FLAT COVERS AND FACTORIZATIONS

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Abstract. The flat cover conjecture, saying that every module has a flat (pre)cover, has been recently proved by Bican, El Bashir and Enochs. We relate flat precovers (and cotorsion preenvelopes) to weak factorizations and prove that flat monomorphisms form a left part of a weak factorization system.

1. Introduction

The flat cover conjecture was formulated by E. Enochs [10] and asks whether every module has a flat cover. Enochs also showed that it is equivalent to the existence of flat precovers. Recall that a flat precover of a module M is a homomorphism $g : F \to M$ where F is flat and for each homomorphism $v : G \to M$ with G flat there is $d : G \to F$ such that $g \cdot d = v$. J. Xu proved that flat covers exist over certain commutative rings and presented the status of the conjecture in [17]. The flat cover conjecture has been recently proved in [5]. This work gives two proofs, one of then (due to Enochs) is based on the result of P. Eklof and P. Trlifaj [9]. We analyse the Enochs' proof from the point of view of the theory of accessible categories (cf. [3]) and prove a stronger result saying that flat monomorphisms form a left part of a weak factorization system on the category of modules. Flat precovers (cotorsion preenvelopes) then arise from a weak factorization of module homomorphisms having domain (codomain) equal to 0. The concept of a weak factorization has originated in homotopy theory (see [7] and [14]).

In more detail, a monomorphism $f : A \to B$ is called *flat* if $\operatorname{coker}(f) : B \to F$ has F flat (see [17]). A module homomorphism $g : C \to D$ is said to have the *right lifting property w.r.t. flat monomorphisms* if in each commutative square

$$\begin{array}{ccc} A & \xrightarrow{u} & C \\ {\scriptstyle f}\downarrow & & \downarrow{\scriptstyle g} \\ B & \xrightarrow[v]{} & D \end{array}$$

with f a flat monomorphism there is a diagonal $d : B \to C$ such that $d \cdot f = u$ and $g \cdot d = v$. We show that any module homomorphism $h : N \to M$ has a

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factorization $h = g \cdot f$ where f is flat and g has the right lifting property w.r.t. flat monomorphisms. By applying it to $h : 0 \to M$, we get

$$h : 0 \longrightarrow F \xrightarrow{\quad g \quad} M$$

and g is a flat precover of M. To verify it, one uses the right lifting property as follows

$$\begin{tikzcd} 0 \arrow[r] \arrow[d] & F \arrow[d, "g"] \\ G \arrow[r, "v"'] \arrow[ur, dashed, "d"] & M \end{tikzcd}$$

Since a homomorphism $g : C \to 0$ has the right lifting property w.r.t. flat monomorphisms iff C is cotorsion, by applying our factorization to $h : N \to 0$ we get a cotorsion preenvelope of N:

$$h : N \xrightarrow{\quad f \quad} C \longrightarrow 0$$

Our approach works for an arbitrary class $\mathcal{F}$ of modules (satisfying conditions of Theorem 4.5) in place of flat modules.

2. Weak factorization systems

As we have mentioned in the introduction, weak factorization systems are well known in homotopy theory (see [12], [13] or [1] for a recent exposition).

Definition 2.1. Let $\mathcal{K}$ be a category and $f : A \to B$, $g : C \to D$ morphisms such that in each commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there is a diagonal $d : B \to C$ with $d \cdot f = u$ and $g \cdot d = v$. Then we say that g has the *right lifting property* w.r.t. f and that f has the *left lifting property* w.r.t g.

