Homotopy locally presentable enriched categories

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In what follows, $\mathcal{V}$ will be a combinatorial monoidal model category such that

(1) it is locally presentable as a closed category,
(2) it has all objects cofibrant.

The main example is **SSet** and another key example is **Cat** with the categorical model structure. We can also take any Cisinski model category.

Our aim is to introduce homotopy locally presentable $\mathcal{V}$-categories and to relate them to combinatorial model $\mathcal{V}$-categories.

Let $\mathcal{V}$ be locally presentable as a closed category. The *trivial* model structure has all morphisms as cofibrations and isomorphisms as weak equivalences. Our assumptions are satisfied and homotopy locally presentable $\mathcal{V}$-categories coincide with locally presentable $\mathcal{V}$-categories.

We have the functors $P : \mathcal{V} \to \mathrm{Ho}\,\mathcal{V}$, $\mathrm{Ho}\,\mathcal{V}(I, -) : \mathrm{Ho}\,\mathcal{V} \to \mathbf{Set}$ and their composition $U : \mathcal{V} \to \mathbf{Set}$.

The *homotopy category* $\mathrm{ho}\,\mathcal{K}$ of a $\mathcal{V}$-category $\mathcal{K}$ has the same objects as $\mathcal{K}$ and

$$\mathrm{ho}\,\mathcal{K}(A, B) = U(\mathcal{K}(A, B))$$

For a model $\mathcal{V}$-category $\mathcal{M}$, we now have the standard homotopy category $\mathrm{Ho}\,\mathcal{M}$, and the homotopy category $\mathrm{ho}\,\mathcal{M}$, and these need not agree. But if $\mathrm{Int}\,\mathcal{M}$ is the full subcategory of $\mathcal{M}$ consisting of the fibrant and cofibrant objects, then $\mathrm{ho}(\mathrm{Int}\,\mathcal{M})$ is equivalent to $\mathrm{Ho}(\mathcal{M})$.

Let $\mathcal{K}_0$ be the underlying category of $\mathcal{K}$. We have a functor $P_{\mathcal{K}} : \mathcal{K}_0 \to \mathrm{ho}\,\mathcal{K}$ and a morphism $f : A \to B$ in $\mathcal{K}$ is called a *homotopy equivalence* if its image in $\mathrm{ho}\,\mathcal{K}$ is invertible.

A $\mathcal{V}$-category $\mathcal{K}$ is called *fibrant* if each hom-object $\mathcal{K}(A, B)$ is fibrant in $\mathcal{V}$.

$\operatorname{Int}\mathcal{M}$ is fibrant for each model $\mathcal{V}$-category $\mathcal{M}$.

In a fibrant $\mathcal{K}$, f is a homotopy equivalence if and only if all $\mathcal{K}(C, f)$ (or all $\mathcal{K}(f, C)$) are weak equivalences in $\mathcal{V}$.

For a trivial model category $\mathcal{V}$, homotopy equivalences in a $\mathcal{V}$-category $\mathcal{K}$ coincide with isomorphisms and any $\mathcal{K}$ is fibrant.

Let $f : A \to B$ be a morphism in a $\mathcal{V}$-category $\mathcal{K}$. Then an object K in $\mathcal{K}$ is called *homotopy orthogonal* to f if $\mathcal{K}(f, K)$ is a weak equivalence.

For a trivial model category $\mathcal{V}$, we get the usual (enriched) orthogonality.

Let $\mathcal{M}$ be a model $\mathcal{V}$-category and $\mathcal{F}$ be a cofibration in $\operatorname{Int}\mathcal{M}$. Then $K \in \operatorname{Int}\mathcal{M}$ is homotopy orthogonal to f iff it is injective to all f-horns, i.e., to pushout-products $i\Box f$ with generating cofibrations i.

An object K is homotopy orthogonal to a class $\mathcal{F}$ of morphisms if it is homotopy orthogonal to each $f \in \mathcal{F}$. The class of all objects homotopy orthogonal to $\mathcal{F}$ is denoted by $\mathsf{HOrt}\,\mathcal{F}$ and is called a *homotopy orthogonality class*. If $\mathcal{F}$ is a set, we speak about *small homotopy orthogonality classes*.

Any homotopy orthogonality class is *homotopy replete*, i.e., it is closed under homotopy equivalent objects.

Theorem 1. Let $\mathcal{M}$ be a tractable left proper model $\mathcal{V}$-category. Assuming Vopěnka's principle, each homotopy orthogonality class in $\mathsf{Int}\,\mathcal{M}$ is a small homotopy orthogonality class.

For a trivial model category $\mathcal{V}$, any locally presentable $\mathcal{V}$-category $\mathcal{M}$ with the trivial model structure is a tractable left proper model $\mathcal{V}$-category. Thus Theorem 1 generalizes the fact that, assuming VP, any orthogonality class in a locally presentable category is small. This is equivalent to VP.

Let $\mathcal{L}$ be a full sub-$\mathcal{V}$-category of a fibrant $\mathcal{V}$-category $\mathcal{K}$. We say that $\mathcal{L}$ is *homotopy reflective* in $\mathcal{K}$ if, for each K in $\mathcal{K}$, there is a morphism $\eta_K \colon K \to K^*$ with K^* in $\mathcal{L}$ such that each L in $\mathcal{L}$ is homotopy orthogonal to η_K.

