When is flatness coherent?

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Abstract

We characterize those small categories with the property that flat (contravariant) functors on them are coherently axiomatized in the language of presheaves on them. They are exactly the categories with the property that every finite diagram into them has a finite set of (weakly) initial cocones.

1 Introduction

Finitely accessible categories are precisely categories of flat presheaves $\mathcal{C}^{op} \to \mathrm{Set}$ on a small category $\mathcal{C}$. They always have a geometric axiomatization in the language of presheaves over $\mathcal{C}$. This language is many sorted and has objects of $\mathcal{C}$ as sorts and morphisms of $\mathcal{C}$ as unary operation symbols. A geometric axiomatization means an axiomatization by sentences

$$\forall x_1, ..., \forall x_n(\phi(x_1, ..., x_n) \to \psi(x_1, ..., x_n))$$

in the logic $\mathcal{L}_{\infty,\omega}$, where ϕ and ψ are positive - existential formulas (see [MP], [AR]).

A natural question is when ϕ and ψ can be taken in the usual first-order logic $\mathcal{L}_{\omega,\omega}$. In this case a geometric axiomatization is called coherent. We will show that this happens

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exactly when $\mathcal{C}$ has finite *fc-colimits.* The latter concept is a generalization of the existence of finite colimits and corresponds to the passage from $\exists! x$ to $\exists x_1, ..., \exists x_n$. This idea is present in [R] and the concept, present in [SGA4], has emerged recently in [B].

In the additive setting presheaves correspond to additive functors $\mathcal{C}^{op} \to \mathrm{AbGr}$ on a small additive category $\mathcal{C}$. In particular when $\mathcal{C}$ has a single object then it corresponds to a ring R, presheaves on $\mathcal{C}$ are right R-modules and flat presheaves are flat modules. Therefore, our question asks when flat right R-modules are coherently axiomatizable in R-modules. The answer is well known: This happens precisely when R is left coherent ([SE], Thm.4). We get this result as a consequence of our Theorem 3.3.

We present two proofs of Theorem 3.3 – one is category-theoretical and uses the machinery of classifying toposes and the second is model-theoretical and uses the compactness theorem. The latter is used to prove Theorem 4.2 which is a generalization of Theorem 3.3 to more general embeddings than that of flat presheaves in presheaves.

2 Flat presheaves

Let $\mathcal{C}$ be a small category. A presheaf $F\colon \mathcal{C}^{op} \to \mathrm{Set}$ is called flat if its left Kan extension $\mathrm{Lan}_Y F\colon [\mathcal{C}, \mathrm{Set}] \to \mathrm{Set}$ along the Yoneda embedding $Y\colon \mathcal{C}^{op} \to [\mathcal{C}, \mathrm{Set}]$ preserves finite limits. This is equivalent to being a filtered colimit of representable functors (see [MP]). Thus the category $\mathrm{Flat}(\mathcal{C}^{op})$ of flat presheaves is the free completion $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$ under filtered colimits (see [SGA4]).

2.1 Definition. A diagram $D\colon \mathbb{I} \to \mathcal{C}$ has an *fc-limit* if there exists a finite family of cones for D which is weakly final, in the sense that every other cone for D factors through one in that family. This is equivalent to saying that the (contravariant) cone functor $\mathrm{cone}(-, D)$ is finitely generated in $[\mathcal{C}^{op}, \mathrm{Set}]$. An fc-colimit of D is defined dually.

2.2 Definition. Given an inclusion $u\colon \mathcal{C} \hookrightarrow \mathcal{D}$ we say that it is an *fc-reflection* if for every $D \in \mathcal{D}$ there is a finite family $\{r_i\colon D \to u(C_i) |\ i = 1, ...n\}$ such that every $D \to u(C)$ factors through one of the r_i's.

