Factorization and local presentability in topological and uniform spaces

M.Hušek[a,b,*], J.Rosický[c,1]

[a] *Charles Univ., Math.Dept., Sokolovská 83, 18675 Prague, Czech Rep.*
[b] *J.E.Purkyně Univ., Math.Dept. České mládeže 8, 40096 Ústı nad Labem, Czech Rep.*
[c] *Masaryk Univ.,Math.Dept., Kotlářská 2, 61137 Brno, Czech Rep.*

Abstract

Investigating dual local presentability of some topological and uniform classes, a new procedure is developed for factorization of maps defined on subspaces of products and a new characterization of local presentability is produced. The factorization is related to large cardinals and deals, mainly, with realcompact spaces. Instead of factorization of maps on colimits, local presentability is characterized by means of factorization on products.

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1. Introduction

During his fruitful mathematical life, W.W.Comfort was often interested in both investigation and applications of dependence of continuous mappings from products on smaller numbers of coordinates – the first such publication seems to be [7] from 1972, the last one [6] from 2012, and many others in between. The so-called dually locally presentable categories deal with more general factorizations of maps on limits of λ-directed inverse systems. We were able to reduce those conditions to special factorizations of maps from products. In this case large cardinals play a role and a new procedure for getting factorizations had to be developed.

Locally presentable categories were defined by P.Gabriel and F.Ulmer in [10] and a new view and comprehensive theory was given by M.Makkai and R.Paré in [18], and by J.Adámek and J.Rosický in [2] (see the last section of the present paper for more details). Most applications were to algebraic structures since topological structures are rarely locally presentable. Because of dualities between topological and algebraic structures (like the Stone or Gelfand dualities between compact spaces and algebras), some categories dual to topological ones were shown to be locally presentable. That lead to a question whether local presentability of dual categories to topological structures can be handled directly without using algebraic structures. The present paper shows that the answer is affirmative and, in fact, more general results can be obtained by those inner topological methods. We shall also deal with a weaker notion, namely near local presentability defined and investigated in [22].

After recalling in the next section some concepts from set theory, category theory and topological structures we describe a characterization of dual (nearly) locally presentable subclasses of Hausdorff topological and uniform spaces (we shall see that for non-Hausdorff spaces dual local presentability is not interesting). Then we apply that characterization to topological and uniform spaces, mainly to classes generated by reals. The last section gives categorical background concerning local presentability and shows that our characterization of dual (nearly) local presentability for topological structures holds in general categories.

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* Corresponding author
Email addresses: mhusek@karlin.mff.cuni.cz (M.Hušek), rosicky@math.muni.cz (J.Rosický)
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2. Generalities

References to basic facts used in the present paper are [1] for category theory, [17] and [8] for set theory, [9] and [16] for topological and uniform or proximity spaces. We now recall some other concepts or those used frequently in the sequel, at first from set theory.

It seems there are various definitions of μ-strongly compact cardinals. We use the one from [4]. A filter $\mathcal{F}$ is κ-complete (κ an infinite cardinal), if $\bigcap \mathcal{F}' \in \mathcal{F}$ for any $\mathcal{F}' \subset \mathcal{F}$ with $|\mathcal{F}'| < \kappa$. Some of the next definitions of large cardinals can be defined for ω, too (e.g., measurable cardinals in [8]). We need those cardinals to be uncountable and add that condition to the definitions.

Definition 1. A cardinal λ with $\lambda > \mu > \omega$ is said to be *μ-strongly compact* if every λ-complete filter on any set extends to a μ-complete ultrafilter.

An uncountable cardinal λ is said to be *strongly compact* if every λ-complete filter on any set extends to a λ-complete ultrafilter.

An uncountable cardinal λ is said to be *measurable* if there is a free λ-complete ultrafilter on λ.

Clearly, if λ is μ-strongly compact and $\lambda' > \lambda$ then λ' is μ-strongly compact, too. If λ is regular ω_1-strongly compact, it is bigger or equal to the first measurable cardinal $\mathfrak{m}_1$. If the existence of ω_1-strongly compact is consistent, it is consistent that $\mathfrak{m}_1$ is ω_1-strongly compact (then it is strongly compact) – see [4].

We use a standard notation $[A]^{<\kappa}$ for the set of all subsets of A of cardinalities smaller than κ.

We shall now briefly recall basics concerning local presentability. For more details and other related notions in category theory see the last section.

For an infinite cardinal λ, an ordered set $(I, <)$ is said to be λ-directed if every $J \subset I$ with $|J| < \lambda$ is followed by some $i_J \in I$, i.e., $j < i_J$ for every $j \in J$. A λ-directed system in a category $\mathcal{K}$ is a collection $\{X_i, \pi_{i,j}\}_I$ where I is a λ-directed ordered set, X_i are objects of $\mathcal{K}$ and the morphisms $\pi_{i,j} : X_i \to X_j, i < j$, have properties: $\pi_{i,i} = 1_{X_i}, \pi_{j,k} \circ \pi_{i,j} = \pi_{i,k}$ whenever $i < j < k$ (i.e., $\{X_i, \pi_{i,j}\}_I$ is a subcategory of $\mathcal{K}$).

Definition 2. A cocomplete category $\mathcal{K}$ is *locally λ-presentable*, where λ is a regular cardinal, if the following two conditions are satisfied:

1. $\mathcal{K}$ has a strong generator $\mathcal{B}$, where $\mathcal{B}$ is a set;
2. for each $B \in \mathcal{B}$, the hom-functor $\hom(B, -) : \mathcal{K} \to \mathtt{Set}$ preserves colimits of λ-directed systems.

A category $\mathcal{C}$ is said to be *locally presentable* if it is locally λ-presentable for some λ.

If, instead of the condition (2) the following weaker condition (2') is satisfied, the category is said to be *nearly locally λ-presentable.*

2'. for each $B \in \mathcal{B}$, the hom-functor $\hom(B, -) : \mathcal{K} \to \mathtt{Set}$ preserves λ-directed colimits expressing a coproduct as a λ-directed colimit of its subcoproducts having index sets of cardinalities smaller than λ.

Section 6 gives more categorical details and relations. It is explained there that $\mathcal{C}$ is locally presentable if it satisfies the condition 1 and, for some regular cardinal μ, the condition 2 for μ-directed systems $\{X_i, \pi_{i,j}\}_I$ with monomorphisms $\pi_{i.j}$. In the next section the previous definitions will be described using topological language in a dual situation.

Objects satisfying the second condition (2) (or (2')) are called λ-presentable (or nearly λ-presentable, resp.).

Recall that a class $\mathcal{B}$ is a generator of $\mathcal{K}$ if for any two different morphisms $f, g : X \to Y$ in $\mathcal{K}$ there exists $B \in \mathcal{B}$ and a morphism $\varphi : B \to X$ with $f\varphi \neq g\varphi$.

A generator $\mathcal{B}$ is strong if for every monomorphism $f : X \to Y$ that is not an isomorphism there exists $B \in \mathcal{B}$ and a morphism $B \to Y$ that does not factorize via f.

The above condition 2 means that for a colimit $k_i : K_i \to K$ in $\mathcal{K}$ of a λ-directed system $\{K_i, k_{ij}\}_I$ and any morphism $f : B \to K$ there exists $i \in I$ and $g : B \to K_i$ (in a sense unique) such that $k_i g = f$.

We say that a category is *dually (nearly) locally presentable* if its dual (i.e., opposite) category is (nearly) locally presentable. The authors of [10] call those categories locally corepresentable.

In the sequel, all subcategories will be full and isomorphism closed. When we write $f : X \to Y$ we have always in mind a morphism from the category we are working in (thus a continuous map when we work in $\mathtt{Top}$).

In topological structures we shall deal with dual notions from the previous part. To specify that we consider limits of directed systems, we use the notion *inverse system* as described in [9]. In topological or uniform spaces and their productive and complete subcategories, a limit $(X, \{\pi_i\})$ of an inverse system $\{X_i, \pi_{i,j}\}_I$ will be regarded as the subspace of $\prod_I X_i$ consisting of all threads, i.e., of points $\{x_i\}$ having the property $\pi_{i,j}(x_i) = x_j$ whenever $i > j$. The maps $\pi_i : X \to X_i$ are the projections. A role of the fact that the inverse system is λ-directed will be used in Lemma 3.3. If J is cofinal in I then the limit of $\{X_i, \pi_{i,j}\}_J$ is isomorphic to X.

