Accessible categories and inaccessible cardinals

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Definition 1. A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-directed colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

An object A is λ-presentable if its hom-functor

$$\hom(A, -) : \mathcal{K} \to \mathbf{Set}$$

preserves λ-directed colimits.

A category is *accessible* if it is λ-accessible for some regular cardinal λ. A cocomplete λ-accessible category is called *locally λ-presentable.* A category is *locally presentable* is it is locally λ-presentable for some regular cardinal λ.

Makkai, Paré 1989
Gabriel, Ulmer 1971; Grothendieck 1972; Lair 1981; JR 1981

Examples 1. (1) $\mathbf{Elem}(T)$ where T is an $L_{\alpha\beta}$-theory; morphisms are $L_{\alpha\beta}$-elementary embeddings. If λ is sufficiently large then K is λ-presentable iff $|K| < \lambda$.

(2) $\mathbf{Mod}(T)$ where T is a basic $L_{\alpha\beta}$-theory; morphisms are homomorphisms. T is basic if it consists of sentences

$$(\forall x)(\varphi(x) \to \psi(x))$$

where $\varphi(x)$ and $\psi(x)$ are positive-existential formulas. Again, for λ sufficiently large, K is λ-presentable iff $|K| < \lambda$. Any accessible category is of this kind.

(3) **Grp**, R-**Mod** are accessible and K is λ-presentable iff it is λ-presentable in a usual sense, i.e., presented by $< \lambda$ elements and $< \lambda$ equations. $\mathbf{Ch}(R)$, **SSet** are accessible. **Top**, Ho **SSet** are not accessible.

(4) Any abstract elementary class is an accessible category.

(5) The category **FAb** of free abelian groups is accessible if there is a compact cardinal and is not accessible under the axiom of constructibility.

Definition 2. For regular cardinals $\lambda < \mu$ we say that $\lambda \lhd \mu$ if $P_\lambda(X)$ has a cofinal set of cardinality $< \mu$ for each set X of cardinality $< \mu$. $P_\lambda(X)$ denotes the set of all subsets of X of cardinality $< \lambda$.

If $\lambda \lhd \mu$ then each λ-accessible category is μ-accessible. The category $\mathbf{Pos}_\lambda$ of λ-directed posets and order-embeddings is λ-accessible and is μ-accessible for $\lambda < \mu$ iff $\lambda \lhd \mu$.

Theorem 1. (Beke, JR) A λ-accessible category with directed colimits is μ-accessible for each $\lambda < \mu$.

A functor $F : \mathcal{K} \to \mathcal{L}$ is called *(strongly)* λ*-accessible* if $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible categories and F preserves λ-directed colimits (and λ-presentable objects). Any λ-accessible functor is μ-accessible for each $\lambda \lhd \mu$. F is called accessible if it is λ-accessible for some regular cardinal λ. By the Uniformization theorem of Makkai, Paré, for each set of accessible functor F_i there is a regular cardinal λ such all F_i are strongly λ-accessible. Theorem 1 holds for strongly λ-accessible functors as well.

A fundamental result of Makkai and Paré is that accessible categories are closed under all constructions of "a limit type". This means that the 2-category of accessible categories and accessible functors has all limits appropriate for 2-categories calculated in the 2-category of categories and functors.

Examples 2. (1) Let $\mathcal{K}$ be an accessible category and $f : A \to B$ a morphism. The (pseudo-)pullback

$$\begin{tikzcd} \mathcal{L} \arrow[r] \arrow[d] & \mathcal{K} \arrow[d, "\mathcal{K}(f,-)"] \\ \mathbf{Epi} \arrow[r] & \mathbf{Set}^{\to} \end{tikzcd}$$

yields the full subcategory $\mathcal{L}$ of $\mathcal{K}$ consisting of objects injective w.r.t. f. Thus any small-injectivity class in an accessible category is accessible. For example, injective R-modules or fibrant objects in a combinatorial model category.

Replacing **Epi** by **Iso** one gets small-orthogonality classes.

One cannot handle projectivity in this way because $\mathcal{K}(-, f)$ is contravariant.

(2) A morphism $g : C \to D$ has a right lifting property w.r.t. f if it is injective in $\mathcal{K} \downarrow D$ to the morphism induced by f. Since $\mathcal{K} \downarrow D$ is accessible, the category $\mathcal{F}^{\Box}$ of morphisms having the right lifting property w.r.t. a set $\mathcal{F}$ of morphisms in $\mathcal{K}$ is accessible.