2.3 Proposition. *If $\mathcal{C}$ is a small category that has finite fc-limits then a functor into a Grothendieck topos, $F\colon \mathcal{C} \to \mathcal{E}$, is flat if and only if it* merges finite fc-limits, *in the sense that if $\{P_k |\ k = 1, ..., n\}$ is an fc-limit for the finite diagram $D\colon \mathbb{I} \to \mathcal{C}$, then the canonically induced $\coprod_{\mathbb{K}} FP_k \to \lim F \circ D$ is epi.*

Proof: Assume that F is flat. We may take this notion to mean what is called *left filtering* in [MM], i.e that it satisfies

- The family of all maps $F(C) \to 1$, for all $C \in \mathcal{C}$, is epimorphic
- For any two objects C, $D \in \mathcal{C}$, the family of maps $F(B) \to F(C) \times F(D)$, where B runs over all the cones $C \leftarrow B \to D$, is epimorphic.

- For any pair of parallel arrows, $u, v: C \rightrightarrows D$ in $\mathcal{C}$, the family of induced maps $F(B) \to Eq(F(u), F(v))$ to the equalizer of $F(u)$, $F(v)$, where B runs over all the cones $B \to C \rightrightarrows D$, is epimorphic.

By ([MM], Theorem VII.9.1) the conjunction of the above conditions is equivalent to the left Kan extension of F along the Yoneda embedding $Y: \mathcal{C} \to [\mathcal{C}^{op}, \mathrm{Set}]$ being left exact. This left Kan extension also has a right adjoint thus it preserves coproducts and epimorphisms. Notice further that, since $\mathcal{C}$ has finite fc-limits, for every finite diagram $D: \mathbb{I} \to \mathcal{C}$ with an fc-limit $\{P_k\}$, we have a pointwise epi $\coprod_k \hom_{\mathcal{C}}(C, P_k) \to \mathrm{cone}(C, D) \cong \lim_i \hom(C, D_i)$, thus the canonical arrow $\coprod_k yP_k \to \lim(y \circ D)$ is an epi. But since

$$\begin{aligned} \coprod_k FP_k &\cong \coprod_k (\mathrm{Lan}_y F \circ y)(P_k) \\ &\cong (\mathrm{Lan}_y F)(\coprod_k yP_k) \end{aligned}$$

while

$$\begin{aligned} \lim(F \circ D) &\cong \lim(\mathrm{Lan}_y F \circ y \circ D) \\ &\cong (\mathrm{Lan}_y F)(\lim(y \circ D)) \end{aligned}$$

we conclude that F merges finite fc-limits.

Conversely assume that F merges finite fc-limits. Then, in order to verify, say, the third clause in the definition of flatness given above, consider a pair of arrows

$$x, y: Eq(Fu, Fv) \rightrightarrows X$$

in $\mathcal{E}$. Assume that whenever they are restricted along any $w: FB \to Eq((Fu, Fv)$, where B is a cone for $u, v: C \rightrightarrows D$, they become equal. In particular they become equal upon restriction to the image under F of the cones $\{P_k\}$ of the fc-limit. But the canonical $\coprod_{\mathbb{K}} FP_k \to \lim F \circ D$ is epi thus x and y are already equal. ∎

3 The characterization theorem

3.1 Theorem. *Let $\mathcal{C}$ be a small category. Then the following conditions are equivalent:*

(i) $\mathcal{C}$ has finite fc-colimits;

(ii) The inclusion $\eta: \mathcal{C} \to \mathrm{colex}(\mathcal{C})$ is fc-reflective;

(iii) There is a coherent theory $\mathbb{T}$ in the language of presheaves on $\mathcal{C}$ such that the category $\mathrm{Mod}_{\mathcal{E}}(\mathbb{T})$ of $\mathbb{T}$-models in any Grothendieck topos $\mathcal{E}$ is equivalent to $\mathrm{Flat}(\mathcal{C}^{op}, \mathcal{E})$, the category of flat functors with values in $\mathcal{E}$.