For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{H}^{\square} = \{g | g \text{ has the right lifting property w.r.t. each } f \in \mathcal{H}\}$$

and

$$^{\square}\mathcal{H} = \{f | f \text{ has the left lifting property w.r.t. each } g \in \mathcal{H}\}.$$

Assume that $\mathcal{K}$ has a terminal object 1. Then an object K of $\mathcal{K}$ is injective to $\mathcal{H}$ iff the morphism $K \to 1$ belongs to $\mathcal{H}^{\square}$. In fact, it exactly means that for every morphisms $f : A \to B$ in $\mathcal{H}$ and every $u : A \to K$ there is $d : B \to K$ with $d \cdot f = u$. We will denote by $\mathcal{H}^{\triangle}$ the full subcategory of $\mathcal{K}$ consisting of all objects injective to $\mathcal{H}$. Dually, if $\mathcal{K}$ has an initial object 0, then K is projective to $\mathcal{H}$ iff the morphism $0 \to K$ belongs to $^{\square}\mathcal{K}$.

Definition 2.2 (see [4])**.** A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{K}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$ such that

(1) $\mathcal{R} = \mathcal{L}^{\square}$, $\mathcal{L} = {}^{\square}\mathcal{R}$

and

(2) any morphism h of $\mathcal{K}$ has a factorization $h = g \cdot f$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

Remark 2.3. (1) Let $(\mathcal{L}, \mathcal{R})$ be a weak factorization system in a category $\mathcal{K}$ having a terminal object 1. Then any morphism $K \to 1$ has a factorization

$$K \xrightarrow{f} \overline{K} \longrightarrow 1$$

with $f \in \mathcal{L}$ and $\overline{K} \in \mathcal{L}^{\triangle}$. It means that K has enough $\mathcal{L}$-injective objects.

Dually, if $\mathcal{K}$ has an initial object 0 then $0 \to K$ has a factorization

$$0 \longrightarrow K^* \xrightarrow{g} K$$

with $g \in \mathcal{R}$ and K^* $\mathcal{R}$-projective. It means that $\mathcal{K}$ has enough $\mathcal{R}$-projective objects.

(2) In general, a morphism $g : C \to D$ lies in $\mathcal{L}^{\square}$ iff as an object of the comma category $\mathcal{K} \downarrow D$, g is $\overline{\mathcal{L}}$-injective for the class $\overline{\mathcal{L}}$ of all morphisms f of $\mathcal{K} \downarrow D$ such that $P(f) \in \mathcal{L}$ where $P : \mathcal{K} \downarrow D \to \mathcal{K}$ is the projection (see [13] 12.4.2). Then, factorizations

$$h : A \xrightarrow{f} C \xrightarrow{g} B$$

yield that $\mathcal{K} \downarrow B$ has enough $\overline{\mathcal{L}}$-injectives for each B in $\mathcal{K}$.

Dually, $A \downarrow \mathcal{K}$ has enough $\overline{\mathcal{R}}$-projectives for each A in $\mathcal{K}$.

For any class $\mathcal{C}$ of morphisms of $\mathcal{K}$ we get $\mathcal{L} = {}^{\square}\mathcal{C}$ and $\mathcal{R} = \mathcal{L}^{\square}$ satisfying 2.2.(1).

Remark 2.4. For any class $\mathcal{C}$ of morphisms of $\mathcal{K}$, ${}^{\square}\mathcal{C}$ contains all isomorphisms and is

(a) stable under pushouts,

(b) closed under transfinite compositions

and

(c) closed under retracts in the comma category $A \downarrow \mathcal{K}$ (of objects under A).

It means that

(a) If

$$\begin{tikzcd} B \arrow[r, "\overline{g}"] & D \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\overline{f}"'] \end{tikzcd}$$

is a pushout and $f \in {}^{\square}\mathcal{C}$ then $\overline{f} \in {}^{\square}\mathcal{C}$.

(b_1) If $f_1 : A \to B$ and $f_2 : B \to C$ are in ${}^{\square}\mathcal{C}$ then $f_2 \cdot f_1 \in {}^{\square}\mathcal{C}$.

(b_2) If $(f_{ij} : A_i \to A_j)_{i \le j \le \lambda}$ is a smooth chain (i.e., λ is a limit ordinal, $(f_{ij} : A_i \to A_j)_{i<j}$ is a colimit for any limit ordinal $j < \lambda$) and $f_{ij} \in {}^{\square}\mathcal{C}$ for $i \le j < \lambda$ then $f_{0\lambda} \in {}^{\square}\mathcal{C}$.

