ON HOMOTOPY VARIETIES

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ABSTRACT. Given an algebraic theory $\mathcal{T}$, a homotopy $\mathcal{T}$-algebra is a simplicial set where all equations from $\mathcal{T}$ hold up to homotopy. All homotopy $\mathcal{T}$-algebras form a homotopy variety. We will give a characterization of homotopy varieties analogous to the characterization of varieties.

1. Introduction

Algebraic theories were introduced by Lawvere (see [31] and also [32]) in order to provide a convenient approach to study algebras in general categories. An *algebraic theory* is a small category $\mathcal{T}$ with finite products. Given a category $\mathcal{K}$ with finite products, a $\mathcal{T}$*-algebra* in $\mathcal{K}$ is a finite product preserving functor $\mathcal{T} \to \mathcal{K}$. Algebras in the category **Set** of sets are usual many-sorted algebras. Algebras in the category **SSet** of simplicial sets are called simplicial algebras and they can be also viewed as simplicial objects in the category of algebras in **Set**. In homotopy theory, one often needs to consider algebras up to homotopy – a homotopy $\mathcal{T}$-algebra is a functor $A : \mathcal{T} \to \mathbf{SSet}$ such that the canonical morphism

$$A(X_1 \times \cdots \times X_n) \to A(X_1) \times \cdots \times A(X_n)$$

is a weak equivalence for each finite product $X_1 \times \cdots \times X_n$ in $\mathcal{T}$. These homotopy algebras have been considered in recent papers [5], [6] and [8] but the subject is much older (see, e.g., [15], [7], [37] or [41]). It is natural to consider *simplicial algebraic theories*, i.e., small simplicial categories $\mathcal{T}$ with finite products. Homotopy algebras are then simplicial functors $\mathcal{T} \to \mathbf{SSet}$ preserving finite products up to a weak equivalence.

Given an algebraic theory $\mathcal{T}$, we get the category $\mathrm{Alg}(\mathcal{T})$ of all $\mathcal{T}$-algebras in **Set**. There is a characterization of categories equivalent to

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$\mathrm{Alg}(\mathcal{T})$ for some algebraic theory $\mathcal{T}$ proved by Lawvere in the single-sorted case which can be immediately extended to the general case (cf. [2], 3.25). Recent papers [3] and [1] characterized $\mathrm{Alg}(\mathcal{T})$ by using sifted colimits. These colimits generalize filtered ones – while a category $\mathcal{D}$ is filtered if colimits over $\mathcal{D}$ commute with finite limits in **Set**, a category $\mathcal{D}$ is *sifted* if colimits over $\mathcal{D}$ commute with finite products in **Set**. The category $\mathrm{Alg}(\mathcal{T})$ precisely consists of sifted colimits of hom-functors in $\mathbf{Set}^{\mathcal{T}}$, i.e., it is a free completion of $\mathcal{T}^{\mathrm{op}}$ under sifted colimits.

Given a simplicial algebraic theory $\mathcal{T}$, homotopy $\mathcal{T}$-algebras form a simplicial category. Our aim is a characterization of simplicial categories of homotopy $\mathcal{T}$-algebras analogous to the just mentioned characterization of categories $\mathrm{Alg}(\mathcal{T})$. To achieve it, we have to use homotopy colimits which can be defined in simplicial categories as a special case of weighted colimits. A category $\mathcal{D}$ is *homotopy sifted* if homotopy colimits over $\mathcal{D}$ commute with finite products in **SSet**. Homotopy sifted categories coincide with totally coaspherical categories in the sense of [36]. Every simplicial category $\mathcal{K}$ has a homotopy category $\mathrm{Ho}(\mathcal{K})$ but these homotopy categories are well behaved only if hom-sets of $\mathcal{K}$ are Kan complexes. We will call such simplicial categories *fibrant*. The category $\mathbf{SSet}^{\mathcal{T}}$ of all simplicial functors $\mathcal{T} \to \mathbf{SSet}$ carries a projective (= Bousfield-Kan) model category structure. We will denote by $\mathrm{HAlg}(\mathcal{T})$ the full subcategory of $\mathbf{SSet}^{\mathcal{T}}$ consisting of those homotopy $\mathcal{T}$-algebras which are both fibrant and cofibrant in this model category structure. Then $\mathrm{HAlg}(\mathcal{T})$ is a fibrant simplicial category and our main result characterizes fibrant simplicial categories weakly equivalent (in the sense of Definition 2.2) to $\mathrm{HAlg}(\mathcal{T})$ for some fibrant simplicial algebraic theory $\mathcal{T}$. It makes possible to recognize that a fibrant simplicial category $\mathcal{K}$ is given by a fibrant simplicial algebraic theory $\mathcal{T}$ and even to reconstruct $\mathcal{T}$ from $\mathcal{K}$.

Categories $\mathrm{Alg}(\mathcal{T})$ of algebras of usual algebraic theories are included in a broader class of locally finitely presentable categories (see [2]). In fact, the later are precisely categories of models of finite limit theories $\mathcal{T}$. It means that $\mathcal{T}$ is a small category with finite limits and a $\mathcal{T}$-model is a functor $\mathcal{T} \to \mathbf{Set}$ preserving finite limits. In the same way, we can include categories of homotopy $\mathcal{T}$-algebras into homotopy locally (finitely) presentable categories. These categories were recently considered in [40], [43], [44], [33] and [34]. Lurie [33] introduced homotopy accessible and homotopy locally presentable categories under the name of accessible ∞-categories and presentable ∞-categories. He worked with CW-complexes instead of simplicial sets and homotopy coherent functors in the sense of [18] and obtained a homotopy Giraud theorem characterizing homotopy Grothendieck toposes. In the recent work [34], he has written his theory in the language of quasi-categories of Joyal (cf. [27] and [28]). Simpson [40] introduced a generalization of homotopy locally presentable categories using the language of Segal categories and characterized them as categories of fibrant and cofibrant objects of cofibrantly generated model categories. By Dugger [22], homotopy locally presentable categories correspond in this way to combinatorial model categories (i.e., cofibrantly generated and locally presentable). Toën and Vezzosi [43] and [44] used the language of Segal categories to deal with homotopy Grothendieck toposes. There is also an unpublished text of Rezk [39] about homotopy toposes.

This work was motivated by the first paper of Lurie [33]. Our simplicial approach could be extended to cover general homotopy accessible categories. They should correspond to categories of homotopy models of theories specified by both homotopy limits and homotopy colimits. In the recent paper [35], Lurie considers homotopy algebras in the context of quasi-categories as well. In particular, his Proposition 12.2 corresponds to our characterization of categories of homotopy algebras (his work appeared only after our paper was completed and submitted for a publication). Let us add that our simplicial approach is very close to a purely model theoretic one. In fact, one cannot expect that the category of homotopy $\mathcal{T}$-algebras carries a model category structure because it is neither complete nor cocomplete. Instead, one can introduce a model category structure on $\mathbf{SSet}^{\mathcal{T}}$ whose fibrant objects are just homotopy $\mathcal{T}$-algebras. This model category structure is a left Bousfield localization of the projective one and was considered by Bergner [8] in the case of ordinary algebraic theories. We show that a simplicial model category $\mathcal{M}$ is Quillen equivalent to this model category of homotopy $\mathcal{T}$-algebras if and only if the simplicial category of fibrant and cofibrant objects in $\mathcal{M}$ is weakly equivalent to $\mathrm{HAlg}(\mathcal{T})$. The author is grateful to the anonymous referee for suggesting a possible model theoretic formulation.

In the second section of this paper, we recall simplicial categories and their homotopy theory. In particular, we introduce the basic concept of a fibrant homotopy colimit. The third section deals with simplicial presheaves, i.e., with simplicial categories $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ where $\mathcal{C}$ is a small simplicial category. This category is equipped with the projective model category structure. We show that if $\mathcal{C}$ is fibrant then simplicial presheaf which is both cofibrant and fibrant in this model category structure is homotopy equivalent to a fibrant homotopy colimit of hom-functors. It is based on Dugger [21] and makes the fibrant simplicial category $\mathrm{Pre}(\mathcal{C})$ of these simplicial presheaves analogous to the usual category of set-valued presheaves in the classical theory. The forth section contains our characterization of fibrant simplicial categories of homotopy $\mathcal{T}$-algebras and, in the last section, we do the same for homotopy models of finite homotopy limit theories.

2. Simplicial categories

A *simplicial category* $\mathcal{K}$ is a category enriched over the category **SSet** of simplicial sets. This means that hom-objects $\hom(K, L)$ are simplicial sets equipped with compositions

$$c_{K,L,M} : \hom(K, L) \times \hom(L, M) \to \hom(K, M)$$

and units

$$\Delta_0 \to \hom(K, K).$$

The underlying category $\mathcal{K}_0$ of $\mathcal{K}$ has the sets $\hom_0(K, L)$ of points (= 0-simplices) of $\hom(K, L)$ as hom-sets. We will often speak about morphisms $K \to L$ having elements of $\hom_0(K, L)$ in mind. Any morphism $f : K \to L$ induces the simplicial map

$$\hom(M, f) : \hom(M, K) \to \hom(M, L)$$

given as the composition of

$$c_{M,K,L}(\mathrm{id}_{\hom(M,K)} \times f) : \hom(M, K) \times \Delta_0 \to \hom(M, L)$$

with the isomorphism

$$\hom(M, K) \cong \hom(M, K) \times \Delta_0.$$

Given simplicial categories $\mathcal{K}$ and $\mathcal{L}$, a *simplicial functor* $F : \mathcal{K} \to \mathcal{L}$ is equipped with simplicial maps

$$F_{K,L} : \hom(K, L) \to \hom(FK, FL)$$

compatible with composition and unit. A *simplicial natural transformation* $\varphi : F \to G$ between simplicial functors is given by morphisms $\varphi_K : FK \to GK$ for each K in $\mathcal{K}$ such that the following diagram commutes for each pair of objects K_1, K_2 of $\mathcal{K}$

$$\begin{tikzcd}
\hom(K_1, K_2) \arrow[r, "F_{K_1,K_2}"] \arrow[d, "G_{K_1,K_2}"'] & \hom(FK_1, FK_2) \arrow[d, "\hom(FK_1,\varphi_{K_2})"] \\
\hom(GK_1, GK_2) \arrow[r, "\hom(\varphi_{K_1},GK_2)"'] & \hom(FK_1, GK_2)
\end{tikzcd}$$

(see [25], or [10] for basic facts about enriched categories in general).

