GENERALIZED BROWN REPRESENTABILITY IN HOMOTOPY CATEGORIES

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Abstract. We show that the homotopy category of a combinatorial stable model category $\mathcal{K}$ is well generated. It means that each object K of $Ho(\mathcal{K})$ is an iterated weak colimit of λ-compact objects for some cardinal λ. A natural question is whether each K is a weak colimit of λ-compact objects. We show that this is related to (generalized) Brown representability of $Ho(\mathcal{K})$.

1. Introduction

Combinatorial model categories were introduced by J. H. Smith as model categories which are locally presentable and cofibrantly generated. The latter means that both cofibrations and trivial cofibrations are cofibrantly generated by a set of morphisms. Most of important model categories are at least Quillen equivalent to a combinatorial one. A natural question is to find properties of homotopy categories of combinatorial model categories. M. Hovey [23], 7.3.1 showed that $Ho(\mathcal{K})$ has a set of weak generators for each cofibrantly generated pointed model category $\mathcal{K}$. We will prove that $Ho(\mathcal{K})$ is well generated whenever $\mathcal{K}$ is a combinatorial pointed model category. It means the existence of a cardinal λ such that $Ho(\mathcal{K})$ has a set of weak λ-compact generators. This concept was introduced by A. Neeman [36] for triangulated categories but it makes sense for homotopy categories of pointed model categories as well. Our result generalizes that of [23], 7.4.3 proved for finitely generated pointed model categories.

If $\mathcal{K}$ is a stable model category then the smallest localizing subcategory containing a set of weak generators $\mathcal{G}$ is $Ho(\mathcal{K})$ itself. It means that each object of $Ho(\mathcal{K})$ is an iterated weak colimit of objects from $\mathcal{G}$. We can ask whether a set $\mathcal{A}$ of objects of $Ho(\mathcal{K})$ can be found such that each object of $Ho(\mathcal{K})$ is a weak colimit of objects from $\mathcal{A}$. This question is related to (generalized) Brown representability of $Ho(\mathcal{K})$.

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Consider the canonical functor

$$E_{\mathcal{A}} : Ho(\mathcal{K}) \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

sending an object K to the restriction

$$E_{\mathcal{A}} K = \hom(-, K)/\mathcal{A}^{op}$$

of its hom-functor $\hom(-, K) : Ho(\mathcal{K})^{\mathrm{op}} \to \mathbf{Set}$ to $\mathcal{A}^{\mathrm{op}}$. Since $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is a free completion of $\mathcal{A}$ under colimits (see [1], 1.45), each object of $Ho(\mathcal{K})$ is a weak colimit of objects from $\mathcal{A}$ if and only if $E_{\mathcal{A}}$ is full. Let $Ho(\mathcal{K})$ be compactly generated and $\mathcal{A}$ consists of $\aleph_0$-compact objects. Then $E_{\mathcal{A}}$ is full if and only if $Ho(\mathcal{K})$ satisfies [BRM] (see [11]), i.e., iff it is Brown representable (for homology) on morphisms. So, our question is whether $Ho(\mathcal{K})$ satisfies [BRM$_\lambda$] for some cardinal λ. A. Beligiannis [4], 11.8 showed that [BRM] implies [BRO], i.e., that every exact functor $\mathcal{A}^{\mathrm{op}} \to \mathbf{Ab}$ ($\mathcal{A}$ still consists of $\aleph_0$-compact objects) is in the image of $E_{\mathcal{A}}$. Since these exact functors form the free completion $Ind(\mathcal{A})$ of $\mathcal{A}$ under filtered colimits, [BRM] is equivalent with

$$E_{\mathcal{A}} : Ho(\mathcal{K}) \to Ind(\mathcal{A})$$

being full and surjective on objects. More precisely, one should say essentially surjective in the sense that each object from $Ind(\mathcal{A})$ is isomorphic to $E_{\mathcal{A}} K$ for some K.

Given a combinatorial stable model category $\mathcal{K}$ such $Ho(\mathcal{K})$ is well λ-generated for a regular cardinal λ, let $\mathcal{A}$ denote the full subcategory consisting of λ-compact objects. Then the image of $E_{\mathcal{A}}$ is contained in the free completion $Ind_\lambda(\mathcal{A})$ of $\mathcal{A}$ under λ-filtered colimits. Our generalized Brown representability thus means the question whether

$$E_{\mathcal{A}} : Ho(\mathcal{K}) \to Ind_\lambda(\mathcal{A})$$

is full and essentially surjective on objects. Previous versions of this paper claimed that for each combinatorial stable model category $\mathcal{K}$ there is a regular cardinal λ such that this is true. Unfortunately, the proofs contain a gap and the author is grateful to R. Jardine and F. Muro for pointing this up. Let us add the one cannot expect $E_{\mathcal{A}}$ being also faithfull. Then $Ho(\mathcal{K})$ would be equivalent to $Ind_\lambda(\mathcal{A})$, i.e., it would be accessible. Even in the compactly generated case, i.e., for $\lambda = \aleph_0$, there are, except trivial situations, phantoms, i.e., non-zero morphisms f in $Ho(\mathcal{K})$ with $E_{\mathcal{A}}(f) = 0$.

2. Basic concepts

A *model structure* on a category $\mathcal{K}$ will be understood in the sense of Hovey [23], i.e., as consisting of three classes of morphisms called weak equivalences, cofibrations and fibrations which satisfy the usual properties of Quillen [39] and, moreover, both (cofibration, trivial fibrations) and (trivial cofibrations, fibration) factorizations are functorial. Recall that trivial (co)fibrations are those (co)fibrations which are in the same time weak equivalences. The (cofibration, trivial fibration) factorization is *functorial* if there is a functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ and natural transformations $\alpha : \mathrm{dom} \to F$ and $\beta : F \to \mathrm{cod}$ such that $f = \beta_f \alpha_f$ is the (cofibration, trivial fibration) factorization of f. Here $\mathcal{K}^{\to}$ denotes the category of morphisms in $\mathcal{K}$ and $\mathrm{dom} : \mathcal{K}^{\to} \to \mathcal{K}$ ($\mathrm{cod} : \mathcal{K}^{\to} \to \mathcal{K}$) asssign to each morphism its (co)domain. The same for (trivial cofibration, fibration) factorization (see [40]) .

A *model category* is a complete and cocomplete category together with a model structure. In a model category $\mathcal{K}$, the classes of weak equivalences, cofibrations and fibrations will be denoted by $\mathcal{W}$, $\mathcal{C}$ and $\mathcal{F}$, resp. Then $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{W}$ and $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{W}$ denote trivial cofibrations and trivial fibrations, resp. We have

$$\mathcal{F}_0 = \mathcal{C}^{\square}, \quad \mathcal{F} = \mathcal{C}_0^{\square}, \quad \mathcal{C} = {}^{\square}\mathcal{F}_0 \quad \text{and} \quad \mathcal{C}_0 = {}^{\square}\mathcal{F}$$

where $\mathcal{C}^{\square}$ denotes the class of all morphisms having the right lifting property w.r.t. each morphism from $\mathcal{C}$ and ${}^{\square}\mathcal{F}$ denotes the class of all morphisms having the left lifting property w.r.t. each morphism of $\mathcal{F}$. $\mathcal{K}$ is called *cofibrantly generated* if there are sets of morphisms $\mathcal{I}$ and $\mathcal{J}$ such that $\mathcal{F}_0 = \mathcal{I}^{\square}$ and $\mathcal{F} = \mathcal{J}^{\square}$. If $\mathcal{K}$ is locally presentable then $\mathcal{C}$ is the closure of $\mathcal{I}$ under pushouts, transfinite compositions and retracts in comma-categories $K \downarrow \mathcal{K}$ and, analogously, $\mathcal{C}_0$ is this closure of $\mathcal{J}$.