(c) If $f : A \to B$ is in ${}^{\square}\mathcal{C}$ and $g : A \to C$ is a retract of f in $A \downarrow \mathcal{K}$ then $g \in {}^{\square}\mathcal{C}$. (The retract assumption means an existence of $s : C \to B$ and $r : B \to C$ with $r \cdot s = \mathrm{id}_c$, $s \cdot g = f$ and $r \cdot f = g$.)

A morphism is called a $\mathcal{C}$-*cofibration* (see [4]) if it is retract (in the sense of (1c)) of an isomorphism or of a transfinite composition of pushouts of elements of $\mathcal{C}$. The class of $\mathcal{C}$-cofibrations is denoted by $\operatorname{cof}(\mathcal{C})$. The following basic result goes back to Quillen [14]. In the present form it can be found in [4] (see also [1]).

Theorem 2.5. *Let $\mathcal{K}$ be a locally presentable category and $\mathcal{C}$ a set of morphisms of $\mathcal{K}$. Then $\mathcal{R} = \mathcal{C}^{\square}$ and $\mathcal{L} = {}^{\square}\mathcal{R}$ form a weak factorization system in $\mathcal{K}$. Moreover, $\mathcal{L} = \operatorname{cof}(\mathcal{C})$.*

A weak factorization system $(\mathcal{L}, \mathcal{R})$ is called *cofibrantly generated* if there exists a set $\mathcal{C}$ of morphisms such that $\mathcal{L} = \operatorname{cof}(\mathcal{C})$ and $\mathcal{R} = \mathcal{C}^{\square}$.

3. Accessible categories of modules

Recall that a category is λ-*accessible*, where λ is a regular cardinal, provided that

(i) $\mathcal{K}$ has λ-directed colimits

and

(ii) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object in $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

$\mathcal{K}$ is called *accessible* if it is λ-accessible for some regular cardinal λ. ω-accessible categories are called *finitely accessible*. A monomorphism $f : A \to B$ is said to be *pure* provided that in each commutative square

$$\begin{tikzcd} X \arrow[r, "h"] \arrow[d, "u"'] & Y \arrow[d, "v"] \\ A \arrow[r, "f"'] & B \end{tikzcd}$$

with X and Y finitely presentable, u factorizes through h, i.e., $u = t \cdot h$ for some $t : Y \to A$ (see [3]).

For a ring R, the category R-**Mod** of R-modules is finitely accessible and pure monomorphisms are the usual ones. Every full subcategory $\mathcal{K}$ of R-**Mod** closed under directed colimits and pure submodules is accessible (see [3] 2.36). A well-known result is that a monomorphism $f : A \to B$ in R-**Mod** is pure if each finitely presentable module is projective w.r.t. $\operatorname{coker}(f)$. Epimorphisms $\operatorname{coker}(f)$, where f is pure, are called *pure epimorphisms* (cf. [15]).

Lemma 3.1. *In R-**Mod**, pure monomorphisms are stable under pullbacks along pure epimorphisms.*

Proof. Consider pure monomorphisms $f : A \to B$, $g : D \to C$ and a pullback

$$\begin{tikzcd} E \arrow[r, "\overline{g}"] \arrow[d, "h"'] & B \arrow[d, "\operatorname{coker}(f)"] \\ D \arrow[r, "g"'] & C \end{tikzcd} \tag{2}$$

We have to prove that $\overline{g}$ is pure. But (2) is a pushout and h is an epimorphism (cf. [6] 1.76). Therefore

$$\operatorname{coker}(\overline{g}) = \operatorname{coker}(g) \cdot \operatorname{coker}(f)$$

(see [16] 12.3.4). Hence $\overline{g}$ is pure (because $\operatorname{coker}(\overline{g})$ is a pure epimorphism). $\square$

The following result presents the core of the Enochs' proof of the flat cover conjecture.