Given simplicial functors $D : \mathcal{D} \to \mathcal{K}$ and $G : \mathcal{D} \to \mathbf{SSet}$, the *limit* K of D *weighted* by G is defined by a simplicial isomorphism natural in X

$$\hom(G, \hom(X, D)) \cong \hom(X, K);$$

hom's are always taken in appropriate simplicial categories. On the left side, it is the simplicial category of simplicial functors from $\mathcal{D}$ to $\mathbf{SSet}$ (see [10]) where $\hom(X, D) : \mathcal{D} \to \mathbf{SSet}$ is the composition of D and $\hom(X, -)$. Analogously, a colimit K of $D : \mathcal{D} \to \mathcal{K}$ weighted by $G : \mathcal{D}^{op} \to \mathbf{SSet}$ is given by a simplicial natural isomorphism

$$\hom(G, \hom(D, X)) \cong \hom(K, X).$$

Recall that a *tensor* of a simplicial set V and an object K of a simplicial category $\mathcal{K}$ is an object $V \otimes K$ given by a simplicial natural isomorphism

$$\hom(V \otimes K, L) \cong \hom(V, \hom(K, L)).$$

Dually, a *cotensor* K^V is given by

$$\hom(L, K^V) \cong \hom(V, hom(L, K)).$$

Model categories are taken in the sense of [26] or [25]. A *simplicial model category* is a model category which is a simplicial category whose simplicial hom-sets are homotopically well behaved (see [25] or [24] for the precise definition). By [22], every combinatorial model category is Quillen equivalent to a simplicial model category. Recall that a model category $\mathcal{M}$ is called *combinatorial* if the category $\mathcal{M}$ is locally presentable (cf. [2]) and its model structure is cofibrantly generated.

There are well developed concepts of simplicial locally presentable categories and simplicial accessible categories (cf. [30], [11] and [12]). Simplicial locally presentable categories are equivalent to categories of models of weighted limit theories while simplicial accessible categories are equivalent to categories of models of theories specified by both weighted limits and weighted colimits. The desired concepts of homotopy locally presentable categories and homotopy accessible categories should be based on homotopy limits and homotopy colimits. In simplicial model categories, the definition of homotopy limits and homotopy colimits adopted in [25], 18.1.8 and 18.1.1 make them a special case of weighted limits and weighted colimits (see [25], 18.3.1); this observation goes back to [13]. The corresponding weights form a homotopy invariant approximations of constant diagrams at a point. The same definitions work in any simplicial category; in what follows, $B(\mathcal{X})$ denotes the nerve of the category $\mathcal{X}$.

Definition 2.1. Let $\mathcal{K}$ be a simplicial category, $\mathcal{D}$ a small category and $D : \mathcal{D} \to \mathcal{K}$ a functor. Then the *simplicial homotopy colimit* $\operatorname{hocolim}_s D$ of D is defined as the colimit of D weighted by

$$B((- \downarrow \mathcal{D})^{op}) : \mathcal{D}^{op} \to \mathbf{SSet}\,.$$

The *simplicial homotopy limit* $\operatorname{holim}_s D$ of D is defined as the limit of D weighted by

$$B(\mathcal{D} \downarrow -) : \mathcal{D} \to \mathbf{SSet}\,.$$

Every simplicial category $\mathcal{K}$ has the *homotopy category* $\operatorname{Ho}(\mathcal{K})$; its objects are the same as that of $\mathcal{K}$ and

$$\hom_{\operatorname{Ho}(\mathcal{K})}(K, L) = \pi_0(\hom_{\mathcal{K}}(K, L)),$$

i.e., the set of morphisms from K to L in $\operatorname{Ho}(\mathcal{K})$ is the set of connected components of the simplicial set of morphisms from K to L in $\mathcal{K}$. A morphism of $\mathcal{K}$ is called a *homotopy equivalence* if it is an isomorphism in $\operatorname{Ho}(\mathcal{K})$. We will use the notation $K \simeq L$ for homotopy equivalent objects while $K \cong L$ will be kept for isomorphic objects.

In **SSet**, the just defined homotopy equivalences coincide with the usual ones. But the homotopy category of **SSet** in our sense is not the usual homotopy category of simplicial sets where isomorphisms are weak equivalences. In order to get the right homotopy category, one has to replace **SSet** by the simplicial category **S** of fibrant simplicial sets (i.e., of Kan complexes). Since homotopy equivalences coincide with weak equivalences here, simplicial $\operatorname{Ho}(\mathbf{S})$ is equivalent to the usual $\operatorname{Ho}(\mathbf{SSet})$.

Every simplicial functor $F : \mathcal{K} \to \mathcal{L}$ induces the functor

$$\operatorname{Ho}(F) : \operatorname{Ho}(\mathcal{K}) \to \operatorname{Ho}(\mathcal{L}).$$

Definition 2.2. A simplicial functor $F : \mathcal{K} \to \mathcal{L}$ is called a *weak equivalence* if

(1) the induced morphisms $\hom(K_1, K_2) \to \hom(F(K_1), F(K_2))$ are weak equivalences for all objects K_1 and K_2 of $\mathcal{K}$ and
(2) each object L of $\operatorname{Ho}(\mathcal{L})$ is isomorphic in $\operatorname{Ho}(\mathcal{L})$ to $\operatorname{Ho}(F)(K)$ for some object K of $\mathcal{K}$.

These weak equivalences are often called DK-equivalences because they were first described by Dwyer and Kan in [23]. They are a part of a model category structure on the category **SCat** of small simplicial categories and simplicial functors (see [9]). Fibrations are simplicial functors $F : \mathcal{C} \to \mathcal{D}$ satisfying two conditions (F1) and (F2) where the first one says that the simplicial maps $\hom(A, B) \to \hom(FA, FB)$ are fibrations of simplicial sets. In the special case when $\mathcal{D}$ is the terminal simplicial category, (F1) says that hom-sets $\hom(A,B)$ are fibrant simplicial sets. Since (F2) is automatic in this case, a small simplicial category $\mathcal{C}$ is fibrant in this model category structure if and only if it has all $\hom(A,B)$ fibrant.

Definition 2.3. A simplicial category $\mathcal{K}$ will be called *fibrant* if all its hom-objects $\hom(A,B)$ are fibrant simplicial sets.

For a simplicial model category $\mathcal{M}$, $\mathrm{Int}(\mathcal{M})$ will denote its full subcategory consisting of objects which are both cofibrant and fibrant. $\mathrm{Int}(\mathcal{M})$ is a fibrant simplicial category and its homotopy category $\mathrm{Ho}(\mathrm{Int}(\mathcal{M}))$ in the simplicial sense is equivalent to $\mathrm{Ho}(\mathcal{M})$ in the model category sense (see [25]). Recall that we have denoted $\mathrm{Int}(\mathbf{SSet})$ by $\mathbf{S}$. Fibrant simplicial categories coincide with categories enriched over $\mathbf{S}$.

The category $\mathbf{S}$ is closed in $\mathbf{SSet}$ under simplicial homotopy limits and under coproducts but it is not closed under simplicial homotopy colimits in general. In order to get an appropriate concept of a homotopy colimit for $\mathbf{S}$, we have to apply a fibrant replacement functor

$$R_f : \mathbf{SSet} \to \mathbf{S}$$

to the simplicial homotopy colimit. We will call this new homotopy colimit *fibrant* and denote it by $\mathrm{hocolim}_f$. Hence, given a diagram $D : \mathcal{D} \to \mathbf{S}$, we have

$$\mathrm{hocolim}_f D = R_f(\mathrm{hocolim}_s D).$$

This definition does not depend on a choice of a fibrant replacement functor because the resulting fibrant homotopy colimits are always homotopy equivalent. From the model category point of view, there is no difference between $\mathrm{hocolim}_s D$ and $\mathrm{hocolim}_f D$ because both objects are weakly equivalent.

Let $\mathcal{M}$ be an arbitrary simplicial model category and consider a diagram $D : \mathcal{D} \to \mathrm{Int}(\mathcal{M})$. We define its *fibrant homotopy colimit* $\mathrm{hocolim}_f D$ as $R_f(\mathrm{hocolim}_s D)$ where R_f is a fibrant replacement functor in $\mathcal{M}$. Since $\mathrm{hocolim}_s D$ is cofibrant (see [25], 18.5.2), its fibrant replacement is both fibrant and cofibrant. Analogously, we define a *fibrant homotopy limit* $\mathrm{holim}_f D$ as a cofibrant replacement $R_c(\mathrm{holim}_s D)$. Since contravariant hom-functors of fibrant objects preserve weak equivalences between cofibrant objects (see [25], 9.3.3), the simplicial sets $\hom(\mathrm{hocolim}_f D, A)$ and $\hom(\mathrm{hocolim}_s D, A)$ are weakly equivalent for any fibrant object A from $\mathcal{M}$. Since both of these simplicial sets are fibrant (see [25], 9.3.1.(2)), they are homotopy equivalent. We get that

$$\hom(\mathrm{hocolim}_f D, A) \simeq \hom(\mathrm{hocolim}_s D, A) \cong \mathrm{holim}_s \hom(D, A)$$

for any fibrant object A of $\mathcal{M}$. Analogously we obtain the formula

$$\hom(A, \operatorname{holim}_f D) \simeq \operatorname{holim}_s \hom(A, D)$$

for any cofibrant object A of $\mathcal{M}$.

Definition 2.4. Let $\mathcal{K}$ be a fibrant simplicial category, $\mathcal{D}$ a category and consider a diagram $D : \mathcal{D} \to \mathcal{K}$. We say that $\operatorname{holim}_f D$ is a *fibrant homotopy limit* of D if there are homotopy equivalences

$$\delta_A : \hom(A, \operatorname{holim}_f D) \to \operatorname{holim}_s \hom(A, D)$$

which are simplicially natural in A.

Analogously, we define *fibrant homotopy colimit* $\operatorname{hocolim}_f D$ of D by the existence of homotopy equivalences

$$\delta_A : \hom(\operatorname{hocolim}_f D, A) \to \operatorname{holim}_s \hom(D, A).$$

which are simplicially natural in A.

In particular, we have the formulas

$$\hom(A, \operatorname{holim}_f D) \simeq \operatorname{holim}_s \hom(A, D)$$

and

$$\hom(\operatorname{hocolim}_f D, A) \simeq \operatorname{holim}_s \hom(D, A).$$

We will see in 3.1(a) that $\operatorname{holim}_f D$ is determined uniquely up to a homotopy equivalence. In the case when $\mathcal{K} = \operatorname{Int}(\mathcal{M})$ for a simplicial model category $\mathcal{M}$, this definition coincides with the previous one.