An object K of a model category $\mathcal{K}$ is called *cofibrant* if the unique morphism $0 \to K$ from an initial object is a cofibration and K is called *fibrant* if the unique morphism $K \to 1$ to a terminal object is a fibration. Let $\mathcal{K}_c$, $\mathcal{K}_f$ or $\mathcal{K}_{cf}$ denote the full subcategories of $\mathcal{K}$ consisting of objects which are cofibrant, fibrant or both cofibrant and fibrant resp. We get the *cofibrant replacement functor* $R_c : \mathcal{K} \to \mathcal{K}$ and the *fibrant replacement functor* $R_f : \mathcal{K} \to \mathcal{K}$. We will denote by $R = R_f R_c$ their composition and call it the *replacement functor*. The codomain restriction of the replacement functors are $R_c : \mathcal{K} \to \mathcal{K}_c$, $R_f : \mathcal{K} \to \mathcal{K}_f$ and $R : \mathcal{K} \to \mathcal{K}_{cf}$.

Let $\mathcal{K}$ be a model category and K an object of $\mathcal{K}$. Recall that a *cylinder object* $C(K)$ for K is given by a (cofibration, weak equivalence) factorization

$$\nabla : K \amalg K \xrightarrow{\gamma_K} C(K) \xrightarrow{\sigma_K} K$$

of the codiagonal ∇. Morphisms $f, g : K \to L$ are *left homotopic* if there is a morphism $h : C(K) \to L$ with

$$f = h\gamma_{1K} \quad \text{and} \quad g = h\gamma_{2K}$$

where $\gamma_{1K} = \gamma_K i_1$ and $\gamma_{2K} = \gamma_K i_2$ with $i_1, i_2 : K \to K \amalg K$ being the coproduct injections. In fact, cylinder objects form a part of the *cylinder functor* $C : \mathcal{K} \to \mathcal{K}$ and $\gamma_1, \gamma_2 : \mathit{Id} \to C$ are natural transformations.

On $\mathcal{K}_{cf}$, left homotopy $\sim$ is an equivalence relation compatible with compositions, it does not depend on a choice of a cylinder object and we get the quotient

$$Q : \mathcal{K}_{cf} \to \mathcal{K}_{cf}/\sim \ .$$

The composition

$$P : \mathcal{K} \xrightarrow{R} \mathcal{K}_{cf} \xrightarrow{Q} \mathcal{K}_{cf}/\sim$$

is, up to equivalence, the projection of $\mathcal{K}$ to the homotopy category $\mathit{Ho}(\mathcal{K}) = \mathcal{K}[\mathcal{W}^{-1}]$ (see [23]). In what follows, we will often identify $\mathcal{K}_{cf}/\sim$ with $\mathit{Ho}(\mathcal{K})$.

A category $\mathcal{K}$ is called λ-*accessible*, where λ is a regular cardinal, provided that

(1) $\mathcal{K}$ has λ-filtered colimits,
(2) $\mathcal{K}$ has a set $\mathcal{A}$ of λ-presentable objects such that every object of $\mathcal{K}$ is a λ-filtered colimit of objects from $\mathcal{A}$.

Here, an object K of a category $\mathcal{K}$ is called λ-*presentable* if its hom-functor $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves λ-filtered colimits; $\mathbf{Set}$ is the category of sets. A category is called *accessible* if it is λ-accessible for some regular cardinal λ. The theory of accessible categories was created in [34] and for its presentation one can consult [1]. We will need to know that λ-accessible categories are precisely categories $\mathit{Ind}_\lambda(\mathcal{A})$ where $\mathcal{A}$ is a small category. If idempotents split in $\mathcal{A}$ then $\mathcal{A}$ precisely consists of λ-presentable objects in $\mathit{Ind}(\mathcal{A})$. In what follows, we will denote by $\mathcal{K}_\lambda$ the full subcategory of $\mathcal{K}$ consisting of λ-presentable objects.

A *locally λ-presentable category* is defined as a cocomplete λ-accessible category and it is always complete. Locally λ-presentable categories are precisely categories $\mathit{Ind}_\lambda(\mathcal{A})$ where the category $\mathcal{A}$ has λ-*small* colimits, i.e., colimits of diagrams $D : \mathcal{D} \to \mathcal{A}$ where $\mathcal{D}$ has less then λ morphisms. In general, the category $\mathit{Ind}_\lambda(\mathcal{A})$ can be shown to be the full subcategory of the functor category $\mathbf{Set}^{\mathcal{A}^{op}}$ consisting of λ-filtered colimits H of hom-functors $\hom(A, -)$ with A in $\mathcal{A}$. In the case that $\mathcal{A}$ has λ-small colimits this is equivalent to the fact that $H : \mathcal{A}^{op} \to \mathbf{Set}$ preserves λ-small limits. More generally, if $\mathcal{A}$ has weak λ-small colimits then $\mathit{Ind}_\lambda(\mathcal{A})$ precisely consists of left λ-covering functors (see [27] 3.2).

Let us recall that a weak colimit of a diagram $D : \mathcal{D} \to \mathcal{A}$ is a cocone from D such that any other cocone from D factorizes through it but not necessarily uniquely. If $\mathcal{X}$ is a category with weak λ-small limits then a functor $H : \mathcal{X} \to \mathbf{Set}$ is *left λ-covering* if, for each λ-small diagram $D : \mathcal{D} \to \mathcal{X}$ and its weak limit X, the canonical mapping $H(X) \to \lim HD$ is surjective (see [9] for $\lambda = \omega$). A left λ-covering functor preserves all λ-small limits which exist in $\mathcal{X}$. Moreover, a functor $H : \mathcal{X} \to \mathbf{Set}$ is left λ-covering iff it is weakly λ-continuous, i.e., iff it preserves weak λ-small limits. This immediately follows from [9], Proposition 20 and the fact that surjective mappings in **Set** split. A functor H is called *weakly continuous* if it preserves weak limits. Hence a weakly continuous functor $H : \mathcal{X} \to \mathbf{Set}$ preserves all existing limits.