Proposition 3.2. *Let $\mathcal{K}$ be a full subcategory of R-**Mod** closed under directed colimits and pure submodules. There is a regular cardinal λ such that every $K \in \mathcal{K}$ is a union of a smooth chain of submodules $K_i \in \mathcal{K}$, $i < \alpha$ such that $|K_0| < \lambda$ and $|K_{i+1}/K_i| < \lambda$ for all $i < \alpha$.*

Proof. Since $|R| < \lambda$, an R-module X is λ-presentable iff $|X| < \lambda$. Following [3] 2.33, there is a regular cardinal $\lambda > |R|$ such that every submodule $A \subseteq B$ with A λ-presentable is contained in a pure submodule $\bar{A} \subseteq B$ with $\bar{A}$ also λ-presentable.

Let $K \in \mathcal{K}$ and construct a smooth chain of pure submodules $K_i \subseteq K$ as follows. Let K_0 be any λ-presentable pure submodule of K. For a limit ordinal i, we take $K_i = \bigcup\limits_{j<i} K_j$. Having i with $K_i \neq K$, take a λ-presentable submodule $0 \neq M \subseteq K/K_i$ and a pullback

$$\begin{tikzcd} K \arrow[r] & K/K_i \\ K_{i+1} \arrow[u] \arrow[r] & M \arrow[u] \end{tikzcd}$$

The pullback property yields the embedding $K_i \subseteq K_{i+1}$ and, by Lemma 3.1, K_{i+1} is pure in K. Clearly $K = \bigcup\limits_{i<\alpha} K_i$ for some α and $K_i \in \mathcal{K}$ for each $i < \alpha$. $\square$

4. Flat monomorphisms

Definition 4.1. Let $\mathcal{F}$ be a full subcategory of R-**Mod**. A monomorphism $f : A \to B$ will be called an *$\mathcal{F}$-monomorphism* if $\operatorname{coker}(f) : B \to F$ has F in $\mathcal{F}$. The class of all $\mathcal{F}$-monomorphisms will be denoted by $\mathcal{F}$-$Mono$.

Some of the following proofs are written in a way showing that they are valid in categories more general than R-**Mod**.

Lemma 4.2. *Let $\mathcal{F}$ a full subcategory of R-**Mod** containing 0 and closed under directed colimits and extensions. Then $\operatorname{cof}(\mathcal{F}\text{-}Mono) = \mathcal{F}\text{-}Mono$.*

Proof. (a) $\mathcal{F}$-$Mono$ is stable under pushouts. Consider a pushout

$$\begin{tikzcd} B \arrow[r, "\overline{g}"] & D \\ A \arrow[u, "f"] \arrow[r, "g"'] & C \arrow[u, "\overline{f}"'] \end{tikzcd}$$

where f is an $\mathcal{F}$-monomorphism. Since $\operatorname{coker}\overline{f} \cdot \overline{g} = \operatorname{coker}(f)$ (see [16] 12.3.4), $\overline{f}$ is an $\mathcal{F}$-monomorphism.

(b) $\mathcal{F}$-$Mono$ is closed under transfinite compositions.

(b$_1$) Let $f : A \to B$ and $g : B \to C$ be $\mathcal{F}$-monomorphisms. Consider the following diagram

$$\begin{tikzcd}
A \arrow[r, "f"] & B \arrow[rr, "g"] \arrow[d, "\mathrm{coker}(f)"'] & & C \arrow[d, "\mathrm{coker}(g)"] \arrow[dl, "\mathrm{coker}(gf)"'] \\
& \overline{B} \arrow[r, "u"'] & \overline{\overline{C}} \arrow[r, "v"'] & \overline{C}
\end{tikzcd}$$

where u and v are the induced homomorphisms. Following [16] 12.3.4, the middle trapezoid is a pushout and thus u is a monomorphism. Hence $v = \mathrm{coker}(u)$. Since $\mathcal{F}$ is closed under extensions and $\overline{B}$ and $\overline{C}$ are in $\mathcal{F}$, $\overline{\overline{C}}$ is in $\mathcal{F}$. Therefore $g \cdot f$ is an $\mathcal{F}$-monomorphism.