A continuous map $f : X \to Y$ on a subspace of a product $\prod_I X_i$ of topological or uniform spaces is said to depend on $J \subset I$ if $f(x) = f(y)$ provided $x, y \in X, \mathrm{pr}_J(x) = \mathrm{pr}_J(y)$. One also says that f depends on $|J|$ coordinates or on less than κ coordinates if $|J| < \kappa$. It is equivalent to existence of a map $g : \mathrm{pr}_J(X) \to Y$ such that $f = g\,\mathrm{pr}_J$ (i.e., f factorizes via $\mathrm{pr}_J(X)$). The map g need not be continuous. If g is continuous one says that the factorization is continuous or that f depends on J continuously. In case $X = \prod_I X_i$, the factorization is always continuous. The factorizations of uniformly continuous mappings which we use are always uniformly continuous.

If $A_i \subset X_i, i \in I$, and $A = \prod_i A_i \subset \prod_I X_i$, then by $R(A)$ we denote the set $\{i \in I; A_i \neq X_i\}$. So, for canonical neighborhoods U of points in products we have $|R(U)| < \omega$.

A productive and complete subcategory $\mathcal{C}$ of Hausdorff topological or uniform spaces is called *simple* if it has a strong cogenerator.

A topological space X is said to be *pseudo-κ-compact* for an infinite cardinal κ if every discrete system of nonvoid open sets in X has cardinality less than κ.

Except at the beginning of Section 4 all the spaces under consideration will be Hausdorff.

3. Dually locally presentable classes of topological structures

We shall now transfer Definition 2 of (near) local presentability to dual situations and, moreover to categories of Hausdorff topological or uniform spaces. In this section we assume $\mathcal{C}$ is a complete and productive subcategory of the category of either all Hausdorff topological spaces ($\mathtt{Top}_2$) or all Hausdorff uniform spaces ($\mathtt{Unif}_2$). In our applications, $\mathcal{C}$ will be usually epireflective, i.e., closed hereditary and productive in $\mathtt{Top}_2$ or $\mathtt{Unif}_2$.

If $\mathcal{C}$ is dually nearly locally presentable it must have a strong cogenerator (Definition 2, item 1) that is a set of objects. In many categories, that set can be assumed to consist of one object (see). In our special case the proof is simple.

Proposition 3.1. *The category $\mathcal{C}$ has a strong cogenerator $\mathcal{A}$ that is a set of spaces iff there exists a space A in $\mathcal{C}$ such that every space from $\mathcal{C}$ can be embedded into a power of A as a closed subspace.*

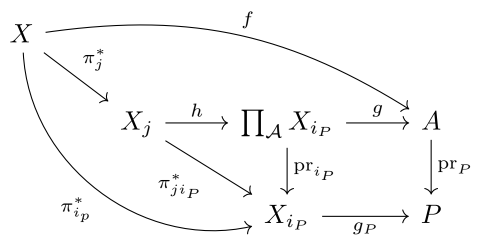
Proof. Assume first that a set $\mathcal{A}$ of spaces is a strong cogenerator of $\mathcal{C}$ and denote $A = \prod_{P \in \mathcal{A}} P$. We want to show that every $X \in \mathcal{C}$ embeds onto a closed subspace of a power of A, in fact into $A^{\mathcal{C}(X,A)}$. Since $\mathcal{A}$ is a cogenerator, morphisms $X \to P, P \in \mathcal{A}$, distinguish points of X, thus $\mathcal{C}(X, A)$ distinguishes points of X. So, the canonical map $\iota : X \to A^{\mathcal{C}(X,A)}$ is an injection. Consider the range-restriction $\iota' : X \to \overline{\iota(X)}$, where the closure is in the power. Then ι' is an epimorphism in $\mathcal{C}$ and every $f : X \to A$ extends to a map $f' : A^{\mathcal{C}(X,A)} \to A$ (the extension is the f-th projection). Consider a map $g : X \to P_0, P_0 \in \mathcal{A}$. The diagonal product of the map g and of some constant maps $X \to P, P \in \mathcal{A}, P \neq P_0$ is a map $f : X \to A$. An extended map $f' : A^{\mathcal{C}(X,A)}$ composed with the projection $A \to P$ gives an extension of g to $A^{\mathcal{C}(X,A)}$ and, thus, a factorization of g via ι'. That implies (strong cogenerator property) that ι' is an isomorphism, i.e., ι is an embedding onto a closed subspace of the power.

Assume conversely that every space from $\mathcal{C}$ can be embedded into a power of A as a closed subspace. We want to show that A is a strong cogenerator of $\mathcal{C}$. Clearly, A is a generator. It remains to show that if $f : X \to Y$ is an epimorhisms (i.e., $f(X)$ is dense in Y) and every $g : X \to A$ facorizes via f then f is an isomorphism. Because of that factorization, we may assume that f is an injection, thus that X is a dense subset of Y with possibly a finer topology than the restriction of Y to X. The factorization then means that every $g : X \to A$ extends uniquely to $f' : Y \to A$, which gives a canonical set-isomorphism between $\mathcal{C}(X, A)$ and $\mathcal{C}(Y, A)$. We get that the composition $X \to Y \to A^{\mathcal{C}(Y,A)}$ is an embedding onto a close set, which implies that the first map $X \to Y$ must be an embedding onto a closed set, thus an isomorphism in $\mathcal{C}$. □

The next assertion concerns the second item of Definition 2 and says that we may again restrict a cogenerator set to a single space. It is a special case of [10], Th.6.2, and [2], Th.1.6). For convenience of readers, a simple proof of our special case is added for item 2, the item 2' is its special case.

Proposition 3.2. *Let $\mathcal{A}$ be a strong cogenerator in $\mathcal{C}$ and for every $P \in \mathcal{A}$ the functor $\mathcal{C}(-, P)$ maps λ-directed limits into colimits in* `Set`*, where $\lambda > |\mathcal{A}|$. Then $\mathcal{C}(-, \prod_{P\in\mathcal{A}} P)$ maps λ-directed limits into colimits in* `Set`*.*

Proof. Our assumption means that for each $P \in \mathcal{A}$, if X is a limit of a λ-directed system $\{X_i, \pi_{i,j}\}_I$, then $\mathcal{C}(X, P)$ is a colimit of the system $\{\mathcal{C}(X_i, P), \pi^*_{i,j}\}$ in `Set`, i.e., $\mathcal{C}(X, P) = \bigcup \mathcal{C}(X_i, P)/\sim$, where $\sim$ is the known equivalence generated by $\{\pi^*_{i,j}\}$. We want to show the same is true for taking $A = \prod_{P\in\mathcal{A}} P$ instead of P. The condition for P means that for every $f : X \to P$ there exists $i \in I$ and $g \in \mathcal{C}(X_i, P)$ such that $f = g \circ \pi_i$. For any $f : X \to A$ and every $P \in \mathcal{A}$ there exists i_P and $g_P \in \mathcal{C}(X_{i_P}, P)$ with $\mathrm{pr}_P \circ f = g_P \circ \pi_i$. There exists $j > i_P, P \in \mathcal{A}$, since $\lambda > |\mathcal{A}|$ and the system $\{X_i, \pi_{i,j}\}_I$ is λ-directed. Take the product $g : \prod_{\mathcal{A}} X_{i_P} \to \prod_{\mathcal{A}} P = A$ of maps $g_P : X_{i_P} \to P$ and the diagonal product $h : X_j \to \prod_{\mathcal{A}} X_{i_P}$. Then $f = g \circ h \circ \pi^*_j$, which completes the proof (see the following commutative diagram).