Remark 2.5. By the enriched Yoneda lemma, the simplicial natural transformation δ in the definition of the fibrant homotopy limit is uniquely determined by the image of $\mathrm{id}_{\operatorname{holim}_f D}$ in the maping

$$\delta_{\operatorname{holim}_f D} : \hom(\operatorname{holim}_f D, \operatorname{holim}_f D) \to \operatorname{holim}_s \hom(\operatorname{holim}_f D, D).$$

This image uniquely corresponds to the morphism

$$\tilde{\delta} : B(\mathcal{D} \downarrow -) \to \hom(\operatorname{holim}_f D, D)$$

which can be understood as an analogy of the limit cone for a usual limit. We will sometimes denote fibrant homotopy limits as pairs $(\operatorname{holim}_f D, \tilde{\delta})$.

Analogously, the simplicial natural transformation δ in the definition of the fibrant homotopy colimit is uniquely determined by the morphism

$$\tilde{\delta} : B(- \downarrow \mathcal{D})^{\mathrm{op}} \to \hom(D, \operatorname{hocolim}_f D)$$

playing the rôle of a colimit cocone.

Definition 2.6. Let $F : \mathcal{K} \to \mathcal{L}$ be a simplicial functor between fibrant simplicial categories. We say that F *preserves* the fibrant homotopy limit of a diagram $D : \mathcal{D} \to \mathcal{K}$ if $(F \operatorname{holim}_f D, F\tilde{\delta})$ is a fibrant homotopy limit of FD.

Analogously we define the preservation of fibrant homotopy colimits.

Definition 2.7. Let $G : \mathcal{K} \to \mathcal{L}$ and $F : \mathcal{L} \to \mathcal{K}$ be simplicial functors between fibrant simplicial categories. We say that F is *homotopy left adjoint* to G if there are morphisms

$$\varphi_{K,L} : \hom(L, GK) \to \hom(FL, K)$$

and

$$\psi_{K,L} : \hom(FL, K) \to \hom(L, GK)$$

which are simplicially natural in K and L and such that $\psi_{K,L}$ is a homotopy inverse to $\varphi_{K,L}$ for each K in $\mathcal{K}$ and L in $\mathcal{L}$.

It implies that the induced functor $\operatorname{Ho}(F)$ is left adjoint to $\operatorname{Ho}(G)$.

3. Simplicial presheaves

Let $\mathcal{C}$ be a small simplicial category and consider the simplicial category $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ of simplicial functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$. We have the Yoneda embedding

$$Y_{\mathcal{C}} : \mathcal{C} \to \mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$$

given by $Y(C) = \hom(-, C)$. The category $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ has all weighted colimits and all weighted limits and the Yoneda embedding $Y_{\mathcal{C}}$ makes it the free completion of $\mathcal{C}$ under weighted colimits. It also preserves all existing weighted limits (cf. [29]). Dually,

$$\overline{Y}_{\mathcal{C}} = Y^{\mathrm{op}}_{\mathcal{C}^{\mathrm{op}}} : \mathcal{C} \to (\mathbf{SSet}^{\mathcal{C}})^{\mathrm{op}}$$

is the free completion of $\mathcal{C}$ under weighted limits and preserves all existing weighted colimits. These free completions exist for an arbitrary simplicial category – one has to take *small* simplicial functors into $\mathbf{SSet}$, i.e., small weighted (co)limits of hom-functors (see [19]).

For a small simplicial category $\mathcal{C}$, $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ is a simplicial combinatorial model category with respect to the projective (= Bousfield-Kan) model category structure. It means that weak equivalences and fibrations are pointwise. (Trivial) cofibrations are then described in the following way. We have the evaluation functors $E_C : \mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{SSet}$, $C \in \mathcal{C}$; $E_C(F) = F(C)$. They are precisely the hom-functors

$$E_C = \hom(\hom(-, C), -).$$

Each evaluation functor E_C has a simplicial left adjoint

$$F_C = - \otimes \hom(-, C).$$

Now, cofibrations are cofibrantly generated by images in F_C, $C \in \mathcal{C}$, of (generating) cofibrations in **SSet** and the same for trivial cofibrations. This procedure is described in [25], 11.6.1, for an ordinary category $\mathcal{C}$ and [17] extends it to the simplicial category of small simplicial functors $\mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$ for an arbitrary simplicial category $\mathcal{C}$. The consequence is that all hom-functors $\hom(-, C)$ are cofibrant.

Remark 3.1. (a) Let $\mathcal{K}$ be a fibrant simplicial category and assume that L_1 and L_2 are fibrant homotopy limits of a diagram $D : \mathcal{D} \to \mathcal{K}$. Let $\mathcal{K}_0$ be a small full subcategory of $\mathcal{K}$ containing both L_1 and L_2. Then the hom-functors $\hom(-, L_1)$ and $\hom(-, L_2)$ are weakly equivalent in the projective model category $\mathbf{SSet}^{\mathcal{K}_0^{\mathrm{op}}}$ with the functor $\operatorname{holim}_s \hom(-, D)$ restricted on $\mathcal{K}_0$. Since they are cofibrant and fibrant, they are homotopy equivalent and thus L_1 and L_2 are homotopy equivalent.

(b) More generally, assume that we have an object K in $\mathcal{K}$ and a morphism

$$k : B(\mathcal{D} \downarrow -) \to \hom(K, D).$$

In the same way as in 2.5, $k = \tilde{\alpha}$ for a simplicial natural transformation

$$\alpha : \hom(-, K) \to \operatorname{holim}_s \hom(-, D).$$

Let $\mathcal{K}_0$ be a small full subcategory of $\mathcal{K}$ containing both $\operatorname{holim}_f D$ and K. Let

$$\gamma : R_c H \to H$$

be a cofibrant replacement in $\mathbf{SSet}^{\mathcal{K}_0^{\mathrm{op}}}$ of the restriction H of the functor $\hom(-, \operatorname{holim}_f D)$ to $\mathcal{K}_0^{\mathrm{op}}$. Since hom-functors are cofibrant, there are simplicial natural transformations

$$\delta' : \hom(-, \operatorname{holim}_f D) \to R_c H$$

and

$$\alpha' : \hom(-, K) \to R_c H$$

such that $\delta = \gamma \cdot \delta'$ and $\alpha = \gamma \cdot \alpha'$. Since γ and δ are weak equivalences, δ' is a weak equivalence and thus a homotopy equivalence because both $\hom(-, \operatorname{holim}_f D)$ and $R_c H$ are cofibrant and fibrant. A homotopy inverse of δ' composed with α' gives a simplicial natural transformation

$$\bar{\alpha} : \hom(-, K) \to \hom(-, \operatorname{holim}_f D)$$

and thus a morphism $K \to \operatorname{holim}_f D$. This justifies our claim (cf. 2.5) that $\tilde{\delta}$ plays the rôle of a limit cone.

(c) Consider diagrams $D_1, D_2 : \mathcal{D} \to \mathcal{K}$ and a natural transformation $\varphi : D_1 \to D_2$. Then the composition

$$\alpha = \operatorname{holim}_s \hom(-, \varphi) \cdot \delta_1 : \hom(-, \operatorname{holim}_f D_1) \to \operatorname{holim}_s \hom(-, D_2)$$

induces (via (b)) a morphism

$$\operatorname{holim}_f \varphi : \operatorname{holim}_f D_1 \to \operatorname{holim}_f D_2.$$

If φ is a pointwise homotopy equivalence then α is a pointwise homotopy equivalence and thus α' and $\bar{\alpha}$ are pointwise homotopy equivalences. Hence $\operatorname{holim}_f \varphi$ is a homotopy equivalence.

We have shown that fibrant homotopy limits are homotopy invariant. Dually, the same is true for fibrant homotopy colimits.

Theorem 3.2. *Let $\mathcal{C}$ be a small simplicial category. Then every object of $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ is weakly equivalent to a simplicial homotopy colimit of hom-functors tensored with Δ_n, $n = 1, 2, \ldots$.*

Proof. Let $\mathcal{C}$ be a small simplicial category and $\hat{\mathcal{C}}$ be the full subcategory of $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ whose objects are functors $\Delta_n \otimes \hom(-, C)$ where $n = 0, 1, 2, \ldots$ and $C \in \mathcal{C}$. Then the codomain restriction

$$G : \mathcal{C} \to \hat{\mathcal{C}}$$

of the Yoneda embedding is a free completion of $\mathcal{C}$ under tensors with Δ_n, $n = 1, 2, \ldots$. There is a one-to-one correspondence between simplicial functors

$$A : \mathcal{C} \to \mathbf{SSet}^{\mathrm{op}}$$

and simplicial functors

$$A' : \hat{\mathcal{C}} \to \mathbf{SSet}^{\mathrm{op}}$$

preserving tensors with Δ_n, $n = 1, 2, \ldots$. It yields a full embedding

$$G_* : \mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}}$$

given by

$$G_*(A) = ((A^{\mathrm{op}})')^{\mathrm{op}}.$$

G_* makes $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ equivalent with the full subcategory of $\mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}}$ consisting of simplicial functors

$$B : \hat{\mathcal{C}}^{\mathrm{op}} \to \mathbf{SSet}$$

with

$$B(\Delta_n \otimes C) = B(C)^{\Delta_n}$$

for $n = 1, 2, \ldots$ and $C \in \mathcal{C}$. Moreover, G_* has a simplicial left adjoint G^* given by restrictions $G^*(B) = B \cdot G^{\mathrm{op}}$.

Consider the underlying ordinary category $\hat{\mathcal{C}}_0$ of $\hat{\mathcal{C}}$ and let

$$F : \hat{\mathcal{C}}_0 \to \hat{\mathcal{C}}$$

be the embedding. It yields a functor

$$F^* : \mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}} \to \mathbf{SSet}^{\hat{\mathcal{C}}_0^{\mathrm{op}}}$$

given by restrictions, i.e., $F^*(B) = BF^{\mathrm{op}}$. The functor F^* has a simplicial left adjoint

$$F_! : \mathbf{SSet}^{\hat{\mathcal{C}}_0^{\mathrm{op}}} \to \mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}}$$

which is the weighted colimit preserving functor induced by the composition

$$Y_{\hat{\mathcal{C}}} \cdot F : \hat{\mathcal{C}}_0 \to \mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}} .$$

Since each simplicial functor $B : \hat{\mathcal{C}}^{\mathrm{op}} \to \mathbf{SSet}$ from the image of G_* preserves cotensors with Δ_n, the action $B_{C,D,n}$ of the simplicial map

$$B_{C,D} : \hom(D, C) \to \hom(B(C), B(D))$$

on n-simplices is equal to the action $B_{C,\Delta_n \otimes D,0}$ of

$$B_{C,\Delta_n \otimes D} : \hom(\Delta_n \otimes D, C) \to \hom(B(C), B(\Delta_n \otimes D))$$

on points. Consequently, F^* is a full embedding on these simplicial functors B, which means that the composition

$$F^* G_* : \mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}} \to \mathbf{SSet}^{\hat{\mathcal{C}}_0^{\mathrm{op}}}$$

is a full embedding. Since this composition has a simplicial left adjoint $G^* F_!$, the category $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ is isomorphic to a reflective full subcategory of $\mathbf{SSet}^{\hat{\mathcal{C}}_0^{\mathrm{op}}}$.