Let $\mathcal{K}$ be a combinatorial model category and $\mathcal{J}$ a generating set of cofibrations. Weak equivalences are given by a pullback

$$\begin{array}{ccc} \mathcal{W} & \longrightarrow & \mathcal{J}^{\Box} \\ \downarrow & & \downarrow \\ \mathcal{K}^{\to} & \xrightarrow[R]{} & \mathcal{K}^{\to} \end{array}$$

where R is the accesible functor sending a morphism h to the fibration in its (trivial cofibration, fibration) factorization. Thus $\mathcal{W}$ is accessible.

(3) Let $P_n : \mathbf{SSet}^{\omega} \to \mathbf{SSet}$ be projections and $\Sigma : \mathbf{SSet} \to \mathbf{SSet}$ be the suspension functor. A spectrum A is given by inserting morphisms $\Sigma P_n A \to P_{n+1} A$, $n \in \omega$. Since Σ, P_n are accessible and inserters are limits, $\mathbf{Sp}$ is accessible.

Makkai, Paré proved that limits of finitely accessible categories and finitely accessible functors are precisely categories $\mathbf{Mod}(T)$ where T is a basic theory in $L^*_{\alpha\omega}$ ($*$ means finitary conjuctions only). An example is $\mathcal{W}$ in $\mathbf{SSet}$.

Accessible categories are not closed under colimits.

Example 3. Let $F : R\text{-}\mathbf{Mod} \to \mathbf{Inj}$ be the accessible functor giving a weak reflection of modules to injective modules and $\varphi : F \to 0$ the unique natural transformation to the zero functor 0. The coinverter of $\varphi : F \to 0$ is the stable category of modules $R\text{-}\mathbf{Mod}\,/\,\mathbf{Inj}$ which makes all injective modules isomorphic to the trivial module 0. In general, the stable category does not have λ-filtered colimits for any regular cardinal λ and thus is not accessible. It can be also viewed as a pseudo-coequalizer of F and 0.

Analogously, we could present $\mathrm{Ho}\,\mathbf{SSet}$ as a colimit of accessible categories and accessible functors.

Paré showed that accessible categories are closed under directed colimits of full embeddings (unpublished).

Theorem 2. (JR) Assuming the existence of arbitrarily large compact cardinals, any directed colimit of accessible categories and accessible embeddings is accessible.

Problem 1. Does one need set theory for this result?

Example 4. Consider the countable chain of finitely accessible categories and finitely accessible functors

$$\mathbf{Set} \xrightarrow{F_{01}} \mathbf{Set}^2 \xrightarrow{F_{12}} \dots \mathbf{Set}^n \xrightarrow{F_{nn+1}} \dots$$

where $F_{nn+1}(X_1, \dots, X_n) = (X_1, \dots, X_n, X_n)$ and $F_{nn+1}(f_1, \dots, f_n) = (f_1, \dots, f_n, f_n)$. The colimit $\mathbf{Set}^{<\omega}$ consists of eventually constant sequences (X_n). $\mathbf{Set}^{<\omega}$ is accessible assuming the existence of an $L_{\omega_1\omega}$-compact cardinal.

Does the accessibility of $\mathbf{Set}^{<\omega}$ depend on set theory?

Vopěnka's principle says that no accessible category has a large rigid class of objects. Many properties of accessible categories are equivalent to VP. For example, the fact that a full subcategory of a locally presentable category closed under colimits is coreflective.

Weak Vopěnka's principle says that $\mathbf{Ord}^{\mathrm{op}}$ cannot be fully embedded into an accessible category. It is equivalent to the fact that a full subcategory of a locally presentable category closed under limits is reflective.

$$VP \Rightarrow WVP$$

Problem 2. Is VP equivalent to WVP?

WVP implies the existence of a proper class of measurable cardinals. VP is stronger then ¬M.

Problem 3. Is WVP equivalent to ¬M?

Semiweak Vopěnka's principle says that no accessible category can contain objects A_i indexed by ordinals such that $\hom(A_i, A_j) \neq \emptyset$ iff $i \geq j$. It is equivalent to the fact that a full subcategory of a locally presentable category closed products and retracts is weakly reflective. sWVP is between WVP and VP.