□

We now know that for $\mathcal{C}$ to be dually nearly locally presentable or dually locally presentable it suffices to consider a single space A as a strong cogenerator. Consequently, the class $\mathcal{C}$ must consist of the so-called A-compact spaces, i.e., of spaces isomorphic to closed subspaces of powers of A. It remains to simplify the condition about preserving limits. At first an almost trivial result about a limit of a λ-directed system expressed by means of products.

Lemma 3.3. *Let X be a limit of a λ-directed inverse system $\{X_i, \pi_{i,j}\}_I$ in $\mathcal{C}$ and $\mathrm{pr}_i : X \to X_i$ be maps onto dense subsets of X_i. Then for any $J \in [I]^{<\lambda}$ and any $i_J \in I$ such that $i_J > j$ for all $j \in J$ one has $\pi_{i_J,J}(X_{i_J}) \subset \overline{\mathrm{pr}_J(X)}$, where $\pi_{i_J,J} : X_{i_J} \to \prod_J X_j$ is the diagonal product of all $\pi_{i_J,j} : X_{i_J,j} \to X_j$.*

Proof. Since X is the set of all threads $x = \{x_i\}_I$ ($\pi_{i,j}(x_i) = x_j$ whenever $i > j$) one has $\mathrm{pr}_J(x) = \{x_j\}_J$ and $\pi_{i_J,J}(x_{i_J}) = \{\pi_{i_J,j}(x_{i_J})\}_J = \{x_j\}_J$. Thus $\pi_{i_J,J}\mathrm{pr}_{i_J} = \mathrm{pr}_J$ and, hence, $\pi_{i_J,J}(\mathrm{pr}_{i_J}(X)) \subset \mathrm{pr}_J(X)$. Consequently, $\pi_{i_J,J}(\overline{\mathrm{pr}_{i_J}(X)}) \subset \overline{\mathrm{pr}_J(X)}$. Since $\overline{\mathrm{pr}_{i_J}(X)} = X_{i_J}$, the proof is finished. □

We can now prove the basic assertion used in the sequel in $\mathtt{Top}_2$ and $\mathtt{Unif}_2$. We say that *λ-directed limits in $\mathcal{C}$ preserve epimorphisms* if $\mathcal{C}$ has the following property:

(*) if X is the limit of a λ-directed inverse system $\{X_i, \pi_{i,j}\}_I$ then $\overline{\pi_i(X)} = X_i, i \in I$, provided $\overline{\pi_{i,j}(X_i)} = X_j$ for all $i > j$;

Clearly, the condition is satisfied for $|I| < \lambda$. If $\mu > \lambda$ and λ-directed limits in $\mathcal{C}$ preserve epimorphisms, then μ-directed limits preserve epimorphisms. The condition (*) for any infinite λ holds in the class of compact spaces (see, e.g., [9], Corollary 3.2.15). It does not hold in the class of realcompact spaces for $\lambda < \mathfrak{m}_1$. To show that, it suffices to take a metric space X of cardinality at least λ and smaller than $\mathfrak{m}_1$ having the property that every complement of a set $P \in [X]^{<\lambda}$ is dense in X. Those complements form a λ-directed inverse system of realcompact spaces with empty limit and dense connecting maps (inclusions). For instance, the Banach space $\ell_2(\lambda)$ satisfies the required conditions.

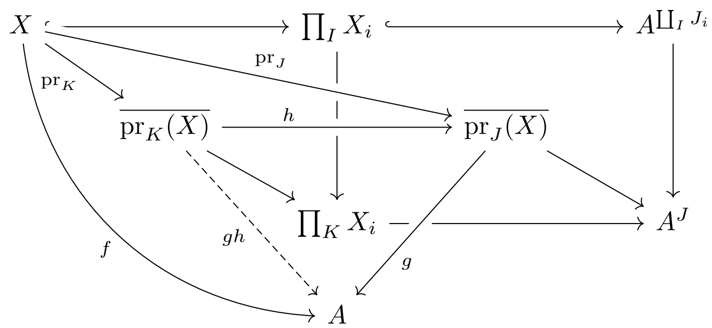
Every dual locally λ-presentable category has the property (*). In fact, it has a stronger property, namely limit of epimorphisms between two λ-directed systems is an epimorphism, as follows from uniqueness of such limits and from the fact the category has a cogenerator satisfying the property 2 of Definition 2 (see the end of Section 6 for proofs).

Theorem 3.4. *The category $\mathcal{C}$ is dually locally presentable iff there exist a regular cardinal λ and $A \in \mathcal{C}$ such that*

1. *λ-directed limits in $\mathcal{C}$ preserve epimorphisms;*
2. *every $X \in \mathcal{C}$ can be embedded as a closed subspace into some power A^κ;*
3. *if $X \in \mathcal{C}$ is a closed subspace of a power A^I then every $f \in \mathcal{C}(X, A)$ (uniformly) continuously factorizes via $\overline{\mathrm{pr}_J(X)}$ for some $J \subset I, |J| < \lambda$.*

Proof. Assume first that $\mathcal{C}$ is dually locally presentable. By the previous consideration there exist a regular cardinal λ such that the condition 1 holds, and a space A satisfying the condition 2 and such that $\mathcal{C}(-, A) : \mathcal{C} \to \mathtt{Set}$ maps λ-directed limits to colimits. If X is a closed subspace of A^I then $(X, \{\mathrm{pr}_J\}_{J\in[I]^{<\lambda}})$ is a limit of the λ-directed system $\{\overline{\mathrm{pr}_J(X)}, \mathrm{pr}_{JK}\}$ where $J, K \in [I[^{<\lambda}, J \supset K$. That implies the validity of the condition 3.

Assume converwsely that the conditions 1–3 hold for a regular cardinal λ. Then A is a strong cogenerator of $\mathcal{C}$ and we must show that $\mathcal{C}(-, A) : \mathcal{C} \to \mathtt{Set}$ sends λ-directed limits to colimits. Let $(X, \{\mathrm{pr}_i\})$ be the limit of a λ-directed inverse system $\{X_i, \pi_{ij}\}$. As explained in Introduction (in more details in Section 6) we may assume that all $\pi_{i,j}$ are epimorphisms, i.e., $\overline{\pi_{i,j}(X_i)} = X_j$. By the condition 2, each X_i may be considered as a closed subspace of some power A^{J_i} and, thus, X is a closed subspace of $\prod_I A^{J_i}$. Take some $f \in \mathcal{C}(X, A)$ – by the condition 3 it depends on some $J \subset \coprod_I J_i, |J| < \lambda$. Then in the original product $\prod_I X_i$ the map f depends on $K = \{i \in I, J \cap J_i \neq \emptyset\}$ that has cardinality smaller than λ. There is a continuous map $g : \overline{\mathrm{pr}_J(X)} \to A$ with $f = g \circ \mathrm{pr}_J(X)$. Since the map $\prod_K X_i \to A^J$ induces the map $h : \overline{\mathrm{pr}_K(X)} \to \overline{\mathrm{pr}_J(X)}$, we get a map $g \circ h : \overline{\mathrm{pr}_K(X)} \to A$ with $g \circ h \circ \mathrm{pr}_K = f$. The condition 1 allows to use the previous Lemma to get that f factorizes via X_{i_K}, where i_K precedes all $i \in K$, which finishes the proof. The procedure is shown in the next commutative diagram.



□

Corollary 3.5. *Let λ-directed limits in $\mathcal{C}$ preserve epimorphisms for some λ. An epireflective subcategory of $\mathcal{C}$ is dually locally presentable iff it is simple, generated by some A, and the condition 3 from Proposition 3.4 holds for A.*

Corollary 3.6. *If the epireflective hull of a space A is dually locally presentable, then any of its productive and complete subcategory is dually locally presentable.*

The situation for dually nearly locally presentable classes is simpler. In this case the condition 2' of Definition 2 means that every $f : \prod_I X_i \to B, B \in \mathcal{B}$, in $\mathcal{C}$ depends on less than λ coordinates. A disadvantage is we cannot restrict factorizations to powers of A as in Theorem 3.4. We get the following characterization.