Proof: (i) $\Rightarrow$ (iii) According to the Proposition above the topos - valued functors out of a category with finite fc-colimits are exactly those $F\colon \mathcal{C}^{op} \to \mathcal{E}$ that *merge finite fc-colimits*, i.e whenever $D\colon \mathbb{I} \to \mathcal{C}^{op}$ is a finite diagram with fc-colimit $P\colon \mathbb{K} \to \mathcal{C}^{op}$ then $\coprod_{\mathbb{K}} FP_k \to \lim FD$ is epi. From this follows that we can axiomatize flat functors by the coherent sentences

$$\forall x_1 : D_{i_1} ... \forall x_n : D_{i_n} (\bigwedge_{\delta_{ij}: D_i \to D_j} (D(\delta_{ij})(x_i) = x_j) \to \bigvee_{k \in \mathbb{K}} \exists y : P_k \bigwedge_i (p_{ki}(y) = x_i)).$$

It is obvious that models of such sentences in the category of sets are just the finite fc-colimit merging functors. Let us say a few more words concerning the interpretation of such sentences in a topos: In the one direction we want, assuming that F merges finite fc-colimits, that any $E \in \mathcal{E}$ forces the above sentences. So, having given E-elements $x_i\colon E \to D_i$ which satisfy the hypothesis of the implication means that we have a cone $(E, x\colon E \to F \circ D)$, thus there is a factorization $E \to \lim(F \circ D)$. Pulling back the epimorphic family $\{FP_k \to \lim(F \circ D)\}$ we obtain a cover $\{E_k \to E\}$ and, for each k, an E_k-element of FP_k with the required property.

Conversely assuming that F satisfies, in the internal logic of $\mathcal{E}$, the above sentence we will show that it merges finite fc-limits, i.e that the canonical $\alpha\colon \coprod_{\mathbb{K}} FP_k \to \lim FD$ is epi. Take $x\colon E \to \lim(F \circ D)$, equivalently a cone $< E, x_i\colon E \to FD_i >$. Then there is a cover $E' \xrightarrow{\epsilon} E$ and an E'-element $y_k\colon E' \to FP_k$ such that, for all i, $x_i \circ \epsilon = p_{k,i} \circ y_k$. In other words we have an E'-element $y\colon E' \to \coprod_{\mathbb{K}} FP_k$ such that $\alpha \circ y = x \circ \epsilon$, i.e α is internally surjective.

(iii) $\Rightarrow$ (ii) Assume that there is a coherent theory $\mathbb{T}$ in the language of presheaves on $\mathcal{C}$ such that $\mathrm{Flat}(\mathcal{C}^{op}, \mathcal{E}) \cong \mathrm{Mod}_{\mathcal{E}}(\mathbb{T})$. Let $B[\mathbb{T}]$ denote the classifying topos for the theory in question. We have

$$\mathrm{Mod}_{\mathcal{E}}(\mathbb{T}) \cong \mathrm{Flat}(\mathcal{C}^{op}, \mathcal{E}) \cong \mathrm{GTop}(\mathcal{E}, [\mathcal{C}, \mathrm{Set}]),$$

where $\mathrm{GTop}(\mathcal{E}, \mathcal{F})$ denotes the category of geometric morphisms and natural transformations between the Grothendieck toposes $\mathcal{E}$, $\mathcal{F}$. Here the first equivalence holds by the assumption while the second one is the content of Diaconescu's theorem. Thus, by the universal property of the classifying topos $B[\mathbb{T}] \cong [\mathcal{C}, \mathrm{Set}]$.

This topos is a subtopos of the presheaf topos $[\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$ that classifies the theory of presheaves on $\mathcal{C}$, and the inclusion is induced in such a way that its direct image is the direct image of the essential geometric morphism induced by the inclusion $\eta\colon \mathcal{C} \to \mathrm{colex}(\mathcal{C})$:

$$W \dashv - \circ \eta \dashv U\colon [\mathcal{C}, \mathrm{Set}] \cong B[\mathbb{T}] \hookrightarrow [\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$$

Since it classifies a coherent extension of the theory of presheaves $\mathbb{T}_{\mathcal{C}}$, $[\mathcal{C}, \mathrm{Set}]$ is closed in $[\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$ under filtered colimits.