(b$_2$) Let $(f_{ij} : A_i \to A_j)_{i \le j \le \lambda}$ be a smooth chain in which f_{ij}, $i \le j < \lambda$ are $\mathcal{F}$-monomorphisms. Consider

$$\begin{tikzcd}
A_i \arrow[r, "f_{ij}"] \arrow[d, "\mathrm{coker}(f_{0i})"'] & A_j \arrow[d, "\mathrm{coker}(f_{0j})"] \\
F_i \arrow[r, "t_{ij}"'] & F_j
\end{tikzcd}$$

where t_{ij} is given by $\mathrm{coker}(f_{0j}) \cdot f_{ij} \cdot f_{0i} = \mathrm{coker}(f_{0j}) \cdot f_{0j} = 0$. Since F_i is in $\mathcal{F}$ for $i < \lambda$, $F_\lambda = \operatorname*{colim}_{i<\lambda} F_i$ and $\mathcal{F}$ is closed under directed colimits, we get that $f_{0\lambda}$ is an $\mathcal{F}$-monomorphism.

(c) $\mathcal{F}$-*Mono* is closed under retracts. Let $F : A \to B$ be an $\mathcal{F}$-monomorphism and $g : A \to C$ a retract of f in $A \downarrow R$-**Mod**. Consider

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[dr, "g"'] & B \arrow[r, "\mathrm{coker}(f)"] \arrow[d, shift left, "r"] & \overline{B} \arrow[d, shift left, "\overline{r}"] \\
& C \arrow[u, shift left, "s"] \arrow[r, "\mathrm{coker}(g)"'] & \overline{C} \arrow[u, shift left, "\overline{s}"]
\end{tikzcd}$$

where r and s make g a retract of f and $\overline{r}$, $\overline{s}$ are the induced homomorphisms. They make $\overline{C}$ a retract of $\overline{B}$ and since $\mathcal{F}$ is closed under retracts in R-**Mod** (see [3] 2.4), we have that $\overline{C}$ is in $\mathcal{F}$, i.e., g is an $\mathcal{F}$-monomorphisms.

(d) $\mathcal{F}$-*Mono* contains all isomorphisms. If $f : A \to B$ is an isomorphism then $\mathrm{coker}(f) : B \to 0$. □

Remark 4.3. We have

$$(\mathcal{F}\text{-Mono})^{\triangle} = \{C \mid \mathrm{Ext}(F, C) = 0 \text{ for all } F \in \mathcal{F}\}.$$

Indeed, $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathcal{F}$ is clearly equivalent to the fact that every $\mathcal{F}$-monomorphism $C \to B$ splits. This property has every $C \in (\mathcal{F}\text{-}Mono)^{\triangle}$. Conversely, consider an R-module C having this property and form a pushout

$$\begin{tikzcd}
A \arrow[r, "f"] \arrow[d, "u"'] & B \arrow[d, "\overline{u}"] \\
C \arrow[r, "\overline{f}"'] & D
\end{tikzcd}$$

where f is an $\mathcal{F}$-monomorphism and u a homomorphism. Since $\overline{f}$ is an $\mathcal{F}$-monomorphism by Lemma 4.2, $\overline{f}$ splits and thus u factorizes through f. Hence $C \in (\mathcal{F}\text{-}Mono)^{\triangle}$.

Lemma 4.4. *Let $\mathcal{F}$ be a full subcategory of R-**Mod** containing all free R-modules. Then $g : C \to D$ belongs to $(\mathcal{F}\text{-}Mono)^{\square}$ iff it is an epimorphism and* $\ker(g) : E \to C$ *has E in $(\mathcal{F}\text{-}Mono)^{\triangle}$.*

Proof. I. Let $g : C \to D$ be in $(\mathcal{F}\text{-}Mono)^{\square}$ and consider

$$E \xrightarrow{\ker(g)} C \xrightarrow{\quad g \quad} D$$

Then $E \in (\mathcal{F}\text{-}Mono)^{\triangle}$.