Theorem 3.7. *The category $\mathcal{C}$ is dually nearly locally λ-presentable iff there exists $A \in \mathcal{C}$ such that*

1. *every $X \in \mathcal{C}$ can be embedded as a closed subspace into some power A^κ;*
2. *for any collection $\{X_i\}_I$ of spaces from $\mathcal{C}$, every $f \in \mathcal{C}(\prod_I X_i, A)$ depends on less than λ coordinates.*

We want to notice that the above results hold for non-Hausdorff spaces, too. Instead of closed subspaces one must take extremal closed subspaces (i.e., the embedding is an extremal monomorphism). The last section contains those results formulated in general categories.

4. Topological spaces

Although we assumed Hausdorff separation in the previous section we start with several easy situations under weaker separations. For the next three paragraphs we assume that $\mathcal{C}$ is a complete and productive subcategory of `Top`. If $\mathcal{C}$ is dually nearly locally presentable, it contains with its strong cogenerator A some subspaces of all powers A^κ. In this section, A will be an at least two-point space and our subcategory consists of extremally closed subspaces of powers A^κ

If $\mathcal{C}$ contains a non-T_0-space then any strong cogenerator of $\mathcal{C}$ contains an at least two-point l indiscrete subspace B. For any product of at least two two-point spaces there exists a map on the product into B that does not factorize via a proper subset of coordinates. Consequently, such a category cannot be dually nearly locally presentable.

If $\mathcal{C}$ contains a T_0-space X that is not a T_1-space then X contains the Sierpinski space S as a subspace (recall, that Sierpinski space is the two-point space $\{0, 1\}$ with open point 0 and not open point 1). Then S is a retract of X and every continuous map from S^I extends continuously to X^I. For every at least two-point set I there exists a continuous map $f : S^I \to S$ not factorizable via a proper subset of I. Indeed, it suffices to take $f(x) = 1$ at the point x having all its coordinates equal to 1, and $f(x) = 0$ for the other cases. Consequently, $\mathcal{C}$ is not dually nearly locally presentable.

Let $\mathcal{C}$ consist of T_1-spaces. If $\mathcal{C}$ contains a class of spaces with cofinite topology of arbitrary large cardinality, then $\mathcal{C}$ has not a cogenerator. We do not know if epireflective hulls of infinite T_1-spaces with cofinite topology are nearly locally presentable.

From now on we assume $\mathcal{C}$ to be a subcategory of Hausdorff spaces. H.Herrlich (see [11]) constructed an example of a reflective subcategory $\mathcal{C}$ of Hausdorff spaces consisting of some but not all closed subspaces of powers of A. That category consists of powers of a strongly rigid compact space and is dually locally presentable. We shall not investigate such extremal classes. We assume $\mathcal{C}$ consists of all spaces embeddable as closed subspaces of powers A^κ for a given at least two-point Hausdorff space A (such spaces are called A-compact according S. Mrówka). In that case we denote the subcategory as $\mathcal{C}(A)$. If $|A| \leq 1$ then $\mathcal{C}(A)$ consists of at most one-point spaces and, thus, is dually locally presentable.

As we see from Theorem 3.4, we need factorization results for continuous maps defined on products or their subspaces. There exist many publications dealing with such factorizations. In addition to the publications by W.W.Comfort mentioned at the beginning, we may mention survey papers [13, 14]. Known results are sufficient to investigate dual local presentability of compact spaces and dual near local presentability of realcompact spaces. For dual local presentability of realcompact spaces, a new factorization procedure must be developed. We shall see that existence of large cardinals is a necessary condition for some classes to be dually locally presentable. Those large cardinals help us to use different factorization procedures than the classical ones. Since our main interest is the class of realcompact spaces, our procedure will be adjusted to that situation. At first we look at the easiest case, namely compact spaces.

4.1. Compact spaces

Let A be a compact space. For some dual categories of compact spaces it is known that they are locally presentable. The proofs use either dualities with algebraic structures (e.g., in [2]) or use inner topological properties only but not Theorem 3.4 (e.g., in [10]). For any compact space A we can prove that any productive and complete class of A-compact spaces is dually locally presentable using the procedure mentioned above. For that we need some factorization results. The first factorization for mappings defined on compact subsets of products was proved by Y.Mibu in [21] by means of the Stone-Weierstrass theorem. So, only maps into products of real lines can be used. We shall use another result proved by G.Vidossich for $w_u(A) = \omega$ in [23]. That result is for uniform spaces but compact spaces form a productive full subcategory of uniform spaces. Recall that uniform weight $w_u(X)$ of a uniform space X is the least infinite cardinal of a base of uniform covers of X (or of uniform neighborhoods of Δ_X).

Theorem 4.1. (G.Vidossich) *For every uniformly continuous map f from a subspace X of a product $\prod_I X_i$ of uniform spaces into a uniform space Y there exists some $J \subset I, |J| \le w_u(Y)$, and a uniformly continuous map $g : \mathrm{pr}_J(X) \to A$ with $f = g \circ \mathrm{pr}_J$.*

As an easy application of the preceding theorem we get the following desired result.

Theorem 4.2. *If A is a compact space then $\mathcal{C}(A)$ is dually locally $w(A)^+$-presentable.*

Proof. As we already mentioned, the condition 1 of Theorem 3.4 is satisfied for compact spaces. The second condition holds because we consider the category $\mathcal{C}(A)$. For the third condition, let X be a compact subspace of a power A^κ and $f : X \to A$ be a continuous map. Then f is uniformly continuous with respect to the unique uniformities on our spaces. By the above Vidossich theorem, f depends continuously on $w(A)$ coordinates. □

Basic examples are compact zero-dimensional spaces for $A = 2$ and all compact spaces for $A = [0, 1]$. There are many other different examples starting with various continua A.

4.2. Realcompact spaces

Take for A a realcompact space. The situation for that case is much more complicated. From that reason we shall restrict our consideration to the categories either of all realcompact Hausdorff spaces (i.e., to $\mathbb{R}$-compact spaces) or to $\mathbb{N}$-compact spaces. We are not aware of a result asserting the class of all zero-dimensional realcompact spaces is simple. A.Mysior proved in [19] that the class is not simple provided $\mathfrak{m}_1$ does not exist. Our next procedures need existence of $\mathfrak{m}_1$ and that is the reason why we take $\mathbb{N}$-compact spaces instead of zero-dimensional realcompact spaces.

One result about dual local presentability of realcompact spaces is known from [2].

Theorem 4.3. *Assuming Vopěnka's principle, the category of realcompact spaces is dually locally presentable.*

Proof. The category of all realcompact spaces is isomorphic to the dual of the full subcategory of the category `Ring` of rings; the full embedding sends a realcompact space X to its ring $C(X)$ of continuous functions $X \to \mathbb{R}$. Thus the claim follows from [2] 6.6 and 6.14. □

Vopěnka's principle is a very strong condition on existence of large cardinals. We shall use our previous procedure to consider validity of Theorem 4.3 under weaker conditions. Nevertheless, some large cardinals will be needed. At first we look at dual near local presentability. A convenient factorization result is the following one due to N.Noble and S.Ulmer in [20]:

Theorem 4.4. [Noble,Ulmer] *The following conditions are equivalent for a nontrivial product of completely regular spaces $\prod_I X_i, |I| \geq \kappa, \kappa$ has uncountable cofinality:*

1. *$\prod_I X_i$ is pseudo-κ-compact;*
2. *every continuous $f : \prod_I X_i \to \mathbb{R}$ depends on less than κ coordinates.*

Instead of $\mathbb{R}$ one can take any space having its diagonal as intersection of countably many closures of open sets ($\overline{G}_\delta$-diagonal). It is easy to modify the proof for Y with $\overline{G}_{<\kappa}$ diagonal. If the range is discrete then the assumption on cofinality of κ is not needed.

Theorem 4.5. *The category of realcompact spaces is dually nearly locally presentable iff measurable cardinals exist.*

Proof. Assume first that $\mathfrak{m}_1$ exists. By Noble-Ulmer theorem, $\mathbb{R}$ satisfies the condition 2 of Theorem 3.7 for $\kappa = \mathfrak{m}_1$ since every realcompact space is pseudo-$\mathfrak{m}_1$-compact. Since $\mathbb{R}$ is a strong generator for the class of realcompact spaces, the result follows.