This, in particular, implies that, given $X \in \mathrm{colex}(\mathcal{C})$, the functor $\hom_{\mathrm{colex}(\mathcal{C})}(X, \eta(-))$ is a finitely presentable object in $[\mathcal{C}, \mathrm{Set}]$ thus

$$\hom_{\mathrm{colex}(\mathcal{C})}(X, \eta(-)) \cong \mathrm{colim}_i \hom_{\mathcal{C}}(C_i, -)$$

for a finite diagram of representables in $[\mathcal{C}, \mathrm{Set}]$. The identity arrows $id_i: C_i \to C_i$ are represented under this isomorphism by arrows $X \to \eta(C_i)$ which form the fc-reflection for $\eta: \mathcal{C} \to \mathrm{colex}(\mathcal{C})$: Given any $X \to \eta(C)$ it has to factor through one of those $X \to \eta(C_i)$, because of the above isomorphism.

(ii) $\Rightarrow$ (i) Let $D: \mathbb{I} \to \mathcal{C}$ be a finite diagram into $\mathcal{C}$ and let X be the colimit of $\eta \circ D$ in $\mathrm{colex}(\mathcal{C})$. Let further $\{X \to \eta(C_i) |\ i = 1, ..., n\}$ be an fc-reflection of X into $\mathcal{C}$. Then any cocone $D \Rightarrow C$ for D in $\mathcal{C}$ induces a cocone $\eta(D) \Rightarrow \eta(C)$ in $\mathrm{colex}(\mathcal{C})$ which factors uniquely through X and then this factorization $X \to \eta(C)$ factors through the fc-reflection $\{X \to \eta(C_i)\}$ and this manifests $\{C_i |\ i = 1, ..., n\}$ as an fc-colimit for D in $\mathcal{C}$. ∎

3.2 Remark. In the above proof we could have derived (i) from (iii) directly once we had identified the classifying topos for $\mathbb{T}$ as $[\mathcal{C}, \mathrm{Set}]$. Then we could have concluded using Exercise 2.17(c) of Expose VI in [SGA4]. We include the full argument above for the sake of completeness and hoping that we have exhibited a clearer picture of the interconnections of the involved concepts.

Up to this point our arguments have been valid in the internal logic of any topos with natural numbers object (so that the notion of classifying topos makes sense). Assuming now the Prime Ideal Theorem (so that coherent toposes have enough points) we may improve the above characterization as follows:

3.3 Theorem. *Let $\mathcal{C}$ be a small category. Then the following conditions are equivalent:*

(i) $\mathcal{C}$ has finite fc-colimits;

(ii) The inclusion $\eta: \mathcal{C} \to \mathrm{colex}(\mathcal{C})$ is fc-reflective;

(iii) $\mathrm{Flat}(\mathcal{C}^{op}, \mathrm{Set})$ is axiomatized by a coherent theory in the language of presheaves on $\mathcal{C}$.

Proof: The extra difficulty now arises in showing that (iii) $\Rightarrow$ (ii). So assume that $\mathrm{Flat}(\mathcal{C}^{op}, \mathrm{Set})$ is axiomatized by a coherent theory $\mathbb{T}$ in the language of presheaves on $\mathcal{C}$. Let $B[\mathbb{T}]$ denote the classifying topos for the theory in question. This topos is a subtopos of the presheaf topos $[\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$ that classifies the theory of presheaves on $\mathcal{C}$. Since it classifies a coherent extension of the theory of presheaves $\mathbb{T}_{\mathcal{C}}$ it is closed in $[\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$ under filtered colimits. We argue that $B[\mathbb{T}]$ is itself a presheaf topos: Take any $\mathbb{T}$-model in Set, in other words an object in $\mathrm{Flat}(\mathcal{C}^{op}, \mathrm{Set})$. It is a filtered colimit of finitely presentable objects in $\mathrm{Flat}(\mathcal{C}^{op}, \mathrm{Set})$. The inclusion

$$\mathrm{Flat}(\mathcal{C}^{op}, \mathrm{Set}) \hookrightarrow \mathrm{Flat}((\mathrm{colex}(\mathcal{C}))^{op}, \mathrm{Set})$$