A functor $F : \mathcal{K} \to \mathcal{L}$ is called *λ-accessible* if $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible categories and F preserves λ-filtered colimits. An important subclass of λ-accessible functors are those functors which also preserve λ-presentable objects. In the case that idempotents split in $\mathcal{B}$, those functors are precisely functors $Ind_\lambda(G)$ where $G : \mathcal{A} \to \mathcal{B}$ is a functor. The uniformization theorem of Makkai and Paré says that for each λ-accessible functor F there are arbitrarily large regular cardinals μ such that F is μ-accessible and preserves μ-presentable objects (see [1] 2.19). In fact, one can take $\lambda \triangleleft \mu$ where $\triangleleft$ is the set theoretical relation between regular cardinals corresponding to the fact that every λ-accessible category is μ-accessible (in contrast to [1] and [34], we accept $\lambda \triangleleft \lambda$). For every λ there are arbitrarily large regular cardinals μ such that $\lambda \triangleleft \mu$. For instance, $\omega \triangleleft \mu$ for every regular cardinal μ.

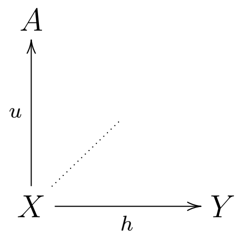
3. Combinatorial model categories

We will follow J. H. Smith and call a model category $\mathcal{K}$ *λ-combinatorial* if $\mathcal{K}$ is locally λ-presentable and both cofibrations and trivial cofibrations are cofibrantly generated by sets $\mathcal{I}$ and $\mathcal{J}$ resp. of morphisms having λ-presentable domains and codomains. Then both trivial fibrations and fibrations are closed in $\mathcal{K}^\to$ under λ-filtered colimits. $\mathcal{K}$ will be called *combinatorial* if it is λ-combinatorial for some regular cardinal λ. Clerly, if $\lambda < \mu$ are regular cardinals and $\mathcal{K}$ is λ-combinatorial then $\mathcal{K}$ is μ-combinatorial.

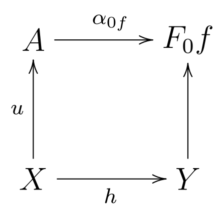
The following result is due to J. H. Smith and is presented in [13], 7.1. We just add a little bit more detail to the proof.

Proposition 3.1 (Smith). *Let $\mathcal{K}$ be a λ-combinatorial model category. Then the functors $\mathcal{K}^\to \to \mathcal{K}$ giving (cofibration, trivial fibration) and (trivial cofibration, fibration) factorizations are λ-accessible.*

Proof. We know that $\mathcal{K}$ is locally λ-presentable and domains and codomains of morphisms from the generating set $\mathcal{I}$ of cofibrations are λ-presentable. For every morphism $f : A \to B$ form a colimit $F_0 f$ of the diagram



consisting of all spans (u, h) with $h : X \to Y$ in $\mathcal{I}$ such that there is $v : Y \to B$ with $vh = fu$. Let $\alpha_{0f} : A \to F_0 f$ denote the component of the colimit cocone (the other components are $Y \to F_0 f$ and they make all squares



to commute). Let $\beta_{0f} : F_0 f \to B$ be the morphism induced by f and v's. Then $F_0 : \mathcal{K}^{\to} \to \mathcal{K}$ is clearly λ-accessible. Let $F_i f, \alpha_{if}$ and β_{if}, $i \leq \lambda$, be given by the following transfinite induction: $F_{i+1} f = F_0 \beta_{if}$, $\alpha_{i+1,f} = \alpha_{0,\beta_{if}} \alpha_{if}$, $\beta_{i+1,f} = \beta_{0,\beta_{if}}$ and the limit step is given by taking colimits. Then all functors $F_i : \mathcal{K}^{\to} \to \mathcal{K}$, $i \leq \lambda$ are λ-accessible and F_λ yields the desired (cofibration, trivial fibration) factorization.

The proof for (trivial cofibration, fibration) factorizations is analogous. □

Remark 3.2. (1) 3.1 implies that, in a λ-combinatorial model category $\mathcal{K}$, weak equivalences are closed under λ-filtered colimits in $\mathcal{K}^{\to}$ (see [13], 7.5).

(2) Following the uniformization theorem ([1] Remark 2.19), there is a regular cardinal μ such that the functors from 3.1 are μ-accessible and preserve μ-presentable objects. This means that the factorizations $A \to C \to B$ of a morphism $A \to B$ have C μ-presentable whenever A and B are μ-presentable. This point is also well explained in [13], 7.2.

Definition 3.3. A λ-combinatorial model category $\mathcal{K}$ will be called *strongly λ-combinatorial* if the functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ giving the (cofibration, trivial fibration) factorization preserves λ-presentable objects.

Remark 3.4. (1) Following 3.2 (2), every combinatorial model category is strongly μ-combinatorial for some regular cardinal μ.

(2) Following [1], 2.20, if $\mathcal{K}$ is strongly λ-combinatorial and $\lambda \triangleleft \mu$ then $\mathcal{K}$ is strongly μ-combinatorial.

(3) In a strongly λ-combinatorial model category $\mathcal{K}$, both the cofibrant replacement functor $R_c : \mathcal{K} \to \mathcal{K}$ and the cylinder functor $C : \mathcal{K} \to \mathcal{K}$ preserve λ-filtered colimits and λ-presentable objects.

Combinatorial model categories form a very broad class. We will give some important examples.

Examples 3.5. (i) The model category **SSet** of simplicial sets is strongly ω_1-combinatorial. Let us add that the functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ giving (trivial cofibration, fibration) factorization preserves ω_1-presentable objects too. This observation can be found in [28], Section 5.

The same is true for the model category $\mathbf{SSet}_*$ of pointed simplicial sets.

(ii) The category $\mathbf{Ch}(R)$ of chain complexes of modules over a ring R is an ω-combinatorial model category (see [23], 2.3.11); fibrations are dimensionwise surjections and weak equivalences are homology isomorphisms. We will show that this model category is strongly ω-combinatorial provided that R is a noetherian ring of a finite projective dimension.

Finitely presentable objects in $\mathbf{Ch}(R)$ are precisely bounded complexes of finitely presentable modules. Each such a complex is clearly finitely presentable. On the other hand, consider a chain complex (A, d); it means that $(d_n : A_n \to A_{n-1})$ for each integer n. For each integer $k \geq 0$, let (A^k, d^k) be the following chain complex: $A^k_n = 0$ for $n > k$ and $n < -k-1$, $A^k_n = A_n$ for $-k \leq n \leq k$, $A^k_{-k-1} = A_{-k}$, $d^k_n = 0$ for $n \leq -k$ and $n > k$, $d^k_n = d_n$ for $-k < n \leq k$ and $d^k_{-k} = \mathrm{id}_{A_{-k}}$. Then (A, d) is a colimit of the chain (A_k, d^k) with colimit components f where $f_n = 0$ for $n < -k-1$ and $n > k$, $f_n = \mathrm{id}_{A_n}$ for $-k \leq n \leq k$ and $f_{-k-1} = d_{-k}$. Thus each finitely presentable complex is bounded and evidently consists of finitely presentable modules.

Following [8], 2.9, $\mathbf{Ch}(R)$ is strongly ω-combinatorial if and only if finitely presentable complexes have finitely presentable cofibrant replacements.

