ARE ALL COFIBRANTLY GENERATED MODEL CATEGORIES COMBINATORIAL?

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Abstract. G. Raptis has recently proved that, assuming Vopěnka's principle, every cofibrantly generated model category is Quillen equivalent to a combinatorial one. His result remains true for a slightly more general concept of a cofibrantly generated model category. We show that Vopěnka's principle is equivalent to this claim. The set-theoretical status of the original Raptis' result is open.

Combinatorial model categories were introduced by J. H. Smith as model categories which are locally presentable and cofibrantly generated. There are of course cofibrantly generated model categories which are not combinatorial – the first example is the standard model category of topological spaces. This model category is Quillen equivalent to the combinatorial model category of simplicial sets. G. Raptis [6] has recently proved a somewhat surprising result saying that, assuming Vopěnka's principle, every cofibrantly generated model category is Quillen equivalent to a combinatorial model category. Vopěnka's principle is a set-theoretical axiom implying the existence of very large cardinals (see [2]). A natural question is whether Vopěnka's principle (or other set theory) is needed for Raptis' result.

A *model category* is a complete and cocomplete category $\mathcal{M}$ together with three classes of morphisms $\mathcal{F}$, $\mathcal{C}$ and $\mathcal{W}$ called *fibrations*, *cofibrations* and *weak equivalences* such that

(1) $\mathcal{W}$ has the 2-out-of-3 property and is closed under retracts in the arrow category $\mathcal{M}^{\to}$, and
(2) $(\mathcal{C}, \mathcal{F} \cap \mathcal{W})$ and $(\mathcal{C} \cap \mathcal{W}, \mathcal{F})$ are weak factorization systems.

Morphisms from $\mathcal{F} \cap \mathcal{W}$ are called *trivial fibrations* while morphisms from $\mathcal{C} \cap \mathcal{W}$ *trivial cofibrations*.

A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in a category $\mathcal{M}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{M}$ such that

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Date: June 26, 2009.

Key words and phrases. cofibrantly generated model category, combinatorial model category, Vopěnka's principle.

* Supported by the Ministry of Education of the Czech republic under the project MSM 0021622409.

(1) $\mathcal{R} = \mathcal{L}^\square$, $\mathcal{L} = {}^\square\mathcal{R}$, and
(2) any morphism h of $\mathcal{M}$ has a factorization $h = gf$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

Here, $\mathcal{L}^\square$ consists of morphisms having the right lifting property w.r.t. each morphism from $\mathcal{L}$ and ${}^\square\mathcal{R}$ consists of morphisms having the left lifting property w.r.t. each morphism from $\mathcal{R}$.

The standard definition of a cofibrantly generated model category (see [5]) is that the both weak factorization systems from its definition are cofibrantly generated in the following sense. A weak factorization system $(\mathcal{L}, \mathcal{R})$ is cofibrantly generated if there exists a set $\mathcal{X}$ of morphisms such that

(1) the domains of $\mathcal{X}$ are small relative to $\mathcal{X}$-cellular morphisms, and
(2) $\mathcal{X}^\square = \mathcal{R}$.

Here, $\mathcal{X}$-cellular morphisms are transfinite compositions of pushouts of morphisms of $\mathcal{X}$. The consequence of this definition is that $\mathcal{L}$ is the smallest cofibrantly closed class containing $\mathcal{X}$. A cofibrantly closed class is defined as a class of morphisms closed under transfinite compositions, pushouts and retracts in $\mathcal{M}^\to$. Moreover, one does not need to assume that $(\mathcal{L}, \mathcal{R})$ is a weak factorization system because it follows from (1) and (2). This observation led to the following more general definition of a cofibrantly generated weak factorization system (see [1]).

A weak factorization system $(\mathcal{L}, \mathcal{R})$ is *cofibrantly generated* if there exists a set $\mathcal{X}$ of morphisms such that $\mathcal{L}$ is the smallest cofibrantly closed class containing $\mathcal{X}$. The consequence is that $\mathcal{X}^\square = \mathcal{R}$. A model category is *cofibrantly generated* if the both weak factorization systems from its definition are cofibrantly generated in the new sense. It does not affect the definition of a combinatorial model category because all objects are small in a locally presentable category. Moreover, the proof of Raptis [6] works for cofibrantly generated model categories in this sense as well.

We will show that Vopěnka's principle follows from the fact that every cofibrantly generated model category (in the new sense) is Quillen equivalent to a combinatorial model category. We do not know whether this is true for standardly defined cofibrantly generated model categories as well. Our proof uses the trivial model structure on a category $\mathcal{M}$ where all morphisms are cofibrations and weak equivalences are isomorphisms.