preserves filtered colimits and finitely presentable objects because it is induced by the universal property of the inductive completion from the inclusion $\eta: \mathcal{C} \to \mathrm{colex}(\mathcal{C})$. This means that the model in question is expressed, as an object in $\mathrm{Flat}((\mathrm{colex}(\mathcal{C}))^{op}, \mathrm{Set})$ as a filtered colimit in $\mathrm{Mod}(\mathbb{T}_{\mathcal{C}})$ of objects that are finitely presentable in $\mathrm{Mod}(\mathbb{T}_{\mathcal{C}})$. Thus the condition of Theorem 1.1 in [B] is satisfied and using the fact that coherent toposes have enough points, we obtain as a consequence we obtain that $B[\mathbb{T}]$ is a presheaf topos.

Inspecting the proof in [B] we see that it is equivalent to $[\mathcal{C}, \mathrm{Set}]$ in such a way that the direct image of its inclusion into $[\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$ is the direct image of the essential geometric morphism induced by the inclusion $\eta: \mathcal{C} \to \mathrm{colex}(\mathcal{C})$:

$$W \dashv - \circ \eta \dashv U: [\mathcal{C}, \mathrm{Set}] \cong B[\mathbb{T}] \hookrightarrow [\mathrm{colex}(\mathcal{C}), \mathrm{Set}]$$

Then the rest of the above proof applies. ∎

4 A generalization and applications

4.1 Definition. An object K of a category $\mathcal{K}$ is said to be injective with respect to a finite cone $(m_i: A \to A_i)_{i=1,\dots,n}$ provided that for each morphism $f: A \to K$ there exists an index i and a morphism $f': A_i \to K$ with $f = f' \circ m_i$.

A full subcategory $\mathcal{X}$ of $\mathcal{K}$ is called an *fc-injectivty class* provided that there exists a class $\mathcal{M}$ of finite cones such that $\mathcal{X}$ precisely consists of objects injective with respect to each cone in $\mathcal{M}$. $\mathcal{X}$ is called an *ω-fc-injectivity class* if all domains and codomain in cones from $\mathcal{M}$ are finitely presentable.

Following [AR] 5.33, any finite cone with finitely presentable domains and codomains gives rise to a coherent sentence α such that, for an A in $\mathcal{K}$, $A \models \alpha$ if and only if A is injective to the cone.

We have mentioned that the category of flat presheaves on a small category $\mathcal{C}$ forms the free completion $\mathrm{Ind}(\mathcal{C})$ of $\mathcal{C}$ under filtered colimits. The category $[\mathcal{C}^{op}, \mathrm{Set}]$ of all presheaves on $\mathcal{C}$ is the free completion of $\mathcal{C}$ under all colimits. One has the formula

$$[\mathcal{C}^{op}, \mathrm{Set}] = \mathrm{Ind}(\mathrm{colex}\mathcal{C}),$$

where $\mathrm{colex}\mathcal{C}$ is the free completion of $\mathcal{C}$ under finite colimits. The inclusion $\mathrm{Ind}\mathcal{C} \hookrightarrow [\mathcal{C}^{op}, \mathrm{Set}]$ is induced by the inclusion $\mathcal{C} \hookrightarrow \mathrm{colex}\mathcal{C}$. In view of these remarks our characterization theorem admits the following generalization:

4.2 Theorem. *Let $\mathcal{A}$ be a small category with finite fc-colimits and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\mathcal{B}$ is fc-reflective in $\mathcal{A}$;*

(ii) $\mathrm{Ind}\mathcal{B}$ *is axiomatized by a coherent theory in the language of presheaves on $\mathcal{A}$;*

(iii) $\mathrm{Ind}\mathcal{B}$ *is axiomatized in the language of presheaves on $\mathcal{A}$;*

(iv) $\mathrm{Ind}\mathcal{B}$ *is closed in* $\mathrm{Ind}\mathcal{A}$ *under ultraproducts;*

(v) $\mathrm{Ind}\mathcal{B}$ *is an ω-fc-injectivity class in* $\mathrm{Ind}\mathcal{A}$.