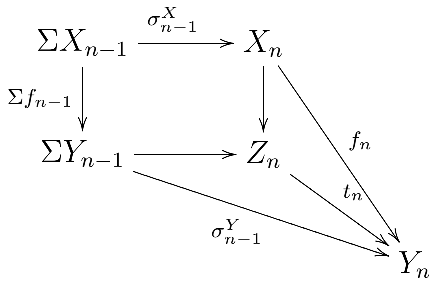
(iii) The category **Sp** of spectra with the strict model category structure (in the sense of [5]) is ω-combinatorial (see [41] A.3). We will show that it is strongly ω_1-combinatorial.

Let us recall that a spectrum X is a sequence $(X_n)^\infty_{n=0}$ of pointed simplicial sets equipped with morphisms $\sigma^X_n : \Sigma X_n \to X_{n+1}$ where Σ is the suspension functor. This means that $\Sigma X_n = S^1 \wedge X_n$ where $S^1 \wedge -$

is the smash product functor, i.e., a left adjoint to

$$-^{S^1} = \hom(S^1, -) : \mathbf{SSet}_* \to \mathbf{SSet}_*.$$

The strict model structure on **Sp** has level equivalences as weak equivalences and level fibrations as fibrations. This means that $f : X \to Y$ is a weak equivalence (fibration) iff all $f_n : X_n \to Y_n$ are weak equivalences (fibrations) in $\mathbf{SSet}_*$. A morphism $f : X \to Y$ is a (trivial) cofibration iff $f_0 : X_0 \to Y_0$ is a (trivial) cofibration and all induced morphisms $t_n : Z_n \to Y_n$, $n \geq 1$, from pushouts are (trivial) cofibrations



(see [5], [26] or [24]). Then a (cofibration, trivial fibration) factorization $X \xrightarrow{g} Z \xrightarrow{h} Y$ of a morphism $f : X \to Y$ is made as follows.

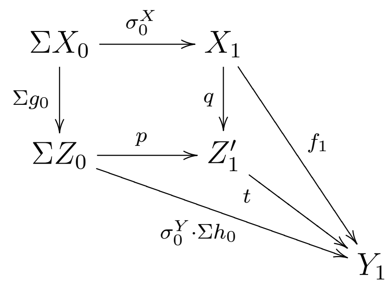
One starts with a (cofibration, trivial fibration) factorization

$$f_0 : X_0 \xrightarrow{g_0} Z_0 \xrightarrow{h_0} Y_0$$

in $\mathbf{SSet}_*$. Then one takes a (cofibration, trivial fibration) factorization

$$t : Z_1' \xrightarrow{u} Z_1 \xrightarrow{h_1} Y_1$$

of the induced morphism from a pushout



and puts $\sigma^Z_1 = up$ and $g_1 = uq$. This yields

$$f_1 : X_1 \xrightarrow{g_1} Z_1 \xrightarrow{h_1} Y_1$$

and one continues the procedure. Analogously, one constructs a (trivial cofibration, fibration) factorization. Since a spectrum X is ω_1-presentable iff all X_n, $n \geq 0$ are ω_1-presentable in $\mathbf{SSet}_*$, it is easy to see that the strict model structure on **Sp** is strongly ω_1-combinatorial. Moreover, the functor $F : \mathcal{K}^{\to} \to \mathcal{K}$ giving (trivial cofibration, fibration) factorization preserves ω_1-presentable objects too.

(iv) The model category **Sp** of spectra with the stable Bousfield-Friedlander model category structure (see [5]) is ω-combinatorial (see [41] A.3). The stable model structure is defined as a Bousfield localization of the strict model structure, i.e., by adding a set of new weak equivalences. Cofibrations and trivial fibrations remain unchanged, which means that the stable model category of spectra is strongly ω_1-combinatorial.

There is well known that the homotopy category of any model category $\mathcal{K}$ has products, coproducts, weak limits and weak colimits. We will recall their constructions.

Remark 3.6. (i) Let K_i, $i \in I$ be a set of objects of $\mathcal{K}$. Without any loss of generality, we may assume that they are in $\mathcal{K}_{cf}$. Then their product in $\mathcal{K}$

$$p_i : K \to K_i$$

is fibrant and let

$$q_K : R_cK \to K$$

be its cofibrant replacement. Then $R_cK \in \mathcal{K}_{cf}$ and

$$Q(p_i q_K) : QR_cK \to QK_i$$

is a product in $Ho(\mathcal{K})$. Recall that $Q : \mathcal{K}_{cf} \to Ho(\mathcal{K}) = \mathcal{K}_{cf}/\sim$ is the quotient functor.

In fact, consider morphisms

$$Qf_i : QL \to QK_i\,, \qquad i \in I$$

in $\mathcal{K}_{cf}/\sim$. Let $f : L \to K$ be the induced morphism and $g : L \to R_cK$ be given by the lifting property:

$$\begin{array}{ccc} 0 & \longrightarrow & R_cK \\ \downarrow & \overset{g}{\nearrow} & \downarrow{\scriptstyle q_K} \\ L & \xrightarrow[f]{} & K \end{array}$$

We have $Q(p_i q_K g) = Qf_i$ for each $i \in I$. The unicity of g follows from the facts that Qq_K is an isomorphism and that left homotopies h_i from $p_i f$ to $p_i f'$, $i \in I$, lift to the left homotopy from f to f'.

Since $\mathcal{K}^{\mathrm{op}}$ is a model category and

$$Ho(\mathcal{K}^{\mathrm{op}}) = (Ho(\mathcal{K}))^{\mathrm{op}}\,,$$

$Ho(\mathcal{K})$ has coproducts.

(ii) In order to show that $Ho(\mathcal{K})$ has weak colimits, it suffices to prove that it has weak pushouts. In fact, a weak coequalizer

$$A \underset{g}{\overset{f}{\rightrightarrows}} B \xrightarrow{\ h\ } D$$

is given by a weak pushout

$$\begin{CD} B @>{h}>> D \\ @A{(f,\mathrm{id}_B)}AA @AA{h}A \\ A \amalg B @>>{(g,\mathrm{id}_B)}> B \end{CD}$$

and weak colimits are constructed using coproducts and weak coequalizers in the same way as colimits are constructed by coproducts and coequalizers. This means that, given a diagram $D : \mathcal{D} \to Ho(\mathcal{K})$, its weak colimit K is a weak coequalizer of f and g constructed as follows

$$\begin{array}{ccccc} Dd & & & & \\ \downarrow{\scriptstyle u_e} & \searrow^{v_d} & & & \\ \coprod\limits_{e:d\to d'} Dd & \underset{g}{\overset{f}{\rightrightarrows}} & \coprod\limits_{d} Dd & \xrightarrow{\ h\ } & K \\ \uparrow{\scriptstyle u_e} & & \uparrow{\scriptstyle v_{d'}} & & \\ Dd & \xrightarrow[De]{} & Dd' & & \end{array}$$

where u_e and v_d are coproduct injections. The weak colimit cocone $\delta_d : Dd \to K$ is given by

$$\delta_d = hv_d$$

for each d in $\mathcal{D}$. We emphasize that the coproduct on the left is over all morphisms of $\mathcal{D}$.

