Towards categorical model theory

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Model theory started with the first order logic, moved to infinitary logic and, following the lead of Shelah, was extended to abstract elementary classes. It happened in 1987, just before the creation of the theory of accessible categories. But it took nearly 25 years to realize that abstract elementary classes are special accessible categories with *concrete directed colimits* (Lieberman, Beke, JR).

These are accessible categories $\mathcal{K}$ with directed colimits equipped with a faithful functor $U : \mathcal{K} \to \mathbf{Set}$ preserving directed colimits.

In an abstract elementary class, U factorizes through a finitely accessible category $\mathcal{L}$ by a functor $H : \mathcal{K} \to \mathcal{L}$ which is *iso-full* and *coherent*. The first property is the fullness w.r.t. isomorphisms and the second is the fullness w.r.t. f such that $H(g)f = H(h)$ for some g and h. Moreover, all morphisms of $\mathcal{K}$ are monomorphisms.

One can take $\mathcal{L} = \mathbf{Emb}\,\Sigma$ for a finitary one-sorted language Σ. This is the category of Σ-structures and substructure embeddings.

In this case, the functor $U : \mathcal{K} \to \mathbf{Set}$ is coherent.

Also, iso-fullness of H can be expressed in the terms of U – *interpretable* finitary function and relation symbols seeing $\mathcal{K}$-morphisms as substructure embeddings are able to detect isomorphisms.

Thus abstract elementary classes can be characterized as certain accessible categories with concrete directed colimits.

Thus they can be made not only syntax-free but even signature-free.

The assumption about monomorphisms is not restrictive. We can pass from a (coherent) accessible category $\mathcal{K}$ with concrete directed colimits to its iso-full subcategory $\mathcal{K}_0$ on the same objects which is a (coherent) accessible category with directed colimits whose morphisms are monomorphisms.

Any finitely accessible category whose morphisms are monomorphisms is an AEC.

More generally, any ∞, ω-*elementary* category whose morphisms are monomorphisms is an AEC. These categories are axiomatizable in $L_{\infty\omega}$ and are precisely limits of finitely accessible categories.

I do not know any example of an abstract elementary class which is not ∞, ω-elementary. With Makkai, we have not succeeded to prove that the category of uncountable sets and monomorphisms is not ∞, ω-elementary.

If there are no measurable cardinals then any (∞, ω)-elementary category is (ω_1, ω)-elementary (Adámek, Johnstone, Makowski, JR, 1997).

Let $\mathcal{K}$ be a λ-accessible category with directed colimits and K an object which is not λ-presentable. Then the smallest regular cardinal κ such that K is κ-presentable is a successor cardinal, i.e., $\kappa = |K|^+$ where $|K|$ is called the *size* of K.

An accessible category with concrete directed colimits has objects of all sizes starting from some cardinal.

If U is coherent then it preserves sizes starting from some cardinal. Thus our internal size $|K|$ coincides with the cardinality of the underlying set of K, starting from some cardinal.

Let $\mathcal{K}$ be an accessible category with directed colimits and λ an infinite cardinal. $\mathcal{K}$ is λ-*categorical* if it has, up to isomorphism, precisely one object of size λ.

Shelah's Categoricity Conjecture claims that for every AEC $\mathcal{K}$ there is a cardinal κ such that $\mathcal{K}$ is either λ-categorical for all $\lambda \geq \kappa$ or $\mathcal{K}$ is not λ-categorical for any $\lambda \geq \kappa$.

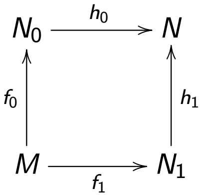
This was conjectured by Loś for first-order theories in a countable language in 1954 and proved by Morley in 1965. In 1970, Shelah extended it for uncountable languages. SCC is the main test question for AECs.

Of course, SCC was formulated using external sizes, i.e., cardinalities of underlying sets. Since they coincide with internal sizes starting from some cardinal, SCC is the property of the category $\mathcal{K}$.

In classical model theory, types are maximal consistent sets of formulas in a single variable.

Shelah introduced (language-free) types for AECs in 1987. His definition makes sense in any accessible category $\mathcal{K}$ with concrete directed colimits.

