathcal{M}$-subobjects.

(c) Let $\mathcal{H}$ be the following collection of cones in $\mathcal{K}$: for each λ-generated object K of $\mathcal{K}$ we choose a cone-reflections $(r_i^K : K \to K_i)_{i \in I(K)}$ of K in $\mathcal{L}$ with $r_i^K \in \mathcal{E}$ for each $i \in I$ and let $\mathcal{H}$ be the set of all these cones. It is sufficient to prove that an object $K \in \mathcal{K}$ lies in $\mathcal{L}$ iff K is injective w.r.t. all cones of $\mathcal{H}$.

Thus, let K be injective w.r.t. $\mathcal{H}$-cones. By the above Lemma, we can express K as a λ-directed union of λ-generated objects $k_t : K_t \to \mathcal{K} \ \ (t \in T)$. Since K is initial w.r.t. a cone-reflection $(r_i^{K_t})$, k_t factors through some $r_i^{K_t}$ which, since $k_t \in \mathcal{M}$ and $r_i^{K_t} \in \mathcal{E}$, implies that $r_i^{K_t}$ is an isomorphism. In other words, $K_t \in \mathcal{L}$ (for each $t \in T$). Since $\mathcal{L}$ is closed under λ-directed unions, this implies $K \in \mathcal{L}$. $\square$

Remark. The factorization of Σ-homomorphisms to surjective Σ-homomorphisms followed by inclusions of substructures yields $\mathbf{Str}\Sigma$ as an (epi, regular mono)-structured category. Then λ-generated Σ-structures are Σ-structures generated by less than λ elements in a usual sense. If a cone $(h_i : A \to B_i)_{i \in I}$ has all h_i surjective and all A and B_i λ-generated then (2) is (equivalent to) a universal sentence of the logic $L_{\infty,\lambda}$. We obtain, from Theorem 2,

Corollary 3. *A class of Σ-structures is axiomatizable by a universal theory of the logic $L_{\infty,\lambda}$ iff it is closed under submodels and λ-directed unions.*

Remark. Corollary 3 has an easy direct proof: any class $\mathcal{L}$ of Σ-structures which is closed under substructures and λ-directed unions can be axiomatized by universal sentences of $L_{\infty,\lambda}$

$$(\forall \overrightarrow{x}) \bigvee_{i\in I} \pi_{A_i}(\overrightarrow{x})$$

where $\{A_i\}_{i\in I}$ is a representative set of Σ-structures $A_i \in \mathcal{L}$ generated by less then λ-elements and π_{A_i} is the diagram of A_i reduced to generators.

Corollary 3 documents the well-known fact that a class of Σ-structures with at most countably many elements, which is axiomatizable in L_{ω_1,ω_1}, cannot be axiomatized by a universal theory in $L_{\omega_1\omega_1}$ (see [D] 2.4.11). An analogous example cannot be found in $L_{\omega,\omega}$: any class of Σ-structures axiomatizable in $L_{\omega,\omega}$ and closed under substructures is axiomatizable by a universal theory in $L_{\omega,\omega}$.

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