Let

$$\begin{CD} B \\ @A{f}AA \\ P @>>{g}> D \end{CD}$$

be a diagram in $\mathcal{K}$. Consider a pushout

$$\begin{array}{ccc} B_1 & \xrightarrow{\overline{g}} & E \\ {\scriptstyle f_1}\uparrow & & \uparrow{\scriptstyle \overline{f}} \\ A & \xrightarrow[g_1]{} & D_1 \end{array}$$

in $\mathcal{K}$ where $f = f_2 f_1$ and $g = g_2 g_1$ are (cofibration, trivial fibration) factorizations. Following the homotopy extension property of cofibrations (see [21], 7.3.12),

$$\begin{array}{ccc} PB_1 & \xrightarrow{P\overline{g}} & PE \\ {\scriptstyle Pf_1}\uparrow & & \uparrow{\scriptstyle P\overline{f}} \\ PA & \xrightarrow[Pg_1]{} & PD \end{array}$$

is a weak pushout in $Ho(\mathcal{K})$ which is called the *homotopy pushout* of the starting diagram. Recall that $P : \mathcal{K} \to Ho(\mathcal{K})$ is the canonical functor.

Following [10], we will call the resulting weak colimits in $Ho(\mathcal{K})$ *standard*. By duality, $Ho(\mathcal{K})$ has weak limits. Since our model categories are functorial, the construction in $\mathcal{K}$ giving standard weak colimits in $Ho(\mathcal{K})$ is functorial in $\mathcal{K}$.

(iii) Consider a diagram $D : \mathcal{D} \to \mathcal{K}$, its colimit $(\bar{\delta}_d : Dd \to \overline{K})$ and $(\delta_d : Dd \to K)$ such that $(P\delta_d : PDd \to PK)$ is a standard weak colimit of PD. There is the comparison morphism $p : K \to \overline{K}$ such that $P(k)\delta_d = P(\overline{\delta}_d)$ for each $d \in \mathcal{D}$. It suffices to find this morphism for a pushout diagram

$$\begin{array}{ccc} B & \xrightarrow{g'} & \overline{E} \\ {\scriptstyle f}\uparrow & & \uparrow{\scriptstyle f'} \\ A & \xrightarrow[g]{} & D \end{array}$$

But it is given by $p\overline{g} = g' f_2$ and $p\overline{f} = f' g_2$; we use the notation from (ii).

(iv) Another, and very important, colimit construction in model categories are *homotopy colimits* (see, e.g., [6], [14], [21]). Both coproducts and homotopy pushouts described above are instances of this concept. While weak colimits correspond to homotopy commutative diagrams, homotopy colimits correspond to homotopy coherent ones. So, one cannot expect that homotopy colimits are weak colimits. There is always a morphism $wcolim\, D \to hocolim\, D$ from the standard weak colimit to the homotopy colimit for each diagram $D : \mathcal{D} \to \mathcal{K}$.

Following 3.2(1), λ-filtered colimits are homotopy λ-filtered colimits in a λ-combinatorial model category.

4. Well generated homotopy categories

Given a small, full subcategory $\mathcal{A}$ of a category $\mathcal{K}$, the *canonical functor*

$$E_{\mathcal{A}} : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$$

assigns to each object K the restriction

$$E_{\mathcal{A}} K = \hom(-, K)\big/\mathcal{A}^{op}$$

of its hom-functor $\hom(-, K) : \mathcal{K}^{\mathrm{op}} \to \mathbf{Set}$ to $\mathcal{A}^{\mathrm{op}}$ (see [1] 1.25). This functor is (a) $\mathcal{A}$-*full* and (b) $\mathcal{A}$-*faithful* in the sense that

(a) for every $f : E_{\mathcal{A}} A \to E_{\mathcal{A}} K$ with A in $\mathcal{A}$ there is $f' : A \to K$ such that $E_{\mathcal{A}} f' = f$ and

(b) $E_{\mathcal{A}} f = E_{\mathcal{A}} g$ for $f, g : A \to K$ with A in $\mathcal{A}$ implies $f = g$.

Let $\mathcal{K}$ be a locally λ-presentable model category and denote by $Ho(\mathcal{K}_\lambda)$ the full subcategory $P(\mathcal{K}_\lambda)$ of $Ho(\mathcal{K})$ consisting of P-images of λ-presentable objects in $\mathcal{K}$ in the canonical functor $P : \mathcal{K} \to Ho(\mathcal{K})$. Let

$$E_\lambda : Ho(\mathcal{K}) \to \mathbf{Set}^{Ho(\mathcal{K}_\lambda)^{\mathrm{op}}}$$

denote the canonical functor $E_{Ho(\mathcal{K}_\lambda)}$.

Theorem 4.1. *Let $\mathcal{K}$ be a strongly λ-combinatorial model category. Then the composition*

$$E_\lambda P : \mathcal{K} \to \mathbf{Set}^{Ho(\mathcal{K}_\lambda)^{\mathrm{op}}}$$

preserves λ-filtered colimits.

Proof. Consider a λ-filtered diagram $D : \mathcal{D} \to \mathcal{K}$ and its colimit $(k_d : Dd \to K)$ in $\mathcal{K}$. Consider $X \in \mathcal{K}_\lambda$ and a morphism $f : PX \to PK$ in $Ho(\mathcal{K})$. Let $u_X : R_c X \to X$ denote the cofibrant replacement and $v_K : K \to R_f K$ the fibrant replacement. Following [23], 1.2.10(ii), there is $\overline{f} : R_c X \to R_f K$ such that $P(v_K) f P u_X = P\overline{f}$. Since $\mathcal{K}$ is strongly λ-combinatorial, the object $R_c X$ is λ-presentable and $(R_f k_d : R_f Dd \to R_f K)$ is a λ-filtered colimit. Thus $\overline{f} = R_f(k_d) g$ for some $g : R_c X \to R_f Dd$ and d in $\mathcal{D}$. We have

$$PR_f(k_d) P(g) (Pu_X)^{-1} = P(\overline{f})(Pu_X)^{-1} = P(v_K) f$$

and thus

$$f = (Pv_K)^{-1} PR_f(k_d)P(g)(Pu_X)^{-1}.$$

Since

$$v_K k_d = R_f(k_d)v_{Dd}$$

where $v_{Dd} : Dd \to R_f Dd$ is the fibrant replacement of Dd, we have

$$f = P(k_d)(Pv_{Dd})^{-1} P(g)(Pu_X)^{-1}.$$

This proves that f factorizes through some Pk_d. In order to verify that $E_\lambda Pk_d : E_\lambda PDd \to E_\lambda PK$ is a λ-filtered colimit, we have to show that this factorization is essentially unique.