Problem 4. Is sWVP equivalent to WVP or to V?

Theorem 3. (Bagaria, Casacuberta, Matthias, JR) Assume the existence of a proper class of supercompact cardinals. Then each $\mathbf{\Sigma}_2$-definable full subcategory of a locally presentable category closed under limits (products and retracts) is (weakly) reflective.

Let $\mathcal{L}$ be the closure of groups $\mathbb{Z}^\kappa/\mathbb{Z}^{<\kappa}$, where κ is a cardinal, under products and retracts in the category of Abelian groups. Then the weak reflectivity of $\mathcal{L}$ lies between the existence of a supercompact cardinal and the existence of a measurable cardinal.

A *full image* of an accessible functor $F : \mathcal{K} \to \mathcal{L}$ is the full subcategory of $\mathcal{L}$ on objects FK, $K \in \mathcal{K}$.

Examples 5. (1) **FAb** is the full image of $F : \mathbf{Set} \to \mathbf{Ab}$.

(2) Homotopy equivalences in a combinatorial model category $\mathcal{K}$ are a full image of an accessible functor into $\mathcal{K}^{\to}$.

(3) Cofibrant objects in a combinatorial model category $\mathcal{K}$ are a split idempotent completion of a full image of an accessible functor into $\mathcal{K}$.

Theorem 4. (JR) Let $\mathcal{M}$ be the full image of an accessible functor into $\mathcal{L}$. Assuming the existence of arbitrarily large compact cardinals, the inclusion of $\mathcal{M}$ into $\mathcal{L}$ preserves existing λ-directed colimits for some regular cardinal $\mathcal{L}$.

Problem 5. Does one need set theory for this result?

A category $\mathcal{K}$ is called *preaccessible* if it satisfies condition (2) from Definition 1 for some regular cardinal λ.

Theorem 5. (Adámek, JR) VP is equivalent to the fact that any full subcategory of an accessible category is preaccessible.

Theorem 6. (JR) Assuming the existence of arbitrarily large compact cardinals, every full image of an accessible category is preaccessible.

Problem 6. Does one need set theory for this result?

A category $\mathcal{K}$ is called *weakly preaccessible* if it has a set $\mathcal{A}$ of objects such that every object K of $\mathcal{K}$ is a canonical λ-filtered colimit of objects from $\mathcal{A}$ (i.e., of the diagram $\mathcal{A} \downarrow K \to \mathcal{K}$).

Theorem 7. (Bagaria, Casacuberta, Matthias, JR) Any $\mathbf{\Sigma}_1$-definable full subcategory of an accessible category is weakly preaccessible.

The model category **Top** of topological spaces is cofibrantly generated but not combinatorial. It is Quillen equivalent to the combinatorial model category **SSet** of simplicial sets.

Theorem 8. VP is equivalent to the fact that every cofibrantly generated model category is Quilen equivalent to a combinatorial one.

$\Rightarrow$ Raptis (2008), $\Leftarrow$ JR

A *left Bousfield localization* of a model category $\mathcal{K}$, $\mathcal{C}$ (cofibrations), $\mathcal{W}$ w.r.t. a class of morphisms $\mathcal{Z}$ is a model category structure $\mathcal{K}\backslash\mathcal{Z}$ on $\mathcal{K}$ such that

(a) $\mathcal{K}\backslash\mathcal{Z}$ has the same cofibrations as $\mathcal{K}$,

(b) weak equivalences of $\mathcal{K}\backslash\mathcal{Z}$ contain $\mathcal{W} \cup \mathcal{Z}$,

(c) each left Quillen functor $H : \mathcal{K} \to \mathcal{L}$ sending morphisms from $\mathcal{Z}$ to weak equivalences is a left Quillen functor $\mathcal{K}\backslash\mathcal{Z} \to \mathcal{L}$.

Theorem 9. (JR, Tholen) Assuming VP, for any left proper, combinatorial model category $\mathcal{K}$ and any class $\mathcal{Z}$ a left Bousfield localization $\mathcal{K}\backslash\mathcal{Z}$ exists.

The proof is based on the absolute result of Smith for a set $\mathcal{Z}$.

Problem 7. Does one need set theory for this result?

Przeździecki showed that it implies WVP.