Given a small full subcategory $\mathcal{A}$ of a category $\mathcal{K}$, the *canonical functor*

$$E_\mathcal{A} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

assigns to each object K the restriction

$$E_{\mathcal{A}}K = \hom(-, K)/\mathcal{A}^{op}$$

of its hom-functor $\hom(-, K) : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ to $\mathcal{A}^{\mathrm{op}}$ (see [2] 1.25).

A small full subcategory $\mathcal{A}$ of a category $\mathcal{K}$ is called *dense* provided that every object of $\mathcal{K}$ is a canonical colimit of objects from $\mathcal{A}$. It is equivalent to the fact that the canonical functor

$$E_{\mathcal{A}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

is a full embedding (see [2], 1.26). A category $\mathcal{K}$ is called *bounded* if it has a (small) dense subcategory (see [2]).

Dense subcategories were introduced by J. R. Isbell [4] and called left adequate subcategories. The following result is easy to prove and can be found in [4].

Lemma 0.1. *Let $\mathcal{A}$ be dense subcategory of $\mathcal{K}$ and $\mathcal{B}$ a small full subcategory of $\mathcal{K}$ containing $\mathcal{A}$. Then $\mathcal{B}$ is dense.*

Proposition 0.2. *Let $\mathcal{K}$ be a cocomplete bounded category. Then $(\mathcal{K}, \mathrm{Iso})$ is a cofibrantly generated weak factorization system.*

Proof. Clearly, $(\mathcal{K}, \mathrm{Iso})$ is a weak factorization system. The canonical functor

$$E_{\mathcal{A}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

has a left adjoint F (see [2], 1.27). The weak factorization system $(\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}, \mathrm{Iso})$ in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is cofibrantly generated (see [9], 4.6). Thus there is a small full subcategory $\mathcal{X}$ of $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ such that each morphism in $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a retract of a $\mathcal{X}$-cellular morphism. Hence each morphism in $\mathcal{K}$ is a retract of a $F(\mathcal{X})$-cellular morphism. Thus $(\mathcal{K}, \mathrm{Iso})$ is cofibrantly generated. $\square$

Given a complete and cocomplete category $\mathcal{K}$, the choice $\mathcal{C} = \mathcal{K}$ and $\mathcal{W} = \mathrm{Iso}$ yields a model category structure on $\mathcal{K}$. The corresponding two weak factorization systems are $(\mathcal{K}, \mathrm{Iso})$ and $(\mathrm{Iso}, \mathcal{K})$ and the homotopy category $\mathrm{Ho}(\mathcal{K}) = \mathcal{K}$. We will call this model category structure *trivial.*

Corollary 0.3. *Let $\mathcal{K}$ be a complete, cocomplete and bounded category. Then the trivial model category structure on $\mathcal{K}$ is cofibrantly generated.*

Proof. Following 0.2, it suffices to add that the weak factorization system $(\mathrm{Iso}, \mathcal{K})$ is cofibrantly generated by $\mathcal{X} = \{\mathrm{id}_O\}$ where O is an initial object of $\mathcal{K}$. $\square$

Theorem 0.4. *Vopěnka's principle is equivalent to the fact that every cofibrantly generated model category is Quillen equivalent to a combinatorial model category.*

Proof. Necessity follows from [6]. Under the negation of Vopěnka's principle, [2], 6.12 presents a complete bounded category $\mathcal{A}$ with the following properties

(1) For each regular cardinal λ, there is a λ-filtered diagram $D_\lambda : \mathcal{D}_\lambda \to \mathcal{K}$ whose only compatible cocones δ_λ are trivial ones with the codomain 1 (= a terminal object),

(2) For each λ, id_1 does not factorize through any component of δ_λ.

Since, following (1), δ_λ is a colimit cocone for each λ, (2) implies that 1 is not λ-presentable for any regular λ. Condition (2) is not stated explicitly in [2] but it follows from the fact that there is no morphism from 1 to a non-terminal object of $\mathcal{A}$. In fact, $\mathcal{A}$ is the full subcategory of the category **Gra** consisting of graphs A without any morphism $B_i \to A$ where B_i is the rigid class of graphs indexed by ordinals (whose existence is guaranteed by the negation of Vopěnka's principle). The existence of a morphism $1 \to A$ means the presence of a loop in A and, consequently, the existence of a constant morphism $B_i \to A$ (having a loop as its value).

Assume that the trivial model category $\mathcal{A}$ is Quillen equivalent to a combinatorial model category $\mathcal{M}$. Since $\mathrm{Ho}\,\mathcal{M}$ is equivalent to $\mathcal{A}$, it shares properties (1) and (2). Moreover, since $\mathrm{Ho}\,\mathcal{K} = \mathcal{K}$, the diagrams D_λ are diagrams in $\mathcal{K}$. It follows from the definition of Quillen equivalence that the corresponding diagrams in $\mathrm{Ho}\,\mathcal{M}$ (we will denote them by D_λ as well) can be rectified. It means that there are diagrams $\overline{D}_\lambda$ in $\mathcal{M}$ such that $D_\lambda = P\overline{D}_\lambda$; here, $P : \mathcal{M} \to \mathrm{Ho}\,\mathcal{M}$ is the canonical functor. Following [3] and [8], there is a regular cardinal λ_0 such that the replacement functor $R : \mathcal{M} \to \mathcal{M}$ preserves λ_0-filtered colimits. R sends each object M to a fibrant and cofibrant object and the canonical functor P can be taken as the composition QR where Q is the quotient functor identifying homotopy equivalent morphisms.