We have

$$(G^* F_!)(\hom(-, \Delta_n \otimes C)) = G^*(\hom(-, \Delta_n \otimes D)) = \Delta_n \otimes \hom(-, C).$$

The second equation follows from the fact that the object $\Delta_n \otimes C$ in $\hat{\mathcal{C}}$ taken as the functor $\mathcal{C}^{\mathrm{op}} \to \mathbf{SSet}$ is precisely $\Delta_n \otimes \hom(-, C)$. Since, following [21], 2.6, each simplicial functor $\hat{\mathcal{C}}_0^{\mathrm{op}} \to \mathbf{SSet}$ is weakly equivalent to a simplicial homotopy colimit of hom-functors, it remains to prove that the composition $G^* F_!$ preserves weak equivalences. The functor G^* preserves weak equivalences because they are pointwise and G^* is given by restrictions. Since $F_!$ is a left Quillen functor, it preserves weak equivalences between cofibrant objects. We will show that it preserves all weak equivalences.

There is another simplicial functor

$$\tilde{F}^* : \mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}} \to \mathbf{SSet}^{\hat{\mathcal{C}}_0^{\mathrm{op}}}$$

with a simplicial left adjoint

$$\tilde{F}_! : \mathbf{SSet}^{\hat{\mathcal{C}}_0^{\mathrm{op}}} \to \mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}}$$

which preserves weak equivalences. These functors are described in [24], Proposition IX.2.10 (in a different notation because the inverse image F^* is denoted by F_* in [24]). In the proof of this proposition, there is found a pointwise homotopy equivalence

$$\varrho : F^* \to \tilde{F}^*;$$

it means that $\varrho_A : F^*(A) \to \tilde{F}^*(A)$ are homotopy equivalences for each A in $\mathbf{SSet}^{\hat{\mathcal{C}}^{\mathrm{op}}}$. The adjunction induces the morphism

$$\sigma : \tilde{F}_! \to F_!$$

such that

$$\hom(\sigma_B, A) \cdot \varphi_{A,B} = \tilde{\varphi}_{A,B} \cdot \hom(B, \varrho_A)$$

where

$$\varphi_{A,B} : \hom(B, F^*(A)) \to \hom(F_!(B), A)$$

and

$$\tilde{\varphi}_{A,B} : \hom(B, \tilde{F}^*(A)) \to \hom(\tilde{F}_!(B), A)$$

denote the adjunction isomorphisms. Since hom-functors both preserve and reflect homotopy equivalences, σ is a pointwise homotopy equivalence. Consequently, $F_!$ preserves weak equivalences. $\square$

We have extended [21], 2.9, from ordinary categories to simplicial categories $\mathcal{C}$.

Lemma 3.3. *Let $G : \mathcal{K} \to \mathcal{L}$ be a simplicial functor between fibrant simplicial categories and $F : \mathcal{L} \to \mathcal{K}$ its homotopy left adjoint. Then F preserves fibrant homotopy colimits and G preserves fibrant homotopy limits.*

Proof. Let $D : \mathcal{D} \to \mathcal{K}$ be a diagram. We get simplicial natural transformations

$$\hom(F \operatorname{hocolim}_f D, -) \to \hom(\operatorname{hocolim}_f D, G(-)),$$

$$\hom(\operatorname{hocolim}_f D, G(-)) \to \operatorname{holim}_s \hom(D, G(-))$$

and

$$\hom(\operatorname{hocolim}_f FD, -)) \to \operatorname{holim}_s \hom(FD, -)$$

whose components are homotopy equivalences. Since compatible weak equivalences between diagrams of fibrant objects induce a weak equivalence of their simplicial homotopy limits, the functors

$$\hom(F \operatorname{hocolim}_f D, -)$$

and

$$\hom(\operatorname{hocolim}_f FD, -)$$

are weakly equivalent and thus homotopy equivalent. This implies that F preserves fibrant homotopy colimits. The statement about G is dual. $\square$

Definition 3.4. Let $\mathcal{C}$ be a small fibrant simplicial category. We put

$$\operatorname{Pre}(\mathcal{C}) = \operatorname{Int}(\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}).$$

$\operatorname{Pre}(\mathcal{C})$ precisely consists of simplicial functors $\mathcal{C}^{op} \to \mathbf{S}$ which are cofibrant objects in the projective model category structure on $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$. Since $\mathcal{C}$ is fibrant, all hom-functors $\hom(-, C)$ belong to $\operatorname{Pre}(\mathcal{C})$ because they are always cofibrant in $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$. Thus we get the Yoneda embedding

$$Y_{\mathcal{C}} : \mathcal{C} \to \operatorname{Pre}(\mathcal{C}).$$

Theorem 3.5. *Let $\mathcal{C}$ be a small fibrant simplicial category. Then every object of* $\operatorname{Pre}(\mathcal{C})$ *is homotopy equivalent to a fibrant homotopy colimit of hom-functors.*

Proof. Let F belong to $\operatorname{Pre}(\mathcal{C})$. By 3.2, F is weakly equivalent to a simplicial homotopy colimit of functors $\Delta_n \otimes \hom(-, C)$. Thus F is cofibrant and, by applying the fibrant replacement functor R_f to this homotopy colimit and using [25], 18.5.3, we get that F is homotopy equivalent to a fibrant homotopy colimit of functors $R_f(\Delta_n \otimes \hom(-, C))$. Since the simplicial maps $u_n : \Delta_0 \to \Delta_n$ sending the unique point of Δ_0 to the point 0 in Δ_n are weak equivalences,

$$u_n \otimes \mathrm{id} : \Delta_n \otimes \hom(-, C) \to \Delta_0 \otimes \hom(-, C) \cong \hom(-, C)$$

are weak equivalences as well (see [25], 9.3.9 (1a)). Hence the morphisms $R_f(u_n \otimes \mathrm{id})$ are homotopy equivalences. We have proved that F is homotopy equivalent to a fibrant homotopy colimit of hom-functors. $\square$

Remark 3.6. We will show that both fibrant homotopy limits and fibrant homotopy colimits in $\operatorname{Pre}(\mathcal{C})$ are are pointwise. Consider a diagram $D : \mathcal{D} \to \operatorname{Pre}(\mathcal{C})$. Then we have

$$\begin{aligned} (\operatorname{holim}_f D)(C) &= \hom(\hom(-, C), \operatorname{holim}_f D) \\ &\simeq \operatorname{holim}_s \hom(\hom(-, C), D) \\ &\simeq \operatorname{holim}_s D(C) = \operatorname{holim}_f D(C). \end{aligned}$$

In the case of colimits, we have

$$(\operatorname{hocolim}_f D)(C) = (R_f \operatorname{hocolim}_s D)(C)$$

and

$$\operatorname{hocolim}_f D(C) = R_f(\operatorname{hocolim}_s D(C)) = R_f((\operatorname{hocolim}_s D)(C)).$$

Since weak equivalences are pointwise in $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$, $(R_f A)(C)$ is weakly equivalent to $R_f(A(C))$ for each A in $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$. Since fibrations are pointwise as well, the both simplicial sets are fibrant and thus they are homotopy equivalent. Hence

$$(\operatorname{hocolim}_f D)(C) \simeq \operatorname{hocolim}_f D(C).$$

Since

$$E_C = \hom(\hom(-, C), -),$$

hom-functors $\hom(-, C)$ are *homotopy absolutely presentable* in the sense that their hom-functors

$$\hom(\hom(-, C), -) : \operatorname{Pre}(\mathcal{C}) \to \mathbf{S}$$

preserve all fibrant homotopy colimits. Consequently, $\operatorname{Pre}(\mathcal{C})$ does not only have fibrant homotopy colimits and but also every object of $\operatorname{Pre}(\mathcal{C})$ is a fibrant homotopy colimit of homotopy absolutely presentable objects.

Definition 3.7. An object K of a fibrant simplicial category $\mathcal{K}$ is called *homotopy finitely presentable* provided that its hom-functor

$$\hom(K, -) : \mathcal{K} \to \mathbf{S}$$

preserves filtered fibrant homotopy colimits.

Recall that filtered homotopy colimits are homotopy colimits of diagrams $D : \mathcal{D} \to \mathcal{K}$ where $\mathcal{D}$ is a filtered category (cf. [2]) and finite homotopy limits are homotopy limits of diagrams $\mathcal{D} \to \mathcal{K}$ where $\mathcal{D}$ has finitely many morphisms.

Definition 3.8. A finite category $\mathcal{D}$ will be called *genuinely finite* if $B(\mathcal{D})$ is a finitely presentable simplicial set.

Proposition 3.9. *In* $\mathbf{S}$*, filtered fibrant homotopy colimits commute with genuinely finite fibrant homotopy limits.*

Proof. The statement means that, given a diagram $D : \mathcal{I} \times \mathcal{J} \to \mathbf{S}$ with $\mathcal{I}$ filtered and $\mathcal{J}$ genuinely finite, the canonical morphism

$$c : \operatorname*{hocolim}_{\mathcal{I}}{}_f \operatorname*{holim}_{\mathcal{J}}{}_f D(i,j) \to \operatorname*{holim}_{\mathcal{J}}{}_f \operatorname*{hocolim}_{\mathcal{I}}{}_f D(i,j)$$

is a homotopy equivalence. By [14], XII., 3.5(ii), filtered simplicial homotopy colimits are weakly equivalent to filtered colimits in $\mathbf{SSet}$. Since $\mathbf{S}$ is closed in $\mathbf{SSet}$ under filtered colimits, filtered fibrant homotopy colimits in $\mathbf{S}$ are homotopy equivalent with filtered colimits. Since each genuinely finite category $\mathcal{D}$ has all simplicial sets $B(\mathcal{D} \downarrow d)$, $d \in \mathcal{D}$ finitely presentable, the result is a consequence of the fact that filtered colimits commute with finite weighted limits in **SSet** (see [11]). $\square$

The published version of this paper claims that 3.9 is true for all finite homotopy limits. The author is grateful to Daniel Davis for a correspondence about this subject and for sending counter-examples due to Takeshi Torii. Later, a simple counter-example was found by Lukáš Vokřínek.