Suppose $\mathfrak{m}_1$ does not exist and take any infinite cardinal κ. If the cofinality of κ is uncountable, the Noble-Ulmer theorem asserts there is a continuous map $f : D^\kappa \to \mathbb{R}$ not depending on less than κ coordinates (we take for D a discrete space of cardinality κ – thus D is realcompact). Since $\mathbb{R}$ is homeomorphic to $(0, 1)$, we may assume the range of f to be a part of $[0, 1]$. Since $[0, 1]$ can be embedded into any strong generator of $\mathcal{C}$ it implies no strong generator satisfies the condition 2 of Theorem 3.7 for any λ. □

Looking at the proof we are able to prove more.

Theorem 4.6. *Let $\mathcal{C}$ be a complete productive subcategory of $\mathcal{C}(A)$, where A is a realcompact non-compact space. Then $\mathcal{C}$ is dually nearly locally presentable iff a measurable cardinal $\mathfrak{m} > |A|$ exists.*

Proof. If such $\mathfrak{m}$ exists then every continuous map $f : \prod_I X_i \to A, X_i \in \mathcal{C}$, depends on less than $\mathfrak{m}$ coordinates. Indeed, $\prod_I X_i$ as a realcompact space is pseudo-$\mathfrak{m}_1$-compact, thus pseudo-$\mathfrak{m}$-compact. By the above Noble-Ulmer theorem, every continuous map $g : \prod_I X_i \to \mathbb{R}$ depends on less than $\mathfrak{m}$ coordinates. Since A embeds into $\mathbb{R}^\kappa$ for some $\kappa < \mathfrak{m}$, our f depends on less than $\mathfrak{m}$ coordinates.

Let there be no measurable cardinal $\mathfrak{m} > |A|$. Since A is non-compact, it contains a copy of $\mathbb{N}$ as a subspace. Take a discrete space D of any cardinality $\kappa > |A|$ with uncountable cofinality. Since the product D^κ is not pseudo-κ-compact, using the Noble-Ulmer theorem again, there exists a continuous map $f : D^\kappa \to \mathbb{N}$ (and thus to A) not depending on less than κ coordinates. Consequently, $\mathcal{C}$ cannot be dually nearly locally presentable. □

Now, we shall investigate dual local presentability of a category $\mathcal{C}$ that is either the category $\mathcal{C}(\mathbb{R})$ of all realcompact spaces or $\mathcal{C}(\mathbb{N})$ of all $\mathbb{N}$-compact spaces. For mappings defined on subspaces of products, the situation is much more complicated and modifications of Theorem 4.4 do not suffice to prove requested results. There are some results that allow one decrease of cardinality of index sets in case the mappings are continuous on the so called (κ, κ)-compact subspaces of products (see [14]). After finitely many steps of using that procedure it stops either at a cardinality we need for our factorization or at a cardinality where assumptions for decreasing the index set fail. The latter situation happens when cofinality of the cardinality of the index set is too small, namely when that cardinality is bigger than some λ and its cofinality is less than λ (here, λ is the cardinal from Theorem 3.4).

We know from Theorem 4.5 that $\mathcal{C}$ is not locally presentable provided no measurable uncountable cardinal exists. Thus in the next part of this section, we assume $\mathfrak{m}_1$ exists. In fact, we shall assume even more, namely existence of ω_1-strongly compact cardinals. Since Vopěnka's principle implies existence of strongly compact and many other large cardinals, our result is a strengthening of Theorem 4.3.

We need some auxiliary results.

Lemma 4.7. *If X is a closed subspace of a product $\prod_\kappa X_\alpha$ then $X = \bigcap\{\mathrm{pr}_\alpha^{-1}(\overline{\mathrm{pr}_\alpha(X)}); \alpha \in S\}$ for any cofinal set S in κ.*

Proof. It suffices to show that for any $x \in \prod_\kappa X_\alpha \setminus X$ there is some $\alpha \in S$ with $\mathrm{pr}_\alpha(x) \notin \overline{\mathrm{pr}_\alpha(X)}$. Take a canonical neighborhood U of x disjoint with X and some $\alpha \in S$ with $R(U) \subset \alpha$. Then $\mathrm{pr}_\alpha U \cap \mathrm{pr}_\alpha(X) = \emptyset$, which implies the requested relation. □

In the next results we must use ω_1-strongly compact cardinals. The result does not hold for $\lambda \le \mathfrak{m}_1$.

Lemma 4.8. *If λ is an ω_1-strongly compact cardinal then λ-directed limits in $\mathcal{C}$ preserve epimorphisms.*

Proof. Let X be the limit of a λ-directed inverse system $\{X_i, \pi_{i,j}\}_I$ of realcompact spaces X_i and continuous maps $\pi_{i,j} : X_i \to X_j$ with $\overline{\pi_{i,j}(X_i)} = X_j$. Assume there exists $i_0 \in I$ such that $\overline{\mathrm{pr}_{i_0}(X)} \ne X_{i_0}$ so that there is a nonvoid open H_{i_0} in X_{i_0} with $\overline{H_{i_0}} \cap \overline{\mathrm{pr}_{i_0}(X)} = \emptyset$. We may assume i_0 is the smallest element of I. For any i take $H_i = \pi_{i,i_0}^{-1}(H_{i_o})$. Then $H_i \ne \emptyset$ and $\overline{H_i} \cap \overline{\mathrm{pr}_i(X)} = \emptyset$ for any i. The collection $\{\mathrm{pr}_i^{-1}(H_i)\}_I$ is a base of a λ-complete filter in $\prod_i X_i$. By our assumption, it can be extended to an ω_1-complete ultrafilter $\mathcal{X}$. Since the product is realcompact, $\mathcal{X}$ converges to a point $x \in \prod_I X_i$. For any i the image $\mathrm{pr}_i(\mathcal{X})$ converges to $\mathrm{pr}_i(x) = x_i$ and similarly for j. Since $\pi_{i,j}\mathrm{pr}_i = \mathrm{pr}_j$ for $i > j$ we get $\pi_{i,j}(x_i) = x_j$, which implies $x \in X$. But $\mathrm{pr}_{i_0}(x) \in \overline{H_{i_0}}$ that is disjoint with $\mathrm{pr}_{i_0}(X)$ – a contradiction. □

Let X be a topological space, $x \in X$ and $\mathcal{F}$ be a filter in X. We say that x is a *G_δ-accumulation point* of $\mathcal{F}$ if it is an accumulation point of $\mathcal{F}$ in the G_δ-topology of X, i.e., if $\{U_n\}_\mathbb{N}$ is a family of neighborhoods of x then $\bigcap U_n$ meets every member of $\mathcal{F}$.

We are now ready to prove the main result of this section. Realize that if X is a realcompact space and $\mathcal{X}$ is an ω_1-complete ultrafilter in X, then $\mathcal{X}$ converges in X. Indeed, $\mathcal{X}$ converges to a point $\xi \in \upsilon(|X|)$ and the continuous extension $\iota : \upsilon(|X|) \to X$ of $1_X : |X| \to X$ maps ξ to a limit of $\mathcal{X}$ in X (here υ is the Hewitt-Nachbin realcompactification).

Theorem 4.9. *If ω_1-strongly compact cardinals exist then the classes of all realcompact or of all $\mathbb{N}$-compact Hausdorff spaces are dually locally presentable.*

Proof. Let A equal either to $\mathbb{N}$ or to $\mathbb{R}$ and take a regular ω_1-strongly compact cardinal λ. We know from Lemma 4.8 that λ-directed limits in $\mathcal{C}(A)$ preserve epimorphisms so that it remains to check the condition 3 from Theorem 3.4. To show that, take some closed subspace X of a power A^κ and some continuous $f : X \to A$.