Let

$$(\overline{\delta}_{\lambda d} : \overline{D}_\lambda d \to M_\lambda)_{d \in \mathcal{D}_\lambda}$$

be colimit cocones. Then

$$(R\overline{\delta}_{\lambda d} : R\overline{D}_\lambda d \to RM_\lambda)_{d \in \mathcal{D}_\lambda}$$

are colimit cocones for each $\lambda > \lambda_0$. Following (1), $RM_\lambda \cong 1$ for each $\lambda > \lambda_0$. The object RM_{λ_0} is μ-presentable in $\mathcal{M}$ for some regular cardinal $\lambda_0 < \mu$. Since RM_{λ_0} and RM_μ are homotopy equivalent, there is a morphism $f : RM_{\lambda_0} \to RM_\mu$. Since f factorizes through some

$R\overline{\delta}_{\mu d}$, id_1 factorizes through some component of δ_μ, which contradicts (2). □

While the weak factorization system $(\mathrm{Iso}, \mathcal{K})$ is cofibrantly generated in the sense of [5], it is not true for $(\mathcal{K}, \mathrm{Iso})$ because the complete, cocomplete and bounded category in [2], 6.12 is not locally presentable just because it contains a non-presentable object. Thus we do not know whether Vopěnka's principle follows from the original result from [6].

The proof above does not exclude that $\mathcal{A}$ *has a combinatorial model*, i.e., that there is a combinatorial model category $\mathcal{M}$ such that $\mathcal{A}$ is equivalent to $\mathrm{Ho}\,\mathcal{M}$.

Proposition 0.5. *Assume the existence of a proper class of compact cardinals and let $\mathcal{K}$ be a complete, cocomplete and bounded category. Then the trivial model category $\mathcal{K}$ has a combinatorial model if and only if $\mathcal{K}$ is locally presentable.*

Proof. If $\mathcal{K}$ is locally presentable the trivial model category $\mathcal{K}$ is combinatorial. Assume that the trivial model category $\mathcal{K}$ is equivalent to $\mathrm{Ho}\,\mathcal{M}$ where $\mathcal{M}$ is a combinatorial model category. Let $\mathcal{X}$ be a dense subcategory of $\mathcal{K}$. Following [8], 4.1, there is a regular cardinal λ such that

(1) $\mathcal{X} \subseteq P(\mathcal{M}_\lambda)$ where $\mathcal{M}_\lambda$ denotes the full subcategory of $\mathcal{M}$ consisting of λ-presentable objects,
(2) The composition $H = E_{P(\mathcal{M}_\lambda)} \cdot P$ preserves λ-filtered colimits.

Since $P(\mathcal{M}_\lambda)$ is dense in $\mathcal{K}$ (see 0.1), $E_{P(\mathcal{M}_\lambda)}$ is a full embedding. Hence $\mathcal{K}$ is the full image of the functor H, i.e., the full subcategory on objects $H(M)$ with M in $\mathcal{M}$. Following [7], Corollary of Theorem 2, $\mathcal{K}$ is locally presentable. □

Vopěnka's principle is stronger than the existence of a proper class of compact cardinals. Thus, assuming the negation of Vopěnka's principle but the existence of a proper class of compact cardinals, there is a cofibrantly generated model category without a combinatorial model.

References

[1] J. Adámek, H. Herrlich, J. Rosický and W. Tholen, *Weak factorization systems and topological functors*, Appl. Cat. Struct. 10 (2002), 237-249.

[2] J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge University Press 1994.

[3] D. Dugger, *Combinatorial model categories have presentations*, Adv. Math. 164 (2001), 177-201.

[4] J. R. Isbell, *Adequate subcategories*, Ill. J. Math. 4 (1960), 541-552.

[5] M. Hovey, *Model categories*, Amer. Math. Soc. 1999.

[6] G. Raptis, *On the cofibrant generation of model categories*, preprint 2008.

[7] J. Rosický, *More on directed colimits of models*, Appl. Cat. Struct. 2 (1994), 71-76.

[8] J. Rosický, *Generalized Brown representability in homotopy categories*, Th. Appl. Cat. 14 (2005), 451-479.

[9] J. Rosický, *On combinatorial model categories*, Appl. Cat. Struct. 17 (2009), 303-316.

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