Proposition 3.10. *Let $\mathcal{K}$ be a fibrant simplicial category. Then a genuinely finite fibrant homotopy colimit of homotopy finitely presentable objects is homotopy finitely presentable.*

Proof. Let $J : \mathcal{J} \to \mathcal{K}$ a genuinely finite diagram with homotopy finitely presentable values. We have to prove that $\operatorname{hocolim}_f J$ is homotopy finitely presentable. Let $I : \mathcal{I} \to \mathcal{K}$ be a filtered diagram. Then, by 3.9, we have

$$\begin{aligned}
\hom(\operatorname{hocolim}_f J, \operatorname{hocolim}_f I) &\simeq \operatorname{holim}_f \hom(J, \operatorname{hocolim}_f I) \\
&\simeq \underset{\mathcal{J}}{\operatorname{holim}_f} \underset{\mathcal{I}}{\operatorname{hocolim}_f} \hom(J, I) \\
&\simeq \underset{\mathcal{I}}{\operatorname{hocolim}_f} \underset{\mathcal{J}}{\operatorname{holim}_f} \hom(J, I) \\
&\simeq \operatorname{hocolim}_f \hom(\operatorname{hocolim}_f J, I).
\end{aligned}$$

Thus $\operatorname{hocolim}_f J$ is homotopy finitely presentable. $\square$

4. Homotopy varieties

Consider a category $\mathcal{K}$ with binary products and diagrams $D_1 : \mathcal{D}_1 \to \mathcal{K}$ and $D_2 : \mathcal{D}_2 \to \mathcal{K}$. We form the diagram

$$D_1 \times D_2 : \mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{K}$$

by means of the formula $(D_1 \times D_2)(d_1, d_2) = D_1 d_1 \times D_2 d_2$ (do not confuse it with the product functor $\mathcal{D}_1 \times \mathcal{D}_2 \to \mathcal{K} \times \mathcal{K}$).

Definition 4.1. Let $\mathcal{K}$ be a fibrant simplicial category having fibrant homotopy colimits and binary products. We say that *fibrant homotopy colimits distribute over binary products* in $\mathcal{K}$ provided that

$$\operatorname{hocolim}_f(D_1 \times D_2) \simeq \operatorname{hocolim}_f D_1 \times \operatorname{hocolim}_f D_2$$

for every pair of diagrams $D_1 : \mathcal{D}_1 \to \mathcal{K}$ and $D_2 : \mathcal{D}_2 \to \mathcal{K}$.

Proposition 4.2. *In* **S**, *fibrant homotopy colimits distribute over binary products.*

Proof. Consider diagrams $D_1 : \mathcal{D}_1 \to \mathbf{S}$ and $D_2 : \mathcal{D}_2 \to \mathbf{S}$. Since the functor $- \times Y : \mathbf{SSet} \to \mathbf{SSet}$ has the simplicial right adjoint $\hom(Y, -)$, it preserves simplicial homotopy colimits. Thus simplicial homotopy colimits distribute over binary products in $\mathbf{SSet}$. Hence it suffices to know that a product $w_1 \times w_2$ of weak equivalences w_1 and w_2 in $\mathbf{SSet}$ is a weak equivalence. This fact can be deduced as follows.

Since the geometric realization functor $|\,| : \mathbf{SSet} \to \mathbf{K}$ to the full subcategory $\mathbf{K}$ of the category of topological spaces consisting of compactly generated spaces preserves finite limits (see [26], 3.2.4), we have $|w_1 \times w_2| = |w_1| \times |w_2|$. Now w is a weak equivalence in $\mathbf{SSet}$ iff $|w|$ is a weak equivalence in $\mathbf{K}$ and the product of weak equivalences in $\mathbf{K}$ is a weak equivalence by [25], 18.5.3 (because products are homotopy limits and all objects are fibrant in $\mathbf{K}$). $\square$

Definition 4.3. A small category $\mathcal{D}$ will be called *homotopy sifted* provided that fibrant homotopy colimits over $\mathcal{D}$ commute in $\mathbf{S}$ with finite products.

Explicitly, $\mathcal{D}$ is homotopy sifted iff it is nonempty (thus fibrant homotopy colimits over $\mathcal{D}$ commute with the empty product) and, given diagrams $D_1, D_2 : \mathcal{D} \to \mathbf{S}$, then the canonical morphism

$$\operatorname{hocolim}_f(D_1 \otimes D_2) \to \operatorname{hocolim}_f D_1 \times \operatorname{hocolim}_f D_2$$

is a homotopy equivalence. Here, the diagram $D_1 \otimes D_2 : \mathcal{D} \to \mathbf{S}$ is given by

$$(D_1 \otimes D_2)(d) = D_1 d \times D_2 d.$$

In fact, $D_1 \otimes D_2$ is the product of D_1 and D_2 in $\mathbf{SSet}^{\mathcal{D}}$.

The following theorem is analogous to the characterization of sifted colimits (see [3]). Recall that a functor $F : \mathcal{K} \to \mathcal{L}$ is called *homotopy final* provided that for every object L of $\mathcal{L}$ the comma-category $L \downarrow F$ is aspherical, i.e., its nerve $B(L \downarrow F)$ is weakly equivalent to the point (see [25], 19.6.1). Every homotopy final functor is final because the latter means that all comma-categories $L \downarrow F$ are non-empty and connected.

Theorem 4.4. *A small category $\mathcal{D}$ is homotopy sifted iff $\mathcal{D}$ is nonempty and the diagonal functor $\Delta : \mathcal{D} \to \mathcal{D} \times \mathcal{D}$ is homotopy final.*

Proof. Given diagrams $D_1, D_2 : \mathcal{D} \to \mathbf{S}$, we have

$$D_1 \otimes D_2 = (D_1 \times D_2)\Delta.$$

By [25], 19.6.7 and 19.6.12, Δ is homotopy final iff the induced map

$$\operatorname{hocolim}_s D\Delta \to \operatorname{hocolim}_s D$$

is a weak equivalence for every diagram $D : \mathcal{D} \times \mathcal{D} \to \mathbf{SSet}$. This is clearly the same as

$$\operatorname{hocolim}_f D\Delta \to \operatorname{hocolim}_f D$$

being a homotopy equivalence for every diagram $D : \mathcal{D} \times \mathcal{D} \to \mathbf{S}$. Consequently, $\mathcal{D}$ is homotopy sifted provided that Δ is homotopy final. Conversely, since the proof of [25], 19.6.12 only uses functors

$$D = \hom((d_1, d_2), -) = \hom(d_1, -) \times \hom(d_2, -),$$

Δ is homotopy final whenever $\mathcal{D}$ is homotopy sifted. □

Remark 4.5. (a) A category $\mathcal{D}$ is homotopy sifted iff all comma-categories $(d_1, d_2) \downarrow \Delta$, where d_1, d_2 are objects from $\mathcal{D}$, are aspherical. Hence $\mathcal{D}$ is homotopy sifted iff $\mathcal{D}^{\mathrm{op}}$ is totally aspherical in the sense of [36], 1.6.3.

(b) By 3.9, each filtered category is homotopy sifted. But it also follows from the fact that every filtered category $\mathcal{D}$ is aspherical because it is a filtered colimit of categories $d \downarrow \mathcal{D}$ having the initial object (see [38]).

(c) Every category $\mathcal{D}$ with finite coproducts is homotopy sifted (see [36], 7.4). It immediately follows from the fact that $d_1 \amalg d_2$ is the initial object in $(d_1, d_2) \downarrow \mathcal{D}$.

(d) Every homotopy sifted category is sifted because Δ is final provided that it is homotopy final.

(e) Recall that a *reflexive coequalizer* is defined a coequalizer of a pair of morphisms $h, k : A \to B$ which have a common section $m : B \to A$, i.e., such that $hm = km = \mathrm{id}_B$; such pairs are called *reflexive* (see [3]). A *fibrant homotopy reflexive coequalizer* is defined as a fibrant homotopy coequalizer of a reflexive pair. Reflexive coequalizers form an important kind of sifted categories (see [3]). But they are not homotopy sifted – a direct inspection shows that the comma category $(A, A) \downarrow \mathcal{D}$ is not aspherical (it is connected but not 2-connected); $\mathcal{D}$ denotes a reflexive pair.

(f) The reflexive pair is the full subcategory of the category $\mathbf{\Delta}^{\mathrm{op}}$ consisting of ordinals $1, 2$. The whole category $\mathbf{\Delta}^{\mathrm{op}}$ is homotopy sifted following [36], 1.6.13. Fibrant homotopy colimits of diagrams over $\mathbf{\Delta}^{\mathrm{op}}$ correspond to geometric realization of simplicial objects in [35].

Definition 4.6. An object K of a fibrant simplicial category $\mathcal{K}$ is called *homotopy strongly finitely presentable* provided that its hom-functor $hom(K, -) : \mathcal{K} \to \mathbf{S}$ preserves homotopy sifted fibrant homotopy colimits.

Remark 4.7. By 4.5(b), every homotopy strongly finitely presentable object is homotopy finitely presentable.

Proposition 4.8. *A finite coproduct of homotopy strongly finitely presentable objects is homotopy strongly finitely presentable.*

Proof. The proof is analogous to that of 3.10. □

Proposition 4.9. *Let $G : \mathcal{K} \to \mathcal{L}$ be a simplicial functor between fibrant simplicial categories which has a homotopy left adjoint $F : \mathcal{L} \to \mathcal{K}$. Then F preserves homotopy strongly finitely presentable objects provided that G preserves homotopy sifted fibrant homotopy colimits.*

Proof. Assume that G preserves homotopy sifted fibrant homotopy colimits. We have to show that for each homotopy strongly finitely presentable object L of $\mathcal{L}$ the object FL is homotopy strongly finitely presentable as well.

Let $\mathcal{D}$ be a homotopy sifted category and consider a diagram $D : \mathcal{D} \to \mathcal{K}$. We have

$$\begin{aligned} \hom(FL, \operatorname{hocolim}_f D) &\simeq \hom(L, G(\operatorname{hocolim}_f D)) \\ &\simeq \hom(L, \operatorname{hocolim}_f GD) \\ &\simeq \operatorname{hocolim}_f \hom(L, GD) \\ &\simeq R_f \operatorname{hocolim}_s \hom(L, GD) \\ &\simeq R_f \operatorname{hocolim}_s \hom(FL, D) \\ &\simeq \operatorname{hocolim}_f \hom(FL, D). \end{aligned}$$

Hence FL is homotopy strongly finitely presentable in $\mathcal{K}$. □

Definition 4.10. A fibrant simplicial category $\mathcal{K}$ will be called a *homotopy variety* provided that it has fibrant homotopy colimits and has a set $\mathcal{A}$ of homotopy strongly finitely presentable objects such that every object of $\mathcal{K}$ is a homotopy sifted fibrant homotopy colimit of objects from $\mathcal{A}$.

