Enriched weakness

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Definition. Let $\mathcal{E}$ be a class of morphisms in a symmetric monoidal closed category $\mathcal{V}$. Let $f : A \to B$ be a morphism in a $\mathcal{V}$-category $\mathcal{K}$. We say that an object C from $\mathcal{K}$ is *f-injective* over $\mathcal{E}$ when the induced morphism

$$\mathcal{K}(f, C) : \mathcal{K}(B, C) \to \mathcal{K}(A, C)$$

is in $\mathcal{E}$.

Given a class $\mathcal{F}$ of morphisms in $\mathcal{K}$, C is $\mathcal{F}$-injective over $\mathcal{E}$ if it is f-injective for all $f \in \mathcal{F}$. $\mathcal{F}$-**Inj** will denote the full subcategory of $\mathcal{K}$ consisting of $\mathcal{F}$-injective objects.

Examples. (1) For $\mathcal{E}$ = isomorphisms, one gets the classical enriched orthogonality.
(2) For $\mathcal{V}$ = **Set** and $\mathcal{E}$ = surjections, one gets the classical injectivity.

Proposition 1. $\mathcal{F}$-**Inj** is closed in $\mathcal{K}$ under any class Φ of limits for which $\mathcal{E}$ is closed in $\mathcal{V}^2$ under Φ-limits.

Proposition 2. $\mathcal{F}$-**Inj** is closed in $\mathcal{K}$ under any class Φ of colimits for which

(1) $\mathcal{E}$ is closed under Φ-colimits;

(2) $\mathcal{K}(A, -)$ preserves Φ-colimits for any object A which is the domain or the codomain of a morphism in $\mathcal{F}$.

Let $G : \mathcal{K} \to \mathcal{L}$ be $\mathcal{V}$-functor. We say that a family of morphisms

$$(\eta_L : L \to UFL)_{L \in \mathcal{L}}$$

makes F a weak left adjoint to G if the induced morphisms

$$\mathcal{K}(FL, K) \xrightarrow{G} \mathcal{K}(GFL, GK) \xrightarrow{\mathcal{K}(\eta_L, GK)} \mathcal{K}(L, GK)$$

are in $\mathcal{E}$.

Of course, F does not need to be a functor.

Given $\mathcal{V}$-functors $D : \mathcal{D} \to \mathcal{K}$ and $H : \mathcal{D}^{\mathrm{op}} \to \mathcal{V}$, $H *_w D$ is a weak colimit of D weighted by H if the induced morphism

$$\mathcal{K}(H *_w D, K) \to [\mathcal{D}^{\mathrm{op}}, \mathcal{V}](H, \mathcal{K}(D, K))$$

is in $\mathcal{E}$.

The right choice for enriching classical injectivity is

$$\mathcal{E} = \text{pure epimorphisms.}$$

The reason is that the latter are precisely filtered colimits of split epimorphisms. One has to assume that $\mathcal{V}$ is locally finitely presentable as a closed category. This means that the underlying ordinary category $\mathcal{V}_0$ is locally finitely presentable and the full subcategory of finitely presentable objects is closed under the monoidal structure.

Theorem 1. The following conditions are equivalent for a full subcategory $\mathcal{A}$ of a locally presentable $\mathcal{V}$-category:

(1) $\mathcal{A} = \mathcal{F}$-**Inj** for a set $\mathcal{F}$;

(2) $\mathcal{A}$ is accessible, accessibly embedded, and closed under products and finite cotensors;

(3) $\mathcal{A}$ is accessibly embedded and weakly reflective.

Theorem 2. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:

(1) $\mathcal{A}$ is accessible and weakly cocomplete;

(2) $\mathcal{A}$ is accessible and has products and finite cotensors;

(3) $\mathcal{A}$ is a small injectivity class in some locally presentable $\mathcal{V}$-category;

(4) $\mathcal{A}$ is weakly reflective, accessibly embedded subcategory of $[\mathcal{C}, \mathcal{V}]$ for some small $\mathcal{V}$-category $\mathcal{C}$;

(5) $\mathcal{A}$ is equivalent to the category of models of a $(\text{limit}, \mathcal{E})$-sketch.

A $\mathcal{V}$-functor is a model of a $(\text{limit}, \mathcal{E})$-sketch if it preserves specified limits and sends specified morphisms to pure epimorphisms.

Now, take $\mathcal{V} = \mathbf{Cat}$ and $\mathcal{E}$ = equivalences.

Theorem 3. The following conditions are equivalent for a full subcategory $\mathcal{A}$ of a locally presentable 2-category:

(1) $\mathcal{A} = \mathcal{F}$-**Inj** for a set $\mathcal{F}$;

(2) $\mathcal{A}$ is accessible, accessibly embedded, closed under flexible limits and 2-replete;

(3) $\mathcal{A}$ is accessible, accessibly embedded, weakly reflective and 2-replete.

Flexible limit $\{H, D\}$ is a limit weighted by a retract $H : \mathcal{D} \to \mathbf{Cat}$ of some G' where $'$ denotes left adjoint to the inclusion

$$[\mathcal{D}, \mathbf{Cat}] \to \mathrm{Psd}[\mathcal{D}, \mathbf{Cat}]$$

where $\mathrm{Psd}[\mathcal{D}, \mathbf{Cat}]$ denotes the 2-category of 2-functors, pseudonatural transformations and modifications.

If $\mathcal{K}$ has all flexible limits then it has all pseudolimits:

$$\{H, D\}_p \cong \{H', D\}$$

Theorem 4. The following are equivalent for a $\mathcal{V}$-category $\mathcal{A}$:

(1) $\mathcal{A}$ is a small injectivity class in some locally presentable 2-category;

(2) $\mathcal{A}$ is equivalent to the category of models of a $(\text{limit}, \mathcal{E})$-sketch.

One does not have the analogy of Theorem 2 here: the full subcategory of **Cat** consisting of the terminal category and the free-living isomorphism is accessible, accessible embedded and weakly reflective but does not have flexible limits.

But one has the analogies of Theorems 1 and 2 for the choice of $\mathcal{V} = \mathbf{Cat}$ and $\mathcal{E} =$ retract equivalences. Products and finite cotensors are replaced by flexible limits.

Notice that both pure epimorphisms and retract equivalences are right parts of weak factorization systems in **Cat** and that retracts equivalences are pure epimorphisms.