Proof: (i)⇒(v) Let $M \in \mathrm{Ind}\mathcal{A}$ be injective with respect to all fc-reflections of $A \in \mathcal{A}$ in $\mathcal{B}$. Then the comma category $\mathcal{B} \downarrow M$ is final in $\mathcal{A} \downarrow M$ and thus $M \in \mathrm{Ind}\mathcal{B}$. Thus $\mathrm{Ind}\mathcal{B}$ precisely consists of those objects from $\mathrm{Ind}\mathcal{A}$ which are injective to these fc-reflections.

The implications (v)⇒(ii)⇒(iii)⇒(iv) are evident.

(iv)⇒(i) Following 2.3 $\mathrm{Ind}\mathcal{A}$ is axiomatized in the language $\mathcal{L}$ of presheaves on $\mathcal{A}$ and thus it is closed in $[\mathcal{A}^{op}, \mathrm{Set}]$ under ultraproducts. Consequently $\mathrm{Ind}\mathcal{B}$ is closed in $[\mathcal{A}^{op}, \mathrm{Set}]$ under ultraproducts. Since $\mathrm{Ind}\mathcal{B}$ is closed in $[\mathcal{A}^{op}, \mathrm{Set}]$ under pure subobjects (see the proof of 2.32 in [AR]), it is closed in $[\mathcal{A}^{op}, \mathrm{Set}]$ under elementary subobjects (see [AR] 5.15). Following [CK] 6.1.15, 4.1.12 and 4.1.13 $\mathrm{Ind}\mathcal{B}$ is axiomatized in $\mathcal{L}$ by a theory $\mathbb{T}$. Let $A \in \mathcal{A}$ and extend $\mathcal{L}$ by a constant c_A of the sort A. Models of the resulting language $\mathcal{L}_A$ are precisely the pairs (M, h) where $h: \hom(-, A) \to M$. Therefore $\mathbb{T}-$models in the language $\mathcal{L}_A$ are precisely the pairs (M, h) with $M \in \mathrm{Ind}\mathcal{B}$. The pairs $(\hom(-, B), \hom(-, f): \hom(-, A) \to \hom(-, B))$, where $f: A \to B$, $B \in \mathcal{B}$ form a weakly initial set in the category of these $\mathbb{T}-$models. There is an $\mathcal{L}_A$ sentence φ_f such that

$$(M, h) \models \varphi_f \;\Leftrightarrow\; \text{there is an } \mathcal{L}_A\text{-homomorphism } \; (\hom(-, B), \hom(-, f)) \to (M, h).$$

Hence the theory $\mathbb{T} \cup \{\neg\varphi_f | \; f: A \to B, \; B \in \mathcal{B}\}$ in the language $\mathcal{L}_A$ is inconsistent. By the compactness theorem there are finitely many morphisms $f_i: A \to B_i$ $B_i \in \mathcal{B}$, $i = 1, ..., n$ such that the theory $\mathbb{T} \cup \{\neg\varphi_{f_i} | i = 1, ..., n\}$ is inconsistent. Thus $f_i: A \to B_i$, $i = 1, ..., n$ form an fc-reflection of A in $\mathcal{B}$. ∎

4.3 Remark. The argument just presented yields another proof of the implication (iii)⇒(i) in Theorem 3.3. The use of classifying topos is now replaced by the compactness theorem. The condition (ii) in Theorem 3.3 is specific for the situation $\mathcal{C} \hookrightarrow \mathrm{colex}\mathcal{C}$ because any object of $\mathrm{colex}\mathcal{C}$ is a finite colimit of objects in $\mathcal{C}$. On the other hand the implication (iii)⇒(i) in the above theorem can be deduced from Theorem 3.3 using the fact that flat presheaves on $\mathcal{A}$ are coherently axiomatizable and that the classifying topos for that theory is $[\mathcal{A}, \mathrm{Set}]$. Then we can conclude using Remark 1.2 , following Theorem 1.1, in [B] using a similar argument like the one given in the proof of Theorem 3.3.

Theorem 4.2 generalizes the following result (see [AR] and [BR]). Let us recall that an ω-injectivity class is an ω-fc-injectivity class such that all cones in $\mathcal{M}$ consist of a single morphism.