To verify it, consider an $\mathcal{F}$-monomorphism $f : A \to B$ and $u : A \to E$. In the diagram

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[dd, "f"'] & E \arrow[d, "\ker(g)"] \\
& C \arrow[d, "g"] \\
B \arrow[r, "0"'] \arrow[uur, dashed, "v"] \arrow[ur, dashed, "t"'] & D
\end{tikzcd}$$

we get a diagonal t with $g \cdot t = 0$ and $t \cdot f = \ker(g) \cdot u$ (because $g \in (\mathcal{F}\text{-}Mono)^{\square}$) and v with $\ker(g) \cdot v = t$ (because $g \cdot t = 0$). Hence

$$\ker(g) \cdot v \cdot f = t \cdot f = \ker(g) \cdot u$$

and thus $v \cdot f = u$.

Consider $d \in D$ and $p : R \to D$ with $p(1) = d$. Since $0 \to R$ belongs to $\mathcal{F}$-Mono, there is a diagonal t in the square

$$\begin{tikzcd}
0 \arrow[r] \arrow[d] & C \arrow[d, "g"] \\
R \arrow[r, "p"'] & D
\end{tikzcd}$$

Hence $d = g(t(1))$ and thus g is an epimorphism.

II. At first, let $g : C \to D$ be an epimorphism such that D is in $\mathcal{F}$ and, in $\ker(g) : E \to C$, E is in $(\mathcal{F}\text{-}Mono)^{\triangle}$. We will prove that $g \in (\mathcal{F}\text{-}Mono)^{\square}$.

Consider an $\mathcal{F}$-monomorphism $f : A \to B$ and a commutative square

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] & D
\end{tikzcd}$$

Since $g = \operatorname{coker}(\ker(g))$, $\ker(g)$ is an $\mathcal{F}$-monomorphism and thus $\ker(g)$ splits (because $E \in (\mathcal{F}\text{-}Mono)^{\triangle}$). Hence g splits; $g \cdot p = \mathrm{id}_C$. Consider the morphism $pvf - u : A \to C$. We have

$$g(pvf - u) = gpvf - gu = vf - gu = 0\,.$$

Thus there is $s : A \to E$ with $\ker(g)s = pvf - u$. Since $E \in (\mathcal{F}\text{-}Mono)^{\triangle}$, there is $t : B \to E$ with $t \cdot f = s$. Then $pv - \ker(g)t : B \to C$ is a required diagonal. Indeed,

$$g(pv - \ker(g)t) = v - 0 = v$$

and

$$(pv - \ker(g)t)f = pvf - \ker(g)s = pvf - (pvf - u) = u\,.$$

Now, let $g : C \to D$ be an arbitrary epimorphism such that $\ker(g) : E \to C$ has $E \in \mathcal{F}\text{-Mono}^{\triangle}$. Consider a commutative square

$$\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] & D
\end{tikzcd}$$

where f is an $\mathcal{F}$-monomorphism. We will complete it to the commutative diagram

$$\begin{tikzcd}
A \arrow[rrr, "u"] \arrow[ddd, "f"'] & & & C \arrow[ddd, "g"] \\
& \bar{A} \arrow[ul, "\bar{p}"'] \arrow[r, "\bar{u}"] \arrow[d, "\bar{f}"] & \bar{C} \arrow[ur, "\bar{q}"] \arrow[d, "\bar{g}"'] & \\
& B^* \arrow[r, "v^*"] \arrow[dl, "p"] \arrow[ur, dashed, "t"] & D^* \arrow[dr, "q"'] & \\
B \arrow[rrr, "v"'] & & & D
\end{tikzcd}$$

where $p : B^* \to B$ and $q : D^* \to D$ are epimorphism with B^* and D^* free, $q \cdot v^* = v \cdot p$, the trapezoids on the left and on the right side are pullbacks and $\bar{u}$ is the induced homomorphism. Since