Claim 1. *The map f depends on less than λ coordinates.*
Suppose f does not depend on less than λ coordinates so that the sets

$$C_J = \{x \in X;\ \text{there is } y \in X \text{ such that } \mathrm{pr}_J(x) = \mathrm{pr}_J(y), f(x) \ne f(y)\}.$$

are nonempty for every $J \in [\kappa]^{<\lambda}$ and $C_J \subset C_K$ for $J \supset K$. The collection $\{C_J; J \in [\kappa]^{<\lambda}\}$ is a base of a λ-complete filter $\mathcal{F}$ on the set X since

$$\bigcap_{i\in I} C_{J_i} \supset C_{\bigcup_I J_i}$$

and $|\bigcup_I J_i| < \lambda$ provided $|I| < \lambda$. Thus $\mathcal{F}$ extends to an ω_1-complete ultrafilter $\mathcal{X}$ on the set X. Since X is realcompact, $\mathcal{X}$ converges in the space X to a point z. Moreover, for every countable collection $\{U_n\}_\mathbb{N}$ of neighborhoods of z the intersection $\bigcap_N U_n \cap P \cap X$ is non-empty for any $P \in \mathcal{X}$. Since A is first countable, there are canonical canonical neighborhoods U_n of z in A^κ that $f(\bigcap U_n \cap X) = \{f(z)\}$ and $\mathrm{pr}_\alpha(\bigcap U_n) = \mathrm{pr}_\alpha(z)$ for $\alpha \in R(\bigcap U_n)$. Denote $J = R(\bigcap U_n)$ and take a point $x \in C_J \cap \bigcap U_n$. There exists some $y \in X$ with $\mathrm{pr}_J(x) = \mathrm{pr}_J(y), f(x) \ne f(y)$. But then $y \in \bigcap U_n$ (since $\mathrm{pr}_J(x) = \mathrm{pr}_J(y)$) and, thus, $f(y) = f(z) = f(x)$ – a contradiction.

Claim 2. *The map f depends continuously on less than λ coordinates.*

By Claim 1 there exists $J_0 \in [\kappa]^{<\lambda}$ such that for each $J \in \mathcal{J} = \{J \in [\kappa]^{<\lambda}, J \supset J_0\}$ there exists a mapping $f_J : \mathrm{pr}_J(X) \to A$ with $f = f_J \circ \mathrm{pr}_J$. Assume that no f_J is continuous and define

$$C_J = \{x \in X; f_J \text{ is not continuous at } \mathrm{pr}_J(x)\}\,.$$

Then $C_J \neq \emptyset, C_J \subset X$ for every $J \in \mathcal{J}$ and $C_J \subset C_K$ for $J \supset K$. The collection $\{C_J\}$ is a base of a λ-complete filter that can be extended to an ω_1-complete ultrafilter $\mathcal{Y}$. Similarly as in the proof of Claim 1 we get an G_δ-accumulation point z of $\mathcal{Y}$ in X, a corresponding G_δ-set U, a countable set J and a point $x \in U \cap C_J$. Since f_J is not continuous at $\mathrm{pr}_J(x)$ there exists a net $\{u_i\}_I$ in X such that $\mathrm{pr}_J(u_i) \to \mathrm{pr}_J(x), f_J(\mathrm{pr}_J(u_i)) \nrightarrow f_J(\mathrm{pr}_J(x))$. The first relation implies $f(u_i) \to f(z)$ and the second one $f(u_i) \nrightarrow f(x)$ – a contradiction since $f(z) = f(x)$.

Claim 3. *There exists some $J \in [\kappa]^{<\lambda}$, such that $f_J : \mathrm{pr}_J(X) \to A$ can be continuously extended to $\overline{\mathrm{pr}_J(X)}$.*

By Claim 2 there exists $J_1 \in [\kappa]^{<\lambda}$ on which f depends continuously. Take the set $\mathcal{J} = \{J \in [\kappa]^{<\lambda}, J \supset J_1\}$ and assume Claim 3 does not hold for any $J \in \mathcal{J}$. We denote

$$C_J = \{x \in \mathrm{pr}_J^{-1}(\overline{\mathrm{pr}_J(X)}); f_J \text{ does not extend continuously to } \mathrm{pr}_J(x)\}\,.$$

Then $C_J \neq \emptyset$ for every $J \in \mathcal{J}$ and $C_J \subset C_K$ for $J \supset K$. Now we have $C_J \cap X = \emptyset$. Nevertheless, the accumulation point z of an ω_1-complete ultrafilter extending $\{C_J\}$ belongs to X since (use Lemma 4.7 for the last equality):

$$z \in \bigcap_{\mathcal{J}} \overline{C_J} = \bigcap_{\mathcal{J}} \mathrm{pr}_J^{-1}(\overline{\mathrm{pr}_J(C_J)}) \subset \bigcap_{\mathcal{J}} \mathrm{pr}_J^{-1}(\overline{\mathrm{pr}_J(X)}) = X\,.$$

Again we find corresponding G_δ-set U as in the proof of Claim 1 and $x \in C_J \cap U$. Since f_J cannot be continuously extended to $\mathrm{pr}_J(x)$ there exists a net $\{u_p\}$ in X such that $\mathrm{pr}_J(u_p) \to \mathrm{pr}_J(x), f_J(\mathrm{pr}_J(u_p))$ does not converge in $\mathbb{N}$. But again the construction of U, x gives $f(u_p) \to f(z)$ – a contradiction. □

5. Uniform and proximity spaces

Subcategories of `Unif` containing a non-Hausdorff space are not dually nearly locally presentable (strong cogenerators contain nontrivial indiscrete subspaces). In this section we restrict our investigation to Hausdorff spaces. We shall identify the category of precompact spaces with proximity spaces and denote it by `Prox`.

The category of all uniform spaces is not simple, thus it is not dually nearly locally presentable. Also the category of all precompact spaces is not simple. To show that, take the topological spaces $P_\kappa = [0,1]^\kappa \setminus \{1\}$ for regular uncountable κ. Those spaces have unique uniformity (thus a unique precompact uniformity). It is shown in [12] that every precompact space embeds as a closed subspace into a power of some P_κ but P_λ does not embed as a closed subspace into power of P_κ for any $\kappa < \lambda$.

We must look for simple epireflective subcategories of uniform spaces, i.e. for classes of uniform spaces embeddable into powers of a given uniform space as closed subspaces. The situation will be simpler than in topological spaces because of the Vidossich factorization theorem 4.1. It implies directly the next result (use Theorem 3.7).

Theorem 5.1. *Every simple epireflective subcategory of Hausdorff uniform spaces is dually nearly locally presentable.*

As a consequence, we get dual near local presentability of some subcategories of $\mathtt{Unif}_2$. To distinguish between closed subspaces of powers A^κ when A is a topological space or a uniform space, we shall use the term A-complete spaces for the latter case instead of A-compact spaces. Thus $\mathbb{R}$-complete spaces are uniform spaces that are uniformly homeomorphic to closed subspaces of powers of the uniform space $\mathbb{R}$. Those spaces coincide with complete spaces having a base composed of countable linear covers – [15]. If we consider $\mathbb{R}$ as a proximity space (i.e., the precompact modification $p\mathbb{R}$ of the metric space $\mathbb{R}$) we shall also use the term $\mathbb{R}$-complete proximity spaces instead of $p\mathbb{R}$-complete spaces. Those proximal spaces X are proximally complete in the sense of J.M.Smirnov, i.e., any Cauchy filter with respect to a uniformity inducing the proximity of X converges in X.

Corollary 5.2. 1. *The class of* $\mathbb{R}$*-complete uniform spaces is dually nearly locally presentable.*

2. *The class of* $\mathbb{R}$*-complete proximity spaces is dually nearly locally presentable.*

For local presentability the situation is not so simple. Unlike the case in topological spaces, the factorization condition 3 in Theorem 3.4 is now easy, but problems with the condition 1 remain.

Using dualities with algebraic structures, one gets the following result corresponding to Theorem 4.3. In the next proof we denote by γX a completion of X, i.e., a compactification of X provided X is precompact.