4.4 Theorem. *Let $\mathcal{A}$ be a small category with weak finite colimits and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\mathcal{B}$ is weakly reflective in $\mathcal{A}$;*

(ii) $\mathrm{Ind}\mathcal{B}$ *is axiomatized by a regular theory in* $\mathrm{Ind}\mathcal{A}$*;*

(iii) $\mathrm{Ind}\mathcal{B}$ *is closed in* $\mathrm{Ind}\mathcal{A}$ *under products;*

(iv) $\mathrm{Ind}\mathcal{B}$ *is an ω-injectivity class in* $\mathrm{Ind}\mathcal{A}$*;*

(v) $\mathrm{Ind}\mathcal{B}$ *is weakly reflective in* $\mathrm{Ind}\mathcal{A}$.

Of course we can further specialize Theorem 4.2 to the following classical fact (see [AR]). In what follows orthogonality is a condition similar to injectivity with the extra requirement that f' is unique.

4.5 Theorem. *Let $\mathcal{A}$ be a small category with finite colimits and $\mathcal{B}$ a full subcategory of $\mathcal{A}$. The following conditions are equivalent:*

(i) *$\mathcal{B}$ is reflective in $\mathcal{A}$;*

(ii) Ind$\mathcal{B}$ *is axiomatized by a limit theory in* Ind$\mathcal{A}$*;*

(iii) Ind$\mathcal{B}$ *is closed in* Ind$\mathcal{A}$ *under limits;*

(iv) Ind$\mathcal{B}$ *is an ω-orthogonality class in* Ind$\mathcal{A}$*;*

(v) Ind$\mathcal{B}$ *is reflective in* Ind$\mathcal{A}$.

4.6 Remark. In Theorem 4.2 one can not add the condition that Ind$\mathcal{B}$ is fc-reflective in Ind$\mathcal{A}$. Consider the language with unary relation symbols R_i and binary relation symbols S_i, $i = 1, 2, \ldots.$ Let $\mathbb{T}$ consist of the sentences

$$\forall x\ (R_i(x) \to \exists y\ S_i(x, y)) \quad i = 1, 2, \ldots$$

Since any finitely presentable $\mathbb{T}-$model is finite and, for each element $a \in A$, there is only finitely many i's with $A \models R_i[a]$, we have $\mathrm{Mod}(\mathbb{T}) = \mathrm{Ind}\mathcal{B}$, $\mathrm{Mod}(\mathcal{L}) = \mathrm{Ind}\mathcal{A}$, $\mathcal{B}$ is fc-reflective in $\mathcal{A}$ and $\mathcal{A}$ has finite colimits. But Ind$\mathcal{B}$ is not fc-reflective in Ind$\mathcal{A}$; it suffices to consider the $\mathcal{L}$ model A with a unique element a such that $A \models R_i[a]$, for each $i = 1, 2, \ldots$ and $A \models \neg S_i(a, a)$, for each $i = 1, 2, \ldots$ Then A does not have an fc-reflection to Ind$\mathcal{B}$.

4.7 Corollary. *Let $\mathcal{A}$ be a small category with finite fc-colimits and finite products and $\mathcal{B}$ a full subcategory of $\mathcal{A}$ closed under finite products. Then* Ind$\mathcal{B}$ *is closed in* Ind$\mathcal{A}$ *under ultraproducts iff it is closed in* Ind$\mathcal{A}$ *under products.*

Proof: Sufficiency is evident because ultraproducts in Ind$\mathcal{B}$ are filtered colimits of products (see [AR]). Conversely, let Ind$\mathcal{B}$ be closed in Ind$\mathcal{A}$ under ultraproducts. Following Theorem 3.1, $\mathcal{B}$ is fc-reflective in $\mathcal{A}$. Since $\mathcal{A}$ has finite products, $\mathcal{B}$ is weakly reflective in $\mathcal{A}$: having an fc-reflection $r_i\colon A \to A_i^*$, $i = 1, \ldots, n$ of A in $\mathcal{B}$ we get the weak reflection $r = < r_1, \ldots, r_n >\colon A \to \prod_{i=1}^n A_i^*$, of A in $\mathcal{B}$. Following Theorem 3.3, Ind$\mathcal{B}$ is closed in Ind$\mathcal{A}$ under products. ∎