$$\begin{tikzcd}
& E \arrow[dl, "\ker(\bar{g})"'] \arrow[dr, "\ker(g)"] & \\
\bar{C} \arrow[rr, "\bar{q}"'] & & C
\end{tikzcd}$$

commutes, $\bar{g}$ is an epimorphism and $D^* \in \mathcal{F}$, we know that $\bar{g} \in (\mathcal{F}\text{-Mono})^{\square}$. Since the trapezoid on the left side is a pushout as well, we get that $\bar{f} \in \mathcal{F}$-Mono. It yields a diagonal $t : B^* \to \bar{C}$ in the square in the middle. Since

$$\bar{q} \cdot t \cdot \bar{f} = u \cdot \bar{p}$$

there is a unique $s : B \to C$ such that

$$s \cdot p = \bar{q} \cdot t \quad \text{and} \quad s \cdot f = u$$

(using the pushout property at B). Since

$$g \cdot s \cdot p = g \cdot \bar{q} \cdot t = q \cdot \bar{g} \cdot t = q \cdot v^* = v \cdot p\,,$$

we have

$$g \cdot s = v\,.$$

Hence s is the needed diagonal. $\square$

Theorem 4.5. *Let $\mathcal{F}$ be a full subcategory of R-**Mod** containing all free modules and closed under directed colimits, pure submodules and extensions. Then $(\mathcal{F}\text{-}Mono, \mathcal{F}\text{-}Mono^{\square})$ is a cofibrantly generated weak factorization system.*

Proof. Following 3.2, there is a regular cardinal λ such that every $F \in \mathcal{F}$ is a union of a smooth chain of submodules $F_i \in \mathcal{F}$, $i < \alpha$, such that $|F_0| < \lambda$ and $|F_{i+1}/F_i| < \lambda$ for all $i < \alpha$. By the standard result (cf. [8], XII.1.14 or [9]), the following statements are equivalent for each R-module C:

(a) $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathcal{F}$;

(b) $\mathrm{Ext}(F, C) = 0$ for all $F \in \mathcal{F}$ with $|F| < \lambda$.

Let $\mathcal{M} = \{\ker(p_F) | F \in \mathcal{F},\ |F| < \lambda\} \cup \{0 \to R\}$ where

$$(*) \qquad 0 \longrightarrow A \xrightarrow{\ker(p_F)} F^* \xrightarrow{p_F} F \longrightarrow 0$$

presents F as a quotient of a free R-module F^*. Since $\mathcal{M} \subseteq \mathcal{F}\text{-}Mono$, we have $(\mathcal{F}\text{-}Mono)^{\triangle} \subseteq \mathcal{M}^{\triangle}$. Assume that $C \in \mathcal{M}^{\triangle}$ and consider the long exact sequence

$$0 \longrightarrow \mathrm{Hom}(F, C) \longrightarrow \mathrm{Hom}(F^*, C) \longrightarrow \mathrm{Hom}(A, C) \longrightarrow \mathrm{Ext}(F, C) \longrightarrow \mathrm{Ext}(F^*, C) \longrightarrow \dots$$

induced by $(*)$. Since $\mathrm{Ext}(F^*, C) = 0$ and $\mathrm{Hom}(F^*, C) \to \mathrm{Hom}(A, C)$ is surjective, we get that $\mathrm{Ext}(F, C) = 0$. Hence C satisfies (b) and, following Remark 4.3, we get that $C \in (\mathcal{F}\text{-}Mono)^{\triangle}$. Hence

$$(\mathcal{F}\text{-}Mono)^{\triangle} = \mathcal{M}^{\triangle}.$$

Following Lemma 4.4,

$$\mathcal{F}\text{-Mono}^{\square} = \mathcal{M}^{\square}\,.$$

In fact, $\mathcal{F}\text{-}Mono^{\square} \subseteq \mathcal{M}^{\square}$ and, following the proof of Lemma 4.4, any $g \in \mathcal{M}^{\square}$ is an epimorphism with $\ker(g) : E \to C$ having E in $\mathcal{F}\text{-}Mono^{\triangle}$.