Assume that $f = P(k_d)g_1 = P(k_d)g_2$ are two such factorizations, i.e., $g_1, g_2 : PX \to PDd$. Again, using [23], 1.2.10(ii), there are $\overline{g_i} : R_cX \to R_f Dd$ such that $P\overline{g}_i = P(v_{Dd})g_i Pu_X$ for $i = 1, 2$. Since $PR_f(k_d)P\overline{g}_1 = PR_f(k_d)P\overline{g}_2$, the morphisms $R_f(k_d)\overline{g}_1$ and $R_f(k_d)\overline{g}_2$ are left homotopic (see [23], 1.2.10(ii) and 1.2.6). Thus there is a morphism $h : CR_cX \to R_f K$ such that $R_f(k_d)\overline{g}_i = h\gamma_{iR_cX}$ for $i = 1, 2$. Since CR_cX is λ-presentable, we can assume without any loss of generality that $h = R_f(k_d)\overline{h}$ for $\overline{h} : CR_cX \to R_f Dd$. Since

$$R_f(k_d)\overline{h}\gamma_{1R_cX} = R_f(k_d)\overline{h}\gamma_{2R_cX}$$

and R_cX is λ-presentable, there is $e : d \to d'$ in $\mathcal{D}$ such that

$$R_f D(e)\overline{h}\gamma_{1R_cX} = R_f D(e)\overline{h}\gamma_{2R_cX}.$$

Thus $R_f D(e)\overline{g}_1$ and $R_f D(e)\overline{g}_2$ are left homotopic. Therefore

$$PR_f D(e)P\overline{g}_1 = PR_f D(e)P\overline{g}_2.$$

Hence

$$PR_f D(e)P(v_{Dd})g_1 Pu_X = PR_f D(e)P(v_{Dd})g_2 Pu_X$$

and thus

$$P(v_{Dd'})PD(e)g_1 Pu_X = P(v_{Dd'})PD(e)g_2 Pu_X.$$

Consequently

$$PD(e)g_1 = PD(e)g_2,$$

which is the desired essential unicity of our factorization. □

Let $P_\lambda : \mathcal{K}_\lambda \to Ho(\mathcal{K}_\lambda)$ denote the domain and codomain restriction of the canonical functor $P : \mathcal{K} \to Ho(\mathcal{K})$. We get the induced functor

$$Ind_\lambda\, P_\lambda : \mathcal{K} = Ind_\lambda\, \mathcal{K}_\lambda \to Ind_\lambda\, Ho(\mathcal{K}_\lambda)\,.$$

Corollary 4.2. *Let $\mathcal{K}$ be a strongly λ-combinatorial model category. Then $E_\lambda P \cong Ind_\lambda\, P_\lambda$.*

Remark. This means that E_λ factorizes through the inclusion

$$Ind_\lambda\, Ho(\mathcal{K}_\lambda) \subseteq \mathbf{Set}^{Ho(\mathcal{K}_\lambda)^{\mathrm{op}}}$$

and that the codomain restriction of E_λ, which we denote E_λ as well, makes the composition $E_\lambda P$ isomorphic to $Ind_\lambda\, P_\lambda$.

Proof. Since both $E_\lambda P$ and $Ind_\lambda\, P_\lambda$ have the same domain restriction on $\mathcal{K}_\lambda$, the result follows from 4.1. □

Corollary 4.3. *Let $\mathcal{K}$ be a strongly λ-combinatorial model category. The the functor*

$$E_\lambda : Ho(\mathcal{K}_\lambda) \to Ind_\lambda\, Ho(\mathcal{K}_\lambda)$$

preserves coproducts.

Proof. Following 3.6 (i) and 4.2, it suffices to show that $Ind_\lambda\, P_\lambda$ preserves coproducts. Since each coproduct is a λ-filtered colimit of λ-small coproducts and $Ind_\lambda\, P_\lambda$ preserves λ-filtered colimits, we have to prove that $Ind_\lambda\, P_\lambda$ preserves λ-small coproducts. Let $\coprod_{i\in I} K_i$ be such a coproduct, i.e., $card I < \lambda$. Each K_i is a λ-filtered colimit $colim\, D_i$ of λ-presentable objects. Let $D_i : \mathcal{D}_i \to \mathcal{K}_\lambda$ denote the corresponding diagrams. Since $\coprod_{i\in I} colim\, D_i$ is isomorphic to a λ-filtered colimit of coproducts $\coprod_{i\in I} D_i d_i$ where $d_i \in \mathcal{D}_i$, $Ind_\lambda\, P_\lambda$ preserves λ-filtered colimits and P_λ preserves λ-small coproducts, the result is proved. □

Definition 4.4. Let $\mathcal{K}$ be a category with coproducts and λ a cardinal. An object A of $\mathcal{K}$ is called λ-*small* if for every morphism $f : A \to \coprod_{i\in I} K_i$ there is a subset J of I of cardinality less than λ such that f factorizes as

$$A \to \coprod_{j\in J} K_j \to \coprod_{i\in I} K_i$$

where the second morphism is the subcoproduct injection.

Remark 4.5. $\aleph_0$-small objects are also called compact or abstractly finite. We use the terminology of A. Neeman [36] who found how compactness should be defined for uncountable cardinals. His definition was simplified by H. Krause in [31]. They considered compactness in additive categories but the definition makes sense in general.

Consider classes $\mathcal{S}$ of λ-small objects of $\mathcal{A}$ such for every morphism $f : S \to \coprod_{i\in I} K_i$ with $S \in \mathcal{S}$ there are morphisms $g_i : S_i \to K_i$ where $S_i \in \mathcal{S}$ for each $i \in I$ such that f factorizes through

$$\coprod_{i\in I} g_i : \coprod_{i\in I} S_i \to \coprod_{i\in I} K_i.$$

Since these classes are closed under unions, there is the greatest class $\mathcal{S}$ with this property. Its objects are called λ-*compact*.

Definition 4.6. Let $\mathcal{K}$ be a category with a zero object 0. A set $\mathcal{G}$ of objects is called *weakly generating* if $\hom(G, K) = \{0\}$ for each $G \in \mathcal{G}$ implies that $K = 0$.

Remark 4.7. A generating set $\mathcal{G}$ of objects is clearly weakly generating. Recall that the former concept means that, given two distinct morphisms $f, g : K_1 \to K_2$, there is a morphism $h : G \to K_1$, $G \in \mathcal{G}$ such that fh and gh are distinct.

M. Hovey proved in [23], 7.3.1 that the homotopy category of a cofibrantly generated pointed model category has a set of weak generators.

The following definition is due to A. Neeman.

Definition 4.8. Let λ be an infinite cardinal. A category $\mathcal{K}$ with coproducts and a zero object is called *well λ-generated* if it has a weakly generating set of λ-compact objects.

$\mathcal{K}$ is called *well generated* if it is well λ-generated for some infinite cardinal λ.

Theorem 4.9. *Let $\mathcal{K}$ be a strongly λ-combinatorial model category. Then $Ho(\mathcal{K})$ is well λ-generated.*

Proof. Following [23], 7.3.1, $Ho(\mathcal{K}_\lambda)$ weakly generates $Ho(\mathcal{K})$. Consider a morphism $f : A \to \coprod_{i \in I} K_i$ where A is in $Ho(\mathcal{K}_\lambda)$. Following 4.3, $E_\lambda f : E_\lambda A \to \coprod_{i \in I} E_\lambda K_i$. Since $E_\lambda A$ is λ-presentable in $Ind_\lambda\, Ho(\mathcal{K}_\lambda)$ and a coproduct is a λ-filtered colimit of λ-small subcoproducts, $E_\lambda f$ factorizes through some $\coprod_{j \in J} E_\lambda K_j$ where J has the cardinality smaller than λ. Since E_λ is $Ho(\mathcal{K}_\lambda)$-full, A is λ-small.