Consider pairs (f, a) where $f : M \to N$ and $a \in UN$. Two pairs (f_0, a_0) and (f_1, a_1) are equivalent it there is an amalgamation



such that $U(h_0)(a_0) = U(h_1)(a_1)$.

The resulting equivalence classes are called *(Galois) types* over M. One needs the amalgamation property to get the equivalence relation.

Let $\mathcal{K}$ be an accessible category with concrete directed colimits, the amalgamation property and the joint embedding property. Then (f_0, a_0) and (f_1, a_1) are equivalent if and only if there is a square

$$\begin{array}{ccc} N_0 & \xrightarrow{g_0} & L \\ {\scriptstyle f_0}\uparrow & & \uparrow{\scriptstyle g_1} \\ M & \xrightarrow[f_1]{} & N_1 \end{array}$$

and an isomorphism $s : L \to L$ such that $sg_0 f_0 = g_1 f_1$ and $U(sg_0)(a_0) = U(g_1)(a_1)$.

Thus types are orbits of automorphism groups.

L can be taken as a λ-saturated object of size λ where λ-*saturated* means to be injective with respect to morphisms between λ-presentable objects. It is often called a *monster model*.

A type (f, a) where $f : M \to N$ is *realized* in K if there is a morphism $g : M \to K$ and $b \in U(K)$ such that (f, a) and (g, b) are equivalent.

Let λ be a regular cardinal. We say that K is λ-*Galois saturated* if for any $g : M \to K$ where M is λ-presentable and any type (f, a) where $f : M \to N$ there is $b \in U(K)$ such that (f, a) and (g, b) are equivalent.

Theorem 1. Let $\mathcal{K}$ be a coherent accessible category with concrete directed colimits, the amalgamation property and the joint embedding property and λ be a sufficiently large regular cardinal. Then K is λ-Galois saturated if and only if it is λ-saturated.

Coherence appears to be indispensable in the "only if" part of the proof, i.e., in the element-by-element construction of morphisms.

Tameness was introduced by Grossberg and VanDieren in 2006 as a smallness property of Galois types for AECs.

Let $\mathcal{K}$ be an accessible category with concrete directed colimits and κ be a regular cardinal. We say that $\mathcal{K}$ is κ-*tame* if for two non-equivalent types (f_0, a_0) and (f_1, a_1) over M there is a morphism $u : X \to M$ with X κ-presentable such that the types $(f_0 u, a_0)$ and $(f_1 u, a_1)$ are not equivalent.
$\mathcal{K}$ is called *tame* if it is κ-tame for some κ.

Theorem.(Grossberg, VanDieren) Let $\mathcal{K}$ be a large, tame AEC with the amalgamation property and the joint embedding property. If $\mathcal{K}$ is λ^+-categorical for a sufficiently large cardinal λ then $\mathcal{K}$ is μ-categorical for all $\mu \geq \lambda^+$.

An uncountable cardinal κ is called *strongly compact* if every κ-complete filter can be extended to a κ-complete ultrafilter on the same set.

Equivalently, $L_{\kappa,\kappa}$ satisfies the compactness theorem.

Theorem.(Boney) Assuming the existence of arbitrarily large strongly compact cardinals, every AEC is tame.

Theorem 2. Assuming the existence of arbitrarily large strongly compact cardinals, every accessible category with concrete directed colimits is tame.

This generalization of Boney's theorem is the consequence of

Theorem.(Makkai, Paré) Assuming the existence of arbitrarily large strongly compact cardinals, every powerful image of an accessible functor is accessible.

The *powerful image* of a functor $G : \mathcal{K} \to \mathcal{L}$ is the smallest full subcategory of $\mathcal{L}$ containing $G(\mathcal{K})$ and closed under subobjects.

We expect that Grossberg-VanDieren theorem is valid for coherent accessible categories with concrete directed colimits, i.e., that one does not need iso-fullness.

Any accessible category $\mathcal{K}$ with concrete directed colimits admits an EM-functor, i.e., a faithful functor $E : \mathbf{Lin} \to \mathcal{K}$ preserving directed colimits.

One does not need coherence for this. In abstract elementary classes one gets this functor from the Shelah's Presentation Theorem which involves both the assumption of coherence and the reintroduction of language into the fundamentally syntax-free world of abstract elementary classes.