Proposition 4.11. *Let $\mathcal{C}$ be a small fibrant simplicial category. Then the category* $\operatorname{Pre}(\mathcal{C})$ *is a homotopy variety.*

Proof. Let $\bar{\mathcal{C}}$ be the closure of $Y(\mathcal{C})$ under finite coproducts in $\operatorname{Pre}(\mathcal{C})$. By 3.6 and 4.8, each object of $\bar{\mathcal{C}}$ is homotopy strongly finitely presentable in $\operatorname{Pre}(\mathcal{C})$. For each object A in $\operatorname{Pre}(\mathcal{C})$, the comma-category $\bar{\mathcal{C}} \downarrow A$ has finite coproducts. By 4.5(c), $\bar{\mathcal{C}} \downarrow A$ is homotopy sifted. Since A is the fibrant homotopy colimit of the projection $\bar{\mathcal{C}} \downarrow A \to \operatorname{Pre}(\mathcal{C})$ (see 3.5), the category $\operatorname{Pre}(\mathcal{C})$ is a homotopy variety. □

Definition 4.12. A *simplicial algebraic theory* is defined as a small fibrant simplicial category $\mathcal{T}$ having finite products.

A *homotopy $\mathcal{T}$-algebra* is a simplicial functor $A : \mathcal{T} \to \mathbf{S}$ belonging to $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ such that the canonical morphism

$$A(X_1 \times \cdots \times X_n) \to A(X_1) \times \cdots \times A(X_n)$$

is a homotopy equivalence for each finite product $X_1 \times \cdots \times X_n$ in $\mathcal{T}$.

We will denote by $\mathrm{HAlg}(\mathcal{T})$ the full subcategory of $\mathrm{Pre}(\mathcal{T}^{op})$ consisting of all homotopy $\mathcal{T}$-algebras.

Example 4.13. Let $\mathcal{T}_0$ be the algebraic theory of one binary operation m. It means that $\mathcal{T}_0$ has objects $X_0, X_1, \dots, X_n, \dots$ and morphisms are generated by $m : X_2 = X_1 \times X_1 \to X_1$. Then a $\mathcal{T}_0$-algebra A is a simplicial set $A(X_1)$ equipped with a binary operation $A(m) : A(X_1) \times A(X_1) \to A(X_1)$. Let $\mathcal{T}_1$ be the simplicial algebraic theory obtained from $\mathcal{T}_0$ by adding a one-dimensional simplex to $\hom(X_3, X_1)$ from the point $m(m \times \mathrm{id})$ to $m(\mathrm{id} \times m)$. It means that we have the corresponding simplicial map

$$h : \Delta_1 \to \hom(X_3, X_1).$$

Given a $\mathcal{T}_1$-algebra A, we get the composition

$$\Delta_1 \to \hom(X_3, X_1) \to \hom(A(X_1)^3, A(X_1))$$

of h with A_{X_3,X_1}. This composition corresponds to the simplicial map

$$\Delta_1 \times A(X_3) \to A(X_1)$$

which is a homotopy from $A(m)(A(m) \times \mathrm{id})$ to $A(m)(\mathrm{id} \times A(m))$. In this way we can get strongly homotopy associative algebras of [41] as algebras for a suitable simplicial algebraic theory. Homomorphisms of these algebras strictly preserve the multiplication.

On the other hand, if $\mathcal{T}_2$ is the algebraic theory of one associative binary operation then homotopy $\mathcal{T}_2$-algebras are simplicial sets equipped with a homotopy associative multiplication and homomorphisms preserve the operation up to homotopy.

Proposition 4.14. *Let $\mathcal{T}$ be a simplicial algebraic theory. Then the simplicial category $\mathrm{HAlg}(\mathcal{T})$ is closed in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ both under fibrant homotopy limits and homotopy sifted fibrant homotopy colimits.*

Proof. Consider a diagram $D : \mathcal{D} \to \mathrm{HAlg}(\mathcal{T})$. Since fibrant homotopy limits in $\mathrm{Pre}(\mathcal{C})$ are pointwise (see 3.6), we have

$$\begin{aligned}(\operatorname*{holim}_{d}{}_f Dd)(X_1 \times \cdots \times X_n) &\simeq \operatorname*{holim}_{d}{}_f Dd(X_1 \times \cdots \times X_n)\\ &\simeq \operatorname*{holim}_{d}{}_f Dd(X_1) \times \cdots \times \operatorname*{holim}_{d}{}_f Dd(X_n).\end{aligned}$$

Thus $\mathrm{HAlg}(\mathcal{T})$ is closed in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ under fibrant homotopy limits. Since homotopy sifted fibrant homotopy colimits commute in $\mathbf{S}$ with finite products, we analogously prove that $\mathrm{HAlg}(\mathcal{T})$ is closed in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ under homotopy sifted fibrant homotopy colimits. $\square$

We are now in a position to characterize simplicial categories which are weakly equivalent (in the sense of Definition 2.2) to some $\mathrm{HAlg}(\mathcal{T})$.

Theorem 4.15. *A fibrant simplicial category $\mathcal{K}$ is a homotopy variety if and only if it is weakly equivalent to* $\mathrm{HAlg}(\mathcal{T})$ *for some simplicial algebraic theory $\mathcal{T}$.*

Proof. I. Let $\mathcal{T}$ be a simplicial algebraic theory. Consider a finite product diagram

$$p_i : X_1 \times \cdots \times X_n \to X_i \quad i = 1, \dots, n$$

in $\mathcal{T}$. Let

$$m_{X_1 \dots X_n} : \hom(X_1, -) \amalg \cdots \amalg \hom(X_n, -) \to \hom(X_1 \times \cdots \times X_n, -)$$

be the morphism induced by

$$\hom(p_i, -) : \hom(X_i, -) \to \hom(X_1 \times \cdots \times X_n, -).$$

Let $A : \mathcal{C} \to \mathbf{S}$ be a functor belonging to $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$. Since

$$\hom(\hom(X_1 \times \cdots \times X_n, -), A) \cong A(X_1 \times \cdots \times X_n)$$

and

$$\hom(\hom(X_1, -) \amalg \cdots \amalg \hom(X_n, -), A) \cong A(X_1) \times \cdots \times A(X_n),$$

the functor A is a homotopy $\mathcal{T}$-algebra iff $\hom(m_{X_1 \dots X_n}, A)$ is a homotopy equivalence for each finite product diagram in $\mathcal{T}$.

Let $\mathcal{Z}$ be the set of all morphisms $m_{X_1 \dots X_n}$. Recall that an object A of $\mathbf{SSet}^{\mathcal{T}}$ is *homotopy orthogonal* to $\mathcal{Z}$ if

$$\mathrm{map}(m_{X_1 \dots X_n}, A)$$

is a weak equivalence for each $m_{X_1 \dots X_n}$ from $\mathcal{Z}$ (see [25], 17.8.5). Here, $\mathrm{map}(B, A)$ denotes a homotopy function complex. Let $\mathcal{Z}^{\perp}$ be the full subcategory of $\mathbf{SSet}^{\mathcal{T}}$ consisting of all fibrant objects homotopy orthogonal to $\mathcal{Z}$. Since $\mathrm{map}(B, A)$ is weakly equivalent to $\hom(B, A)$ whenever B is cofibrant and A is fibrant and all morphisms from $\mathcal{Z}$ have cofibrant domains and codomains, we have

$$\mathrm{HAlg}(\mathcal{T}) = \mathrm{Pre}(\mathcal{T}^{\mathrm{op}}) \cap \mathcal{Z}^{\perp}.$$

By [16], 1.1, there is a functor $L : \mathbf{SSet}^{\mathcal{T}} \to \mathcal{Z}^{\perp}$ preserving weak equivalences and equipped with a simplicial natural transformation

$$\eta : \mathrm{Id} \to L$$

which is idempotent up to homotopy and, moreover, $\mathrm{map}(\eta_K, M)$ is a weak equivalence for all K in $\mathbf{SSet}^{\mathcal{T}}$ and M in $\mathcal{Z}^{\perp}$.

Consider a diagram $D : \mathcal{D} \to \mathrm{HAlg}(\mathcal{T})$ and a $\mathcal{T}$-algebra A. We have (where R_c denotes a cofibrant replacement functor in $\mathbf{SSet}^{\mathcal{T}}$)

$$\begin{aligned} \hom(R_c L(\mathrm{hocolim}_f\, D), A) &\simeq \mathrm{map}(R_c L(\mathrm{hocolim}_f\, D), A) \\ &\simeq \mathrm{map}(L(\mathrm{hocolim}_f\, D), A) \\ &\simeq \mathrm{map}(\mathrm{hocolim}_f\, D, A) \\ &\simeq \hom(\mathrm{hocolim}_f\, D, A) \\ &\simeq \mathrm{holim}_s \hom(D, A). \end{aligned}$$

Thus $R_c L(\mathrm{hocolim}_f\, D)$ is a fibrant homotopy colimit of D in $\mathrm{HAlg}(\mathcal{T})$. Hence $\mathrm{HAlg}(\mathcal{T})$ has fibrant homotopy colimits.

Since $\mathrm{HAlg}(\mathcal{T})$ is closed in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ under homotopy sifted fibrant homotopy colimits (see 4.14) and hom-functors are homotopy absolutely presentable in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ (see 3.6), hom-functors are homotopy strongly finitely presentable in $\mathrm{HAlg}(\mathcal{T})$. By repeating the argument from the proof of 4.11, we show that $\mathrm{HAlg}(\mathcal{T})$ is a homotopy variety.