4.8 Remark. The proof of Corollary 4.7 shows that a small category with finite fc-colimits and finite products has finite weak colimits. Hence we get the following consequence of Proposition 24 in [BR]:

4.9 Proposition. *If a small category $\mathcal{A}$ is finitely complete and finitely fc-cocomplete then it is finitely cocomplete.*

The equivalence of conditions (ii), (iii) and (iv) in Theorem 4.2 is a consequence of the following result of Volger ([V], see [H] as well).

4.10 Theorem. *Let $\mathcal{L}$ be a language and $\mathcal{C}$ a class of $\mathcal{L}-$models. The following are equivalent*

(i) $\mathcal{C}$ can be axiomatized by a coherent theory;

(ii) $\mathcal{C}$ is closed under ultraproducts and pure subobjects;

(iii) $\mathcal{C}$ is closed under ultraproducts and directed colimits.

Since (ii) implies that $\mathcal{C}$ is axiomatizable (see the proof of Theorem 4.2), the equivalence of (i) and (ii) is a standard preservation theorem which can be deduced using [CK] 3.2.1. In order to apply [V], one uses the fact that $f: A \to B$ is a pure monomorphism iff there is a homomorphism $g: B \to C$ such that $g \circ f: A \to C$ is an elementary emedding (see [CK]); Volger speaks of h-sandwiches.

As a consequence of Theorem 4.2 we obtain the following result, extending [SE], Theorem 4, of Sabbagh and Eklof:

4.11 Theorem. *Let R be a ring. Working in the category of right R-modules and with the one-sorted language of R-modules (with each element of R serving as a unary function symbol for multiplication), the following are equivalent:*

(i) *R is left coherent.*

(ii) *Any product of flat right R-modules is flat.*

(iii) *The category of flat right R-modules is a weakly reflective subcategory of all right R-modules.*

(iv) *The category of finitely generated free right R-modules has weak cokernels.*

(v) *Flat right R-modules have a coherent axiomatization.*

(vi) *Flat right R-modules form a first-order axiomatizable class.*

(vii) *Flat right R-modules are closed under ultraproducts.*

Proof: (i)$\Leftrightarrow$(ii) is a classical fact (see [C]). Since flat right R-modules are precisely filtered colimits of finitely presentable projective right R-modules, they form the category Ind($\mathcal{P}$), where $\mathcal{P}$ is the full subcategory of finitely presentable projective right R-modules. Since all right R-modules form Ind($\mathcal{F}$), where $\mathcal{F}$ denotes the full subcategory of finitely presentable right R-modules, the equivalence (ii)$\Leftrightarrow$(iii) follows from Theorem 4.4. Since every finitely presentable right R-module is the cokernel of a homomorphism between finitely generated free right R-modules, (iv) is equivalent to $\mathcal{P}$ being weakly reflective in $\mathcal{F}$ (modules from $\mathcal{P}$ are precisely direct summands of finitely generated free right R-modules). Hence, following again Theorem 4.4 (iv)$\Leftrightarrow$(iii). Moreover $\mathcal{P}$ is weakly reflective in $\mathcal{F}$ if and only if $\mathcal{P}$ is fc-reflective in $\mathcal{F}$. The latter happens because K is injective to a finite cone $(m_i: A \to A_i)_{i=1,\dots,n}$ if and only if K is injective to the morphism $< m_1, ..., m_n >: A \to \bigoplus_{i=1}^{n} A_i$. Thus following Theorem 4.2 (iv)$\Leftrightarrow$(v)$\Leftrightarrow$(vi)$\Leftrightarrow$(vii). In more detail, Theorem 4.2 yields an axiomatization in the language of presheaves over finitely presentable R-modules. But the latter is first-order (even coherently) definable in our language of R-modules. ∎

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