We have

$$\mathcal{F}\text{-Mono} \subseteq {}^{\square}(\mathcal{M}^{\square}) = \mathrm{cof}(\mathcal{M}) \subseteq \mathcal{F}\text{-Mono}$$

by Lemma 4.2. Since $\mathcal{M}$ is a set, $(\mathcal{F}\text{-Mono}, \mathcal{F}\text{-Mono}^{\square})$ is a cofibrantly generated weak factorization system. $\square$

We will denote by $Flat$ the class of all flat monomorphisms and by Cot the class of all epimorphisms $g : C \to D$ having, in $\ker(g) : E \to C$, $E \in Flat^{\triangle}$, (i.e., C cotorsion).

Corollary 4.6. *Let R be a ring. Then $(Flat, Cot)$ is a weak factorization system in R-**Mod**.*

Proof. The full subcategory of R-**Mod** consisting of flat R-modules is closed under directed colimits, extensions and contains all free R-modules. Since flat R-modules are precisely directed colimits of finitely presentable projective R-modules, any pure submodule of a flat R-module is flat (cf. [3], the proof of Proposition 2.32). Hence the result follows from Theorem 4.5 and Lemma 4.4. □

Remark 4.7. We have shown that Theorem 2.5 can replace [10], Theorem 10 in the proof of the flat cover conjecture.

5. Flat precovers in varieties

Under a *variety* we mean an equationally defined class of finitary universal algebras (cf. [3]). R-modules form a variety for any ring R. Even the comma category $A \downarrow R$-**Mod** is a variety for any R-module A. It suffices to take elements of A as new constant symbols and equalities true in A as new equations. $A \downarrow R$-**Mod** is no longer an additive category.

In any variety $\mathcal{V}$ we can define *flat algebras* as directed colimits of (regular) projective algebras. Then, in R-**Mod**, we get the usual flat modules.

Lemma 5.1. *Let A be an R-module. Then $f : A \to X$ is flat in $A \downarrow R$-**Mod** iff it is a flat monomorphism in R-**Mod**.*

Proof. Since the projection $P : A \downarrow R\text{-}\mathbf{Mod} \to R\text{-}\mathbf{Mod}$ preserves colimits, it preserves regular epimorphisms. Conversely, consider

$$\begin{tikzcd} & A \arrow[dl, "x"'] \arrow[dr, "y"] & \\ X \arrow[rr, "f"'] & & Y \end{tikzcd}$$

where $f : X \to Y$ is an epimorphism in R-**Mod**. Hence f is a coequalizer of its kernel pair

$$Z \underset{v}{\overset{u}{\rightrightarrows}} X$$

in R-**Mod**. It is easy to see that f is a coequalizer of the pair

$$\begin{tikzcd} & A \arrow[dl, "i_1"'] \arrow[dr, "x"] & \\ A + Z \arrow[rr, shift left, "\langle x, u\rangle"] \arrow[rr, shift right, "\langle x, v\rangle"'] & & X \end{tikzcd}$$

in $A \downarrow R$-**Mod**. Hence f is a regular epimorphism in $A \downarrow R$-**Mod** iff $P(f)$ is an epimorphism in R-**Mod**.

Since (monomorphisms with a projective kernel, epimorphisms) is a weak factorization system in R-**Mod** (see [2]), $p : A \to X$ is projective in $A \downarrow R$-**Mod** iff it is a monomorphism and $\operatorname{coker}(p) : X \to P$ has P projective (cf. Remark 2.3 (2)). Consequently, $p : A \to Y$ is flat in $A \downarrow R$-**Mod** iff it is a monomorphism and $\operatorname{coker}(p) : Y \to Q$ has Q flat. □

Corollary 5.2. *Let R be a ring and A an R-module. Then any algebra in $A \downarrow R$-**Mod** has a flat precover.*

Proof immediatelly follows from 4.1 and 5.1.

The author does not know any example of a variety in which flat precovers do not exist.

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