II. Let $\mathcal{K}$ be a homotopy variety and $\mathcal{A}$ be a set from 4.10. Let $\bar{\mathcal{A}}$ be the closure of $\mathcal{A}$ (considered as the full subcategory of $\mathcal{K}$) under finite coproducts in $\mathcal{K}$. By 4.8, each object of $\bar{\mathcal{A}}$ is homotopy strongly finitely presentable in $\mathcal{K}$. Put $\mathcal{T} = (\bar{\mathcal{A}})^{op}$. Then $\mathcal{T}$ is a simplicial algebraic theory. Let

$$E : \mathcal{K} \to \mathbf{SSet}^{\mathcal{T}}$$

be the simplicial functor given by

$$E(K) = \hom(-, K)$$

where the hom-functor is restricted to $\bar{\mathcal{A}}$. Since $\mathcal{K}$ is fibrant, E has fibrant values. Let K be an object of $\mathcal{K}$ and express it as a homotopy sifted fibrant homotopy colimit of a diagram $D : \mathcal{D} \to \mathcal{A}$. Then, for each A in $\bar{\mathcal{A}}$ we have

$$\begin{aligned} E(K)(A) &= \hom(A, \mathrm{hocolim}_f\, D) \simeq \mathrm{hocolim}_f \hom(A, D) \\ &= R_f\, \mathrm{hocolim}_s \hom(A, D) = R_f\, \mathrm{hocolim}_s\, ED(A) \end{aligned}$$

and

$$(\mathrm{hocolim}_f\, ED)(A) = (R_f\, \mathrm{hocolim}_s\, ED)(A) \simeq R_f\, \mathrm{hocolim}_s\, ED(A)$$

(see 3.6 for the last step). Hence E preserves homotopy sifted fibrant homotopy colimits of objects from $\bar{\mathcal{A}}$. It implies that E has cofibrant values as well and that the codomain restriction of E is the functor

$$\mathcal{K} \to \mathrm{HAlg}(\mathcal{T})$$

which, by 3.5, satisfies condition (2) from 2.2. We will show that

$$E : \mathcal{K} \to \mathrm{HAlg}(\mathcal{T})$$

is a weak equivalence.

Consider objects K_1 and K_2 from $\mathcal{K}$ and express them as homotopy sifted fibrant homotopy colimits $K_i = \mathrm{hocolim}_f\, D_i$ of $D_i : \mathcal{D}_i \to \mathcal{A}$ where $i = 1, 2$. Then we have

$$\begin{aligned}
\hom(K_1, K_2) &\simeq \hom(\mathrm{hocolim}_f\, D_1, \mathrm{hocolim}_f\, D_2)\\
&\simeq \mathrm{holim}_s \hom(D_1, \mathrm{hocolim}_f\, D_2)\\
&\simeq \mathrm{holim}_s\, \mathrm{hocolim}_f \hom(D_1, D_2)\\
&\simeq \mathrm{holim}_s\, \mathrm{hocolim}_f \hom(\hom(-, D_1), \hom(-, D_2))\\
&\simeq \mathrm{holim}_s \hom(\hom(-, D_1), \mathrm{hocolim}_f \hom(-, D_2))\\
&\simeq \hom(\mathrm{hocolim}_f \hom(-, D_1), \mathrm{hocolim}_f \hom(-, D_2))\\
&\simeq \hom(\hom(-, \mathrm{hocolim}_f\, D_1), \hom(-, \mathrm{hocolim}_f\, D_2))\\
&\simeq \hom(EK_1, EK_2).
\end{aligned}$$

Here, we have used the homotopy invariance of simplicial homotopy colimits, the enriched Yoneda lemma, the homotopy absolute presentability of hom-functors in $\mathrm{Pre}(\bar{\mathcal{A}})$ (see 3.6) and homotopy strong finite presentability of objects from $\mathcal{A}$. Hence E satisfies condition (1) from 2.2. □

Definition 4.16. Let $\mathcal{C}$ be a small fibrant simplicial category. Then $\mathrm{HSind}(\mathcal{C})$ will denote the full subcategory of $\mathrm{Pre}(\mathcal{C})$ consisting of homotopy sifted fibrant homotopy colimits of hom-functors.

Theorem 4.17. *Let $\mathcal{C}$ be a small fibrant simplicial category having finite coproducts. Then the simplicial categories* $\mathrm{HAlg}(\mathcal{C}^{op})$ *and* $\mathrm{HSind}(\mathcal{C})$ *are weakly equivalent.*

Proof. Since homotopy sifted fibrant homotopy colimits commute with finite products in $\mathbf{S}$, we always have

$$\mathrm{HSind}(\mathcal{C}) \subseteq \mathrm{HAlg}(\mathcal{C}^{op}).$$

Conversely, we know that each object from $\mathrm{HAlg}(\mathcal{C}^{\mathrm{op}})$ is a homotopy sifted fibrant homotopy colimit of finite coproducts of hom-functors (see the proof of 4.15). Since $L(m_{X_1 \dots X_n})$ is a weak equivalence for each morphism $m_{X_1\dots X_n}$ from this proof (see [20], 1.C.5), each object from $\mathrm{HAlg}(\mathcal{C}^{\mathrm{op}})$ is homotopy equivalent to an object from $\mathrm{HSind}(\mathcal{C})$. $\square$

As a consequence, we get that $\mathrm{HSind}(\mathcal{C})$ has all homotopy sifted fibrant homotopy colimits. $\mathrm{HSind}(\mathcal{C})$ is analogous to the free completion $\mathrm{Sind}(\mathcal{C})$ of a category $\mathcal{C}$ under sifted colimits introduced in [3].

Let $\mathcal{T}$ be a simplicial algebraic theory. Consider the left Bousfield localization of the projective model category structure on $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ with respect to the set $\mathcal{Z}$ from the proof of 4.15. The resulting model category will be called the *model category for homotopy $\mathcal{T}$-algebras.* It was considered in [8] in the case of an ordinary algebraic theory. There is proved in [5] and [8] that each homotopy $\mathcal{T}$-algebra is weakly equivalent to a strict $\mathcal{T}$-algebra in this model category structure.

As a consequence of 4.15 we get the following characterization of model categories for homotopy algebras.

Corollary 4.18. *A simplicial model category $\mathcal{M}$ is Quillen equivalent to the model category for homotopy $\mathcal{T}$-algebras for some simplicial algebraic theory $\mathcal{T}$ if and only if* $\mathrm{Int}(\mathcal{M})$ *is a homotopy variety.*

Proof. Let $\mathcal{T}$ be a simplicial algebraic theory and $\mathcal{M}$ the model category for homotopy $\mathcal{T}$-algebras. Then $\mathcal{M}$ is simplicial and $\mathrm{Int}(\mathcal{M}) = \mathrm{HAlg}(\mathcal{T})$ is a homotopy variety. Conversely, let $\mathcal{M}$ be a simplicial model category such that $\mathrm{Int}(\mathcal{M})$ is a homotopy variety. Let $\mathcal{A}$ be a set from 4.10 considered as the full subcategory of $\mathcal{M}$ and put $\mathcal{T} = \mathcal{A}^{\mathrm{op}}$. Then $\mathcal{T}$ is a simplicial algebraic theory. Since $\mathbf{SSet}^{\mathcal{A}}$ is the free completion of $\mathcal{T}$ under weighted colimits and $\mathcal{M}$ has all weighted colimits (cf. [10], 6.6.14), there is a unique simplicial functor

$$F : \mathbf{SSet}^{\mathcal{A}} \to \mathcal{M}$$

such that $FY_{\mathcal{T}}$ is the embedding of $\mathcal{A}$ to $\mathcal{M}$. Moreover, F is simplicially left adjoint to

$$E : \mathcal{M} \to \mathbf{SSet}^{\mathcal{A}}$$

where $E(M)$ is the restriction of $\hom(-, M)$ to $\mathcal{A}$. E is a right Quillen functor because, for a (trivial) fibration $h : M_1 \to M_2$,

$$\hom(A, h) : \hom(A, M_1) \to \hom(A, M_2)$$

is a (trivial) fibration for each A from $\mathrm{Int}(\mathcal{M})$ (see [25], 9.3.1 and 9.3.2). We know from the second part of the proof of 4.15 that E induces a weak equivalence of simplicial categories $\mathrm{Int}(\mathcal{M})$ and $\mathrm{HAlg}(\mathcal{T})$. Consequently, E induces an equivalence of their homotopy categories, which implies that E is a Quillen equivalence. $\square$

Using 3.6, one gets the following results which are analogous to 4.15 and 4.18.

Theorem 4.19. *A fibrant simplicial category $\mathcal{K}$ is weakly equivalent to* $\mathrm{Pre}(\mathcal{C})$ *for some small fibrant simplicial category $\mathcal{C}$ if and only if it has fibrant homotopy colimits and has a set $\mathcal{A}$ of homotopy absolutely presentable objects such that every object of $\mathcal{K}$ is a fibrant homotopy colimit of objects from $\mathcal{A}$.*

Corollary 4.20. *A simplicial model category $\mathcal{M}$ is Quillen equivalent to the model category* $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ *for some small fibrant simplicial category $\mathcal{C}$ if and only* $\mathrm{Int}(\mathcal{M})$ *has a set $\mathcal{A}$ of homotopy absolutely presentable objects such that every object of* $\mathrm{Int}(\mathcal{M})$ *is a fibrant homotopy colimit of objects from $\mathcal{A}$.*

5. Homotopy locally finitely presentable categories

Definition 5.1. A fibrant simplicial category $\mathcal{K}$ will be called *homotopy locally finitely presentable* provided that it has fibrant homotopy colimits and has a set $\mathcal{A}$ of homotopy finitely presentable objects such that every object of $\mathcal{K}$ is a filtered fibrant homotopy colimit of objects from $\mathcal{A}$.

Proposition 5.2. *Let $\mathcal{C}$ be a small fibrant simplicial category. Then the category* $\mathrm{Pre}(\mathcal{C})$ *is homotopy locally finitely presentable.*

Proof. Following 3.5, each object K in $\mathrm{Pre}(\mathcal{C})$ is a fibrant homotopy colimit of objects from $\mathcal{C}$. Since a fibrant homotopy colimit can be expressed as a filtered fibrant homotopy colimit of finite fibrant homotopy colimits, K is a filtered fibrant homotopy colimit of finite fibrant homotopy colimits of objects from $\mathcal{C}$.

Let $B_n(\mathcal{X})$ be the n-truncated nerve of a category $\mathcal{X}$, which is the simplicial subset of $B(\mathcal{X})$ containing all non-degenerated simplices Δ_k, $k \leq n$ of $B(\mathcal{X})$ and all degenerated ones. In fact, $B_n(\mathcal{X})$ is a retract of $B(\mathcal{X})$ and and $B(\mathcal{X})$ is a filtered colimit of $B_n(\mathcal{X})$, $n = 1, 2 \dots$. Consider a finite diagram $\mathcal{D} \to \mathcal{K}$. Then the weight $G = B((- \downarrow \mathcal{D})^{op})$ giving the simplicial homotopy colimit of D is a filtered colimit of weights $G_n = B_n((- \downarrow \mathcal{D})^{op})$, $n = 1, 2 \dots$. Let $\mathrm{colim}_{G_n} D$ be the colimit of D weighted by G_n, $n = 1, 2 \dots$. Since each $B_n(\mathcal{X})$ is a retract of $B(\mathcal{X})$, $\mathrm{colim}_{G_n} D$ is cofibrant and $\mathrm{colim}_G D$ is a colimit of the chain of $\mathrm{colim}_{G_n} D$, $n = 1, 2, \dots$. Since the replacement functor R_f preserves filtered colimits, $M = \mathrm{hocolim}_f D$ is a colimit of the chain of objects $M_n = R_f(\mathrm{colim}_{G_n} D)$ belonging to $\mathrm{Pre}(\mathcal{C})$. We will show that each object M_n is homotopy finitely presentable in $\mathrm{Pre}(\mathcal{C})$.