Analogously, the proof of 4.3 also yields that objects from $Ho(\mathcal{K}_\lambda)$ are λ-compact. The reason is that a morphism $f : A \to \coprod_{i \in I} K_i$ with $A \in Ho(\mathcal{K}_\lambda)$ is sent by E_λ to the morphism whose codomain is a λ-filtered colimit of coproducts of objects from $Ho(\mathcal{K}_\lambda)$. □

As a corollary we get the result of A. Neeman [37] that, for any Grothendieck abelian category $\mathcal{K}$, the derived category $D(\mathcal{K})$ is well generated.

5. Brown representability

Definition 5.1. A locally λ-presentable model category $\mathcal{K}$ will be called λ-*Brown on morphisms* provided that the functor

$$E_\lambda : Ho(\mathcal{K}) \to Ind_\lambda\, Ho(\mathcal{K}_\lambda)$$

is full. $\mathcal{K}$ will be called λ-*Brown on objects* provided that E_λ is essentially surjective. Finally, $\mathcal{K}$ is λ-*Brown* if it is λ-Brown both on objects and on morphisms.

Remark 5.2. (i) Recall that E_λ is essentially surjective if each object in $Ind_\lambda\, Ho(\mathcal{K}_\lambda)$ is isomorphic to $E_\lambda K$ for some K in $Ho(\mathcal{K})$.

(ii) Whenever $\mathcal{K}$ is strongly ω-combinatorial and E_ω is full then it is essentially surjective on objects as well. In fact, by 4.2, $Ind_\omega\, P_\omega$ is full. Since each object of $Ind_\omega(\mathcal{K}_\omega)$ can be obtained by an iterative taking of colimits of smooth chains (see [1]) and P_ω is essentially surjective on objects, $Ind_\omega\, P_\omega$ is essentially surjective on objects as well. Hence $\mathcal{K}$ is ω-Brown on objects. This argument does not work for $\lambda > \omega$ because, in the proof, we need colimits of chains of cofinality ω. This result corresponds to [4], 11.8.

(iii) If $\mathcal{K}$ is a locally finitely presentable model category such that $Ho(\mathcal{K})$ is a stable homotopy category in the sense of [25] then $\mathcal{K}$ is ω-Brown in our sense iff $Ho(\mathcal{K})$ is Brown in the sense of [25].

(iv) Let $\mathcal{K}$ be a strongly λ-combinatorial model category which is λ-Brown on morphisms. Consider an object K in $\mathcal{K}$. We can express K as a λ-filtered colimit of a diagram $D : \mathcal{D} \to Ho(\mathcal{K}_\lambda)$ with a colimit cocone $(\delta_d : Dd \to K)_{d\in\mathcal{D}}$. We get the cone $(P\delta_d : PDd \to PK)_{d\in\mathcal{D}}$ and, following 4.2, $(E_\lambda P\delta_d : E_\lambda PDd \to E_\lambda PK)_{d\in\mathcal{D}}$ is a colimit cocone. Let $\varphi_d : PDd \to L$ be another cocone. There is a unique morphism $t : E_\lambda PK \to E_\lambda L$ such that $tE_\lambda P\delta_d = E_\lambda\varphi_d$ for each $d \in \mathcal{D}$. Since $\mathcal{K}$ is λ-Brown on morphisms, we have $t = E_\lambda \bar{t}$ where $\bar{t} : PK \to L$. Since E_λ is $Ho(\mathcal{K}_\lambda)$-faithful, $\bar{t}P\delta_d = \varphi_d$ for each $d \in \mathcal{D}$. Hence $P\delta_d : PDd \to PK$ is a weak colimit. Hence each object of $Ho(\mathcal{K})$ is a weak λ-filtered colimit of objects from $Ho(\mathcal{K}_\lambda)$.

Consider a morphism $f : PK \to PK$ such that $fP\delta_d = P\delta_d$ for each d in $\mathcal{D}$. Then $E_\lambda f$ is an isomorphism and, if E_λ reflects isomorphisms, f is an isomorphism as well. This means that each object in $Ho(\mathcal{K})$ is a *minimal weak colimit* (in the sense of [25]) of objects from $Ho(\mathcal{K}_\lambda)$. Minimal colimits are determined uniquely up to an isomorphism. Another possible terminology, going back to [20], is a *stable weak colimit*.

(v) Let $\mathcal{K}$ be a strongly λ-combinatorial model category which is λ-Brown. Consider a λ-filtered diagram $D : \mathcal{D} \to Ho(\mathcal{K}_\lambda)$ and let $(\delta_d : E_\lambda Dd \to K)_{d\in\mathcal{D}}$ be a colimit of $E_\lambda D$ in $Ind_\lambda\, Ho(\mathcal{K}_\lambda))$. Since $\mathcal{K}$

is λ-Brown on objects, we can assume that $K = E_\lambda \overline{K}$. Since E_λ is $Ho(\mathcal{K}_\lambda)$-full and faithful, there is a cocone $\overline{\delta}_d : Dd \to \overline{K}$ such that $E_\lambda \overline{\delta}_d = \delta_d$ for each d in $\mathcal{D}$. By the same argument as in (iv), we get that $\overline{\delta}_d : Dd \to \overline{K}$ is a weak colimit cocone. Hence $Ho(\mathcal{K})$ has weak λ-filtered colimits of objects from $Ho(\mathcal{K}_\lambda)$.

(vi) $\mathcal{K}$ being λ-Brown can be viewed as a weak λ-accessibility of $Ho(\mathcal{K})$ because $Ho(\mathcal{K})$ is λ-accessible with $Ho(\mathcal{K})_\lambda = Ho(\mathcal{K}_\lambda)$ iff

$$E_\lambda : Ho(\mathcal{K}) \to Ind_\lambda\, Ho(\mathcal{K}_\lambda)$$

is an equivalence. This means that $\mathcal{K}$ is λ-Brown and E_λ is faithful. But this happens very rarely.

Examples 5.3. We will show that the homotopy categories

$$Ho(\mathbf{SSet}_n)$$

are finitely accessible for each $n = 1, 2, \dots$, i.e., that E_ω is an equivalence in this case. Recall that $\mathbf{SSet}_n = \mathbf{Set}^{\mathbf{\Delta}_n}$ where $\mathbf{\Delta}_n$ is the category of ordinals $\{1, 2, \dots, n\}$. The model category structure is the truncation of that on simplicial sets, i.e., cofibrations are monomorphisms and trivial cofibrations are generated by the horn inclusions

$$j_m : \Delta_m^k \to \Delta_m \qquad 0 < k \leq m \leq n.$$

Here, $\Delta_m = Y_n(m+1)$ where $Y_n : \mathbf{\Delta}_n \to \mathbf{SSet}_n$ is the Yoneda embedding for $m < n$ and Δ_n is $Y_n(n+1)$ without the $(n+1)$-dimensional simplex $\{0, 1, \dots, n\}$.