Theorem 5.3. *Assuming Vopěnka's principle the class of* $\mathbb{R}$*-complete proximity spaces is dually locally presentable.*

Proof. Any $\mathbb{R}$-complete proximity space X induces a realcompact topological space. The category of proximity spaces is isomorphic to the category $\mathcal{K}$ whose objects are triples $(X, \gamma X, f)$ where $f : X \to \gamma X$ is the embedding of X into its completion, i.e., f makes X a dense subspace of a compact space γX. Morphisms $(X, \gamma X, f) \to (X', \gamma X', f')$ of $\mathcal{K}$ are uniformly continuous maps $g : X \to X'$. Consider the functor $G : \mathcal{K}^{\mathrm{op}} \to \mathtt{Ring}^{\to}$ sending $(X, \gamma X, f)$ to the monomorphism $C(f) : C(\gamma X) \to C(X)$. This makes $\mathcal{K}^{\mathrm{op}}$ isomorphic to a full subcategory of the category $\mathtt{Ring}^{\to}$ of morphisms of rings. Since the latter is locally presentable, the result follows from [2] 6.6 and 6.14. □

We shall now improve the previous result assuming existence of ω_1-strongly compact cardinal instead of Vopěnka's principle. It remains to show that the condition 1 of Theorem 3.4 holds. The proof is similar to that of Lemma 4.8.

Lemma 5.4. *Let* $\mathcal{C}$ *be either the category of* $\mathbb{R}$*-complete uniform spaces or the category of* $\mathbb{R}$*-complete proximity spaces. If* λ *is an* ω_1*-strongly compact cardinal then* λ*-directed limits in* $\mathcal{C}$ *preserve epimorphisms.*

Proof. Let X be the limit of a λ-directed inverse system $\{X_i, \pi_{i,j}\}_I$ in $\mathcal{C}$, where $\pi_{i,j} : X_i \to X_j$ satisfy $\overline{\pi_{i,j}(X_i)} = X_j$ whenever $i > j$. Assume there exists $i_0 \in I$ such that $\overline{\mathrm{pr}_{i_0}(X)} \neq X_{i_0}$ so that there is a nonvoid open H_{i_0} in X_{i_0} with $\overline{H_{i_0}} \cap \overline{\mathrm{pr}_{i_0}(X)} = \emptyset$. We can assume i_0 is the smallest element of I. For any i take $H_i = \pi^{-1}_{i,i_0}(H_{i_o})$. Then $H_i \neq \emptyset$ and $\overline{H_i} \cap \overline{\mathrm{pr}_i(X)} = \emptyset$. The collection $\{\mathrm{pr}_i^{-1}(H_i)\}_I$ is a base of a λ-complete filter in $\prod_i X_i$. By our assumption, it can be extended to a ω_1-complete ultrafilter $\mathcal{X}$. Assume $\mathcal{X}$ converges to a point $x \in \prod_I X_i$. For any i the image $\mathrm{pr}_i(\mathcal{X})$ converges to x_i Since $\pi_{i,j}\mathrm{pr}_i = \mathrm{pr}_j$ for $i > j$, we get $\pi_{i,j}(x_i) = x_j$, which implies $x \in X$ and that contradicts the assumption $x_{i_0} \notin \mathrm{pr}_{i_0}(X)$.

If X_i are $\mathbb{R}$-complete uniform spaces, the product $\prod_I X_i$ is complete and has a base of at most countable uniform covers. Since $\mathcal{X}$ is ω_1-complete, it is Cauchy in the product and, thus converges.

The case of $\mathbb{R}$-complete proximity spaces can be proved similarly using proximal completeness. Since proximal completeness is not so standard as uniform completeness, we shall use another procedure dealing with `Unif` instead of `Prox`. Let X_i be closed subspaces of powers $(p\mathbb{R})^{J_i}$ and denote $J = \coprod J_i$.. Then $\prod_I X_i$ is a closed subspace of the power $(p\mathbb{R})^J$. The embedding $X_i \to (p\mathbb{R})^{J_i}$ is uniformly continuous if regarded as the map of the uniformly discrete space $|X_i| \to \mathbb{R}^{J_i}$ so that the product map $\prod_I |X_i| \to \mathbb{R}^J$ is uniformly continuous and so is that map $e(\prod_I |X_i|) \to \mathbb{R}^J$, where e is the modification taking all countable uniform covers as the new base. Since $\mathcal{X}$ is ω_1-complete, it is a Cauchy filter in $e(\prod_I |X_i|)$ and, thus, converges in $\mathbb{R}^J$ and in $(p\mathbb{R})^J$. Since $\prod_I X_i$ is closed in $(p\mathbb{R})^J$, $\mathcal{X}$ converges in $\prod_I X_i$, which was to prove.

□

Using Lemma 5.4 and the Vidossich Theorem 4.1 we get the next result.

Theorem 5.5. *Assume an* ω_1*-strongly compact cardinal exists.*

1. *The class of* $\mathbb{R}$*-complete uniform spaces is dually locally presentable.*
2. *The class of* $\mathbb{R}$*-complete proximity spaces is dually locally presentable.*

Remark. There is the question whether the assumption on existence of an ω_1-strongly compact cardinal can be removed in the preceding theorem. The corresponding situation in topological spaces had two reasons why large cardinals had to be used. The first one was Theorem 4.5 showing that existence of measurable cardinals is needed. That reason cannot be used in uniform spaces because the corresponding Theorem 5.1 holds without assuming existence of large cardinals. The second reason follows from the fact that for $\lambda < \mathfrak{m}_1$ the condition 1 in Theorem 3.4 is not satisfied for realcompact spaces, as described in the paragraph preceding that theorem. That procedure cannot be directly modified to $\mathbb{R}$-complete uniform spaces since every complete space is closed in any bigger space.

6. Locally presentable categories

Recall that a cocomplete category $\mathcal{K}$ is locally λ-presentable, where λ is a regular cardinal, if it has a strong generator consisting of λ-presentable objects. An object A is λ-presentable if its hom-functor

$$\hom(A, -) : \mathcal{K} \to \mathtt{Set}$$

preserves λ-directed colimits. This means that for any λ-directed colimit $k_i : K_i \to K$ in $\mathcal{K}$ and any morphism $f : A \to K$ there exists $i \in I$ and $g : A \to K_i$ such that $k_i g = f$. Moreover, this factorization is essentially unique in the sense that if $f = k_i g$ and $f = k_i h$, then $k_{ij} g = k_{ij} h$ for some $i \leq j$ (where $k_{ij} : K_i \to K_j$ is a morphism of our directed diagram).

An object is presentable if it is λ-presentable for some regular cardinal λ. Similarly, a category is locally presentable if it is locally λ-presentable for some regular cardinal λ. Locally presentable categories are precisely those categories which can be axiomatized by limit sentences in an infinitary first-order logic. They include varieties and quasivarieties of algebras. More can be found in [2], the original reference is [10].

Recall that a strong generator is a small full subcategory $\mathcal{A}$ of $\mathcal{K}$ such that the functor $E_{\mathcal{A}} : \mathcal{K} \to \mathtt{Set}^{\mathcal{A}^{\mathrm{op}}}$, $EK = \mathcal{K}(-, K)$, is faithful and conservative (= reflects isomorphisms). A generator $\mathcal{A}$ of $\mathcal{K}$ is strong if and only if for each object K and each proper subobject of K there exists a morphism $A \to K$ with $A \in \mathcal{A}$ which does not factorize through that subobject.

Recall that an epimorphism $f : K \to L$ is strong if each commuting square

$$\begin{array}{ccc} L & \xrightarrow{\;v\;} & B \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle g} \\ K & \xrightarrow[\;u\;]{} & A \end{array}$$

such that g is a monomorphism has a diagonal fill-in, i.e., a morphism $t : L \to A$ with $tf = u$ and $gt = v$. A category is weakly co-wellpowered if any object has only a set of strong quotients. Any locally presentable category is weakly co-wellpowered, in fact, even co-wellpowered.

In a category with coproducts and pullbacks, $\mathcal{A}$ is a strong generator if and only if every object is a strong quotient of a coproduct of objects from $\mathcal{A}$ (following, e.g., [3] 6.3 and the fact that strong and extremal epimorphisms coincide).