Let $J : \mathcal{J} \to \mathcal{K}$ be a filtered diagram. We have

$$\begin{aligned}
\hom(M_n, \operatorname{hocolim}_f J) &\simeq \hom(\operatorname{colim}_{G_n} D, \operatorname{hocolim}_f J)\\
&\simeq \hom(\operatorname{colim}_{G_n} D, \operatorname{colim} J)\\
&\cong \lim_{G_n} \hom(D, \operatorname{colim} J)\\
&\cong \lim_{G_n} \operatorname{colim} \hom(D, J)\\
&\cong \operatorname{colim} \lim_{G_n} \hom(D, J)\\
&\simeq \operatorname{colim} \hom(\operatorname{colim}_{G_n} D, J)\\
&\simeq \operatorname{hocolim}_f \hom(\operatorname{colim}_{G_n} D, J)\\
&\simeq \operatorname{hocolim}_f \hom(M_n, J).
\end{aligned}$$

Here, we used the fact that simplicial sets $B_n((- \downarrow \mathcal{D})^{op})$ are finitely presentable.

We have proved that K is a fibrant homotopy filtered colimit of fibrant homotopy filtered colimits of homotopy finitely presentable objects M_n. It is easy to see that K is a filtered fibrant homotopy colimit of M_n. □

Definition 5.3. A *finite homotopy limit theory* is defined as a small fibrant simplicial category $\mathcal{T}$ having all genuinely finite fibrant homotopy limits.

A *homotopy $\mathcal{T}$-model* is a simplicial functor $A : \mathcal{T} \to \mathbf{S}$ belonging to $\operatorname{Pre}(\mathcal{T}^{\mathrm{op}})$ and preserving genuinely finite fibrant homotopy limits.

We will denote by $\operatorname{HMod}(\mathcal{T})$ the full subcategory of $\operatorname{Pre}(\mathcal{T}^{\mathrm{op}})$ consisting of all homotopy $\mathcal{T}$-models.

Proposition 5.4. *Let $\mathcal{T}$ be a finite homotopy limit theory. Then the simplicial category* $\operatorname{HMod}(\mathcal{T})$ *is closed in* $\operatorname{Pre}(\mathcal{T}^{\mathrm{op}})$ *both under fibrant homotopy limits and filtered fibrant homotopy colimits.*

Proof. It is analogous to that of 4.14 (using 3.9). □

Theorem 5.5. *A fibrant simplicial category $\mathcal{K}$ is homotopy locally finitely presentable if and only if it is weakly equivalent to* $\operatorname{HMod}(\mathcal{T})$ *for some finite homotopy limit theory $\mathcal{T}$.*

Proof. We proceed analogously as in the proof of 4.15.

I. Let $\mathcal{T}$ be a finite homotopy limit theory. We replace the morphisms $m_{X_1 \dots X_n}$ from the proof of 4.15 by morphisms

$$m_D : \operatorname{hocolim}_f \hom(D, -) \to \hom(\operatorname{holim}_f D, -)$$

for each genuinely finite diagram $\mathcal{D} \to \mathcal{T}$. By the dual of 3.1(b), m_D corresponds to the morphism

$$\bar{m}_D : B((- \downarrow \mathcal{D}^{\mathrm{op}})^{\mathrm{op}}) \to \hom(\hom(D, -), \hom(\operatorname{holim}_f D, -)).$$

Since the domain of $\bar{m}_D$ is isomorphic to $B(\mathcal{D} \downarrow -)$ and the codomain to $\hom(\operatorname{holim}_f D, D)$, $\bar{m}_D$ corresponds to the morphism

$$\tilde{m}_D : B(\mathcal{D} \downarrow -) \to \hom(\operatorname{holim}_f D, D).$$

Now, in order to define m_D, we take the morphism $\tilde{\delta}_D$ from 2.5 for $\tilde{m}_D$. Like in the proof of 4.15, we know that the functor R_cL preserves fibrant homotopy colimits. Because of 5.2, it suffices to show that objects R_cLM_n are homotopy finitely presentable in $\mathrm{HMod}(\mathcal{T})$. Since $\mathrm{HMod}(\mathcal{T})$ is closed in $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ under a fibrant homotopy colimit of each filtered diagram $J : \mathcal{J} \to \mathrm{HMod}(\mathcal{T})$ (see 5.4), we have

$$\begin{aligned}
\hom(R_cLM_n, \operatorname{hocolim}_f J) &\simeq \hom(M_n, \operatorname{hocolim}_f J) \\
&\simeq \operatorname{hocolim}_s \hom(M_n, J) \\
&\simeq \operatorname{hocolim}_s \hom(R_cLM_n, J) \\
&\simeq \operatorname{hocolim}_f \hom(R_cLM_n, J).
\end{aligned}$$

Here, the first and the third weak equivalence is analogous to the calculation at the end of I. in the proof of 4.15 and the second equivalence follows from M_n being homotopy finitely presentable (see the proof of 5.2).

II. Let $\mathcal{K}$ be homotopy locally finitely presentable simplicial category and $\mathcal{A}$ be the set from 5.1. Let $\bar{\mathcal{A}}$ be the closure of $\mathcal{A}$ under genuinely finite fibrant homotopy colimits in $\mathcal{K}$. By 3.10, each object from $\bar{\mathcal{A}}$ is homotopy finitely presentable in $\mathcal{K}$. Now, we put $\mathcal{T} = (\bar{\mathcal{A}})^{\mathrm{op}}$ and proceed analogously as in the proof of 4.15. ☐

Corollary 5.6. *A homotopy locally finitely presentable category has all fibrant homotopy limits.*

Proof. It follows from 5.5 and 5.4. ☐

Definition 5.7. By a *homotopy finite limit sketch* is meant a triple $\mathcal{H} = (\mathcal{T}, \mathbf{L}, \sigma)$ consisting of a small fibrant simplicial category $\mathcal{T}$, a set $\mathbf{L}$ of genuinely finite diagrams in $\mathcal{T}$ and an assignment σ of a morphims

$$\sigma(D) : B(\mathcal{D} \downarrow -) \to \hom(X_D, D)$$

in $\mathbf{SSet}^{\mathcal{D}}$ to each diagram $D \in \mathbf{L}$.

By a *homotopy model* of $\mathcal{H}$ is meant a simplicial functor $A : \mathcal{T} \to \mathbf{S}$ belonging to $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ and sending $\sigma(D)$ to $\tilde{\delta}_D$ for each $D \in \mathbf{L}$.

We will denote by $\mathrm{HMod}(\mathcal{H})$ the full subcategory of $\mathrm{Pre}(\mathcal{T}^{\mathrm{op}})$ consisting of all homotopy models of $\mathcal{H}$.

Remark 5.8. Every homotopy finite limit theory is a homotopy finite limit sketch. Since the part I. of the proof of 5.5 is valid for each homotopy finite limit sketch $\mathcal{H}$, $\mathrm{HMod}(\mathcal{H})$ is always homotopy locally finitely presentable.

Proposition 5.9. *Every homotopy variety is homotopy locally finitely presentable.*

Proof. Since each simplicial algebraic theory is a homotopy finite limit sketch, the result follows from 5.5. □

Definition 5.10. Let $\mathcal{C}$ be a small fibrant simplicial category. Then $\mathrm{HInd}(\mathcal{C})$ will denote the full subcategory of $\mathrm{Pre}(\mathcal{C})$ consisting of filtered fibrant homotopy colimits of hom-functors.

Theorem 5.11. *Let $\mathcal{C}$ be a small fibrant simplicial category having genuinely finite fibrant homotopy colimits. Then the simplicial categories* $\mathrm{HInd}(\mathcal{C})$ *and* $\mathrm{HMod}(\mathcal{C}^{op})$ *are weakly equivalent.*

Proof. Since filtered fibrant homotopy colimits commute with genuinely finite fibrant homotopy limits in $\mathbf{S}$ (see 3.9), we always have

$$\mathrm{HInd}(\mathcal{C}) \subseteq \mathrm{HMod}(\mathcal{C}^{op}).$$

Conversely, we know that each object from $\mathrm{HMod}(\mathcal{C}^{\mathrm{op}})$ is a filtered fibrant homotopy colimit of genuinely finite fibrant homotopy colimits of hom-functors (using 3.5). Since $L(m_D)$ is a weak equivalence for each morphisms m_D from the proof of 5.5 (see [20], 1.C.5), each object from $\mathrm{HMod}(\mathcal{C}^{\mathrm{op}})$ is homotopy equivalent to an object from $\mathrm{HInd}(\mathcal{C})$. □

As a consequence, we get that $\mathrm{HInd}(\mathcal{C})$ has all filtered fibrant homotopy colimits. Hence it is analogous to the free completion $\mathrm{Ind}(\mathcal{C})$ of a category $\mathcal{C}$ under filtered colimits introduced in [4].

Let $\mathcal{T}$ be a finite homotopy limit theory. Consider the left Bousfield localization of the projective model category structure on $\mathbf{SSet}^{\mathcal{C}^{\mathrm{op}}}$ with respect to the set $\mathcal{Z}$ consisting of morphisms m_D from the proof of 5.5. The resulting model category will be called the *model category for homotopy $\mathcal{T}$-models.* As a consequence of 5.5 we get the following characterization of these model categories.

Corollary 5.12. *A simplicial model category $\mathcal{M}$ is Quillen equivalent to the model category for homotopy $\mathcal{T}$-models for some finite homotopy limit theory $\mathcal{T}$ if and only if* $\mathrm{Int}(\mathcal{M})$ *is homotopy locally finitely presentable.*

Remark 5.13. Everything in this section can be done for an arbitrary regular cardinal λ instead of ω. It means that we work with homotopy λ-filtered fibrant homotopy colimits and compare homotopy locally λ-presentable categories with categories of models of λ-small homotopy limit theories. The distinction between finite and genuinely finite homotopy limits will disappear here.

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