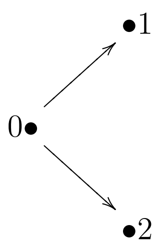
For example $\mathbf{SSet}_1 = \mathbf{Set}$ and trivial cofibrations are generated by $j_1 : 1 \to 2$. Then weak equivalences are precisely mappings between non-empty sets and $Ho(\mathbf{SSet}_1)$ is the category $\mathbf{2}$; all non-empty sets are weakly equivalent. $\mathbf{SSet}_2$ is the category of oriented multigraphs with loops. Trivial cofibrations are generated by the embedding j_1 of

$$\bullet 0$$

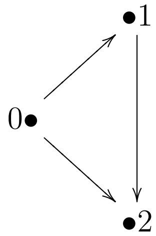
to

$$0\bullet \longrightarrow \bullet\, 1$$

(degenerated loops are not depicted), by the embedding j_2 of



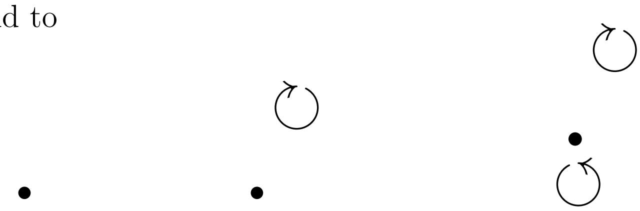
to



and their orientation variants. This makes all connected multigraphs weakly equivalent and $Ho(\mathbf{SSet}_2)$ is equivalent to **Set**; the cardinality of a set corresponds to the number of connected components.

In the case of $\mathbf{SSet}_3$, 1-connected objects cease to be weakly equivalent and their contribution to $Ho(\mathbf{SSet}_3)$ are trees (with a single root) of height ≤ 2. For example,

correspond to



(degenerated loops are not depicted). Therefore $Ho(\mathbf{SSet}_3)$ is equivalent to the category of forests of height ≤ 2. Analogously $Ho(\mathbf{SSet}_n)$ is equivalent to the category of forests of height $\leq n$. Hence it is finitely accessible.

Let us add that $\mathbf{SSet}_2$ is a natural model category of oriented multigraphs with loops (cf. [32]) and that the symmetric variants $\mathbf{Set}^{\mathbf{F}_n^{op}}$, where $\mathbf{F}_n$ is the category of cardinals $\{1, \dots, n\}$, are Quillen equivalent to $\mathbf{SSet}_n$ and left-determined by monomorphisms in the sense of [40].

Definition 5.4. Let $\mathcal{K}$ be a model category. Morphisms $f, g : K \to L$ in $Ho(\mathcal{K})$ will be called λ-*phantom equivalent* if $E_\lambda f = E_\lambda g$.

This means that $f, g : K \to L$ are λ-phantom equivalent iff $fh = gh$ for each morphism $h : A \to K$ with $A \in Ho(\mathcal{K}_\lambda)$.

Proposition 5.5. *Let $\mathcal{K}$ be a strongly λ-combinatorial model category which is λ-Brown on morphisms. Then for each object X in $Ho(\mathcal{K})$ there exists a weakly initial λ-phantom equivalent pair $f, g : X \to L$.*

Proof. We have $X = PK$. Let $(\delta_d : Dd \to K)_{d\in\mathcal{D}}$ be a canonical λ-filtered colimit of objects from $\mathcal{K}_\lambda$. Following 5.2 (iv), $(P\delta_d : PDd \to X)_{d\in\mathcal{D}}$ is a weak λ-filtered colimit. Take the induced morphism $p : \coprod_{d\in\mathcal{D}} PDd \to X$ and its weak cokernel pair f, g

$$\coprod PDd \xrightarrow{p} X \underset{g}{\overset{f}{\rightrightarrows}} L.$$

Since the starting colimit $(\delta_d : Dd \to K)_{d\in\mathcal{D}}$ is canonical, $E_\lambda p$ is an epimorphism in $Ind_\lambda\, Ho(\mathcal{K}_\lambda)$. Thus f and g are λ-phantom equivalent.

Let $f', g' : K \to L'$ be a λ-phantom equivalent. Then $f'p = g'p$ and thus the pair f', g' factorizes through f, g. Thus f, g is a weakly initial λ-phantom equivalent pair. □

For $\lambda < \mu$ we get a unique functor

$$F_{\lambda\mu} : Ind_\mu(Ho(\mathcal{K}_\mu)) \to Ind_\lambda(Ho(\mathcal{K}_\lambda))$$

which preserves μ-filtered colimits and whose domain restriction on $Ho(\mathcal{K}_\mu)$ coincides with that of E_λ.

Proposition 5.6. *Let $\mathcal{K}$ be a locally λ-presentable strongly μ-combinatorial model where $\lambda < \mu$ are regular cardinals. Then $F_{\lambda\mu}E_\mu \cong E_\lambda$.*

Proof. Following 4.2, we have $E_\mu P \cong Ind_\mu(P_\mu)$ and thus the functors $F_{\lambda\mu}E_\mu P \cong F_{\lambda\mu}\, Ind_\mu(P_\mu)$ and $E_\lambda P$ have the isomorphic domain restrictions on $\mathcal{K}_\mu$. We will show that the functor $E_\lambda P$ preserves μ-filtered colimits. Since $F_{\lambda\mu}\, Ind_\mu(P_\mu)$ has the same property, we will obtain that $F_{\lambda\mu}E_\mu P \cong E_\lambda P$ and thus $F_{\lambda\mu}E_\mu \cong E_\lambda$.

The functor $E_\lambda P$ preserves μ-filtered colimits iff for every object A in $\mathcal{K}_\lambda$ the functor

$$hom(PA, P-) : \mathcal{K} \to \mathbf{Set}$$

preserves μ-filtered colimits. Since $\mathcal{K}_\lambda \subseteq \mathcal{K}_\mu$, this follows from 4.1. □

Corollary 5.7. *Let $\mathcal{K}$ be a locally λ-presentable strongly μ-combinatorial model category where $\lambda < \mu$ are regular cardinals. Then E_μ reflects isomorphisms provided that E_λ reflects isomorphisms.*

Proof. It follows from 5.6. □

M. Hovey [23] introduced the concept of a pre-triangulated category (distinct from that used in [36]) and showed that the homotopy category of every pointed model category is pre-triangulated in his sense. He calls a pointed model category $\mathcal{K}$ *stable* if $Ho(\mathcal{K})$ is triangulated. In particular, $\mathcal{K}$ is stable provided that $Ho(\mathcal{K})$ is a stable homotopy category in the sense of [25].

Proposition 5.8. *Let $\mathcal{K}$ be a strongly combinatorial stable model category. Then E_λ reflects isomorphisms for arbitrarily large regular cardinals λ.*

Proof. Following [23] 7.3.1, every combinatorial pointed model category $\mathcal{K}$ has a set $\mathcal{G}$ of weak generators. Let $\Sigma^* = \{\Sigma^n Z | Z \in \mathcal{G}, n \in \mathbf{Z}\}$. Following [36] 6.2.9, there is a regular cardinal λ such that E_λ reflects isomorphisms. Thus the result follows from 5.7. □

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