An object A is λ-generated if its hom-functor $\hom(A, -)$ preserves λ-directed colimits of monomorphisms. Here, the essential unicity of a factorization is automatic. An object is generated if it is λ-generated for some λ. A cocomplete category is locally λ-generated if it is strongly co-wellpowered and has a strong generator consisting of λ-generated objects. A locally generated category is locally λ-generated for some λ. Any locally λ-presentable category is clearly locally λ-generated. Conversely and not so evidently, a locally λ-generated category is locally presentable but not necessarily for the same λ (see [10], or [2]).

Nearly locally presentable categories were introduced in [22] as a generalization of locally presentable ones. The idea is to replace λ-presentable objects by nearly λ-presentable ones. This means the their hom-functor $\hom(A, -)$ preserves only very special λ-directed colimits, namely those expressing a coproduct $\coprod_I K_i$ as a λ-directed colimit of its λ-small subcoproducts $\coprod_J K_j$, i.e., $|J| < \lambda$. Then a cocomplete category is nearly locally λ-presentable if it is strongly co-wellpowered and has a strong generator consisting of nearly λ-presentable objects. Any locally λ-presentable category is nearly locally λ-presentable.

This definition simplifies if coproduct injections are monomorphisms. Then A is nearly λ-presentable if every for every morphism $f : A \to \coprod_I K_i$ there is a subset J of I of cardinality less that λ such that f factorizes through subcoproduct injection $\coprod_J K_j \to \coprod_I K_i$.

6.1. More about locally generated categories

Following [1] (and the fact that strong and extremal epimorphisms coincide), every locally generated category has a (strong epimorphism, monomorphism) factorization (by such factorizations we mean a factorization system in the sense of [1]).

Lemma 6.1. *Let $\mathcal{K}$ be a locally λ-generated category, $k_i : K_i \to K$ a λ-directed colimit of monomorphisms and $l_i : K_i \to L$ a compatible cocone of monomorphisms. Then the induced morphism $t : K \to L$ is a monomorphism.*

Proof. It suffices to show that, for two morphisms $u, v : A \to K$ with a λ-generated domain, $tu = tv$ implies $u = v$. This follows from the fact that there exists $i \in I$ and $u', v' : A \to K_i$ such that $u = k_i u'$ and $v = k_i v'$. Since $l_i u' = tk_i u' = tk_i v' = l_i v'$, we have $u' = v'$ and thus $u = v$. □

Lemma 6.2. *Let $\mathcal{K}$ be a locally λ-generated category, $k_i : K_i \to K$ a λ-directed colimit of monomorphisms, $h : K \to L$ a strong epimorphism and $K_i \xrightarrow{h_i} L_i \xrightarrow{l_i} L$ (strong epimorphism, monomorphism) factorization of hk_i. Then h is a λ-directed colimit of h_i in the category $\mathcal{K}^{\to}$ of morphisms in $\mathcal{K}$*

Proof. Any morphism $k_{ij} : K_i \to K_j$ of the starting diagram induces the unique monomorphism $l_{ij} : L_i \to L_j$ such that $l_{ij} h_i = h_j k_{ij}$ and $l_j l_{ij} = l_i$ for each $i \leq j$ in I. Thus $l_i : L_i \to L$ is a cocone on a λ-directed diagram. Let $\bar{l}_i : L_i \to \operatorname{colim} L_i$ be a colimit of this diagram and $t : \operatorname{colim} L_i \to L$ the induced morphism. Since $\bar{l}_i h_i$ is a cocone of the starting diagram, there is the unique morphism $g : K \to \operatorname{colim} L_i$ such that $gk_i = \bar{l}_i h_i$ for each i in I. Thus $g = \operatorname{colim} h_i$ in $\mathcal{K}^{\to}$. Since $tgk_i = t\bar{l}_i h_i = l_i h_i = hk_i$ for each i in I, we have $tg = h$. Following 6.1, t is a monomorphism. Since h is an extremal epimorphism, t is an isomorphism. Thus $h = \operatorname{colim} h_i$. □

Definition 3. *Let $\mathcal{K}$ be a cocomplete category having a (strong epimorphism, monomorphism) factorizations and λ be a cardinal. We say that an object A has the λ-factorization property if for any strong epimorphism $h : \coprod\limits_{i \in I} K_i \to K$ and any morphism $f : A \to K$ there is a subset $J \subseteq I$, $|J| < \lambda$ and a morphism $g : A \to K_J$ such that $f = m_J g$; here*

$$\coprod_{j \in J} K_j \xrightarrow{e_J} K_J \xrightarrow{m_J} K$$

is a (strong epimorphism, monomorphism) factorization of the composition hu_J where $u_J : \coprod\limits_{j \in J} K_j \to \coprod\limits_{i \in I} K_i$ is the subcoproduct injection.

Proposition 6.3. *Let $\mathcal{K}$ be a locally λ-generated category. Then an object A has the λ-factorization property if and only if it is λ-generated.*

Proof. Assume that A has the λ-factorization property and consider a λ-directed colimit $k_i : K_i \to K$ of monomorphisms and a morphism $f : A \to K$. Since $\mathcal{K}$ is locally λ-generated, $K_i : K_i \to K$ are monomorphisms. We have a strong epimorphism $h : \coprod\limits_{i \in I} K_i \to K$ giving the colimit as a quotient of a coproduct. This means that $hu_i = k_i$ for each i in I; here $u_i : K_i \to \coprod\limits_{i \in I} K_i$ are coproduct injections.

Let $f = m_J g$ be given by the factorization property. Since I is λ-directed, J has an upper bound i_J in I. We get the induced morphism $t : \coprod\limits_{j \in J} K_j \to K_{i_J}$. This means that $tv_j = k_{j i_J}$ for each coproduct injection $v_j : K_j \to \coprod\limits_{j \in J} K_j$. We have

$$hu_J v_j = hu_j = k_j = k_{i_J} k_{j i_J} = hu_{i_J} k_{j i_j} = hu_{i_J} t v_j$$

for each j in J. Hence $hu_J = hu_{i_J}t$. Thus

$$m_J e_J = hu_J = hu_{i_J}t = k_{ij}t$$

Following the diagonalization property, there is $p : K_J \to K_{i_J}$ such that $k_{i_J}p = m_J$ and $pe_J = t$. Thus $f = m_J g = k_{i_J}pg$. Consequently A is λ-generated.

Conversely, assume that A is λ-generated and consider a strong epimorphism $h : \coprod_{i\in I} K_i \to K$ and a morphism $f : A \to K$. Express $\coprod_{i\in I} K_i$ as a λ-directed colimit of its λ-small subcoproducts $u_J : \coprod_{j\in J} K_j \to \coprod_{i\in I} K_i$. Let $e_J h_J$ be a (strong epimorphism, monomorphism) factorization of $hu_J : \coprod_{j\in J} K_j \to \coprod_{i\in I} K_i$. Following 6.2, $h = \operatorname{colim} h_J$ in $\mathcal{K}^{\to}$. Thus f factorizes through some m_J, which proves that A has the λ-factorization property. □

Remark 6.4. The implication that an object with a λ-factorization property is λ-generated only needs that *λ-directed colimits preserve monomorphisms* in the sense that, given a λ-directed diagram $(k_{ij} : K_i \to K_j)_{i\leq j\in I}$ of monomorphisms, then the colimit cocone $k_i : K_i \to K$ consists of monomorphisms. Any locally λ-generated category even has a stronger property: given λ-directed diagrams $(k_{ij} : K_i \to K_j)_{i\leq j\in I}$ and $(l_{ij} : L_i \to L_j)_{i\leq j\in I}$ of monomorphisms and compatible monomorphisms $h_i : K_i \to L_i$ then the induced morphism $h : K \to L$ on colimits is a monomorphism.

Corollary 6.5. *A cocomplete category $\mathcal{K}$ is locally λ-generated if and only if*

1. *λ-directed colimits preserve monomorphisms and*
2. *$\mathcal{K}$ contains a strong generator $\mathcal{A}$ consisting of objects having the λ-factorization property.*

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