Theorem 2. Let $\mathcal{M}$ be a tractable left proper model $\mathcal{V}$-category. Then each small homotopy orthogonality class in $\operatorname{Int} \mathcal{M}$ is homotopy reflective.

For a trivial model category $\mathcal{V}$, homotopy reflective means reflective. Thus Theorem 2 generalizes the fact that small orthogonality classes in locally presentable categories are reflective.

Let $\mathcal{K}$ a fibrant $\mathcal{V}$-category, $S\colon \mathcal{D} \to \mathcal{K}$ a diagram, and $G\colon \mathcal{D}^{\mathsf{op}} \to \mathcal{V}$ a cofibrant weight. Then a *homotopy colimit* of S weighted by G is an object $G *_h S$ equipped with a natural transformation $\beta\colon \mathcal{K}(G *_h S, -) \to [\mathcal{D}^{\mathsf{op}}, \mathcal{V}](G, \mathcal{K}(S, -))$ whose components are weak equivalences.

β corresponds to a cocone $\delta : G \to \mathcal{K}(S, G *_h S)$.

For an arbitrary weight, we can define the homotopy colimit by taking its cofibrant replacement.

For a trivial model category $\mathcal{V}$, every weight is cofibrant and we get usual weighted colimits.

If G is cofibrant and the weighted colimit $G * S$ exists, then it is a homotopy colimit $G *_h S$.

Weighted homotopy colimits are determined up to homotopy equivalence.

Weighted homotopy limits $\{G, S\}_h$ are defined dually.

Ordinary colimits in a $\mathcal{V}$-category can be understood as weighted colimits $\Delta I * S$ where ΔI is constant at I and S is the extension of the starting diagram on the free $\mathcal{V}$-category.

Theorem 3. Let $\mathcal{V}$ be λ-combinatorial, $\mathcal{M}$ be a λ-combinatorial model $\mathcal{V}$-category, $\mathcal{I}$ a λ-filtered category and $S : \mathcal{I} \to \operatorname{Int} \mathcal{M}$. Then the canonical comparison $\operatorname{hocolim} S \to \operatorname{colim} S$ is a weak equivalence.

A $\mathcal{V}$-functor $F : \mathcal{K} \to \mathcal{L}$ between fibrant $\mathcal{V}$-categories *preserves* the homotopy weighted colimit $G *_h S$ when the composite

$$G \xrightarrow{\ \delta\ } \mathcal{K}(S, G *_h S) \xrightarrow{\ F\ } \mathcal{L}(FS, F(G *_h S)).$$

exhibits $F(G *_h S)$ as the homotopy colimit $G *_h FS$.

Let $\mathcal{K}$ be a fibrant $\mathcal{V}$-category. Then $\mathcal{K}(A, -) : \mathcal{K} \to \operatorname{Int} \mathcal{V}$ preserves weighted homotopy limits for each A in $\mathcal{K}$.

Theorem 4. Let $\mathcal{M}$ be a model $\mathcal{V}$-category. Then $\operatorname{Int}\mathcal{M}$ has weighted homotopy colimits and weighted homotopy limits.

In fact, if G is cofibrant then $G * S$ is cofibrant and $G *_h S$ is its fibrant replacement. Dually, $\{G, S\}$ is fibrant and $\{G, S\}_h$ is its cofibrant replacement.

Proposition 1. Let $\mathcal{K}$ be a fibrant $\mathcal{V}$-category. Then homotopy orthogonality classes in $\mathcal{K}$ are closed under existing weighted homotopy limits.

Theorem 5. Let $\mathcal{M}$ be a tractable left proper model $\mathcal{V}$-category. Then a full subcategory of $\operatorname{Int}\mathcal{M}$ is a small homotopy orthogonality class iff it is homotopy reflective and closed under homotopy λ-filtered colimits for some regular cardinal λ.

For a trivial model category $\mathcal{V}$, we get the usual characterization of small orthogonality classes.

An object A of a fibrant $\mathcal{V}$-category $\mathcal{K}$ is called *homotopy λ-presentable* when $\mathcal{K}(A,-) : \mathcal{K} \to \operatorname{Int}\mathcal{V}$ preserves homotopy λ-filtered colimits.

A small full subcategory $\mathcal{A}$ of a fibrant $\mathcal{V}$-category $\mathcal{K}$ is called *homotopy dense* if the induced functor

$$F : \mathcal{K} \xrightarrow{E} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}] \xrightarrow{Q} [\mathcal{A}^{\mathrm{op}}, \mathcal{V}]$$

is *locally a weak equivalence*, i.e. $\mathcal{K}(K, K') \to [\mathcal{A}^{\mathrm{op}}, \mathcal{V}](FK, FK')$ are weak equivalences in $\mathcal{V}$.

For a trivial model category $\mathcal{V}$, we get the usual dense subcategory.

A fibrant $\mathcal{V}$-category having a homotopy dense subcategory is called *homotopy bounded*.

A fibrant $\mathcal{V}$-category is called *homotopy locally λ-presentable* if it has homotopy weighted colimits and a homotopy dense subcategory consisting of homotopy λ-presentable objects.

For a trivial model category $\mathcal{V}$, we get locally λ-presentable categories.

A fibrant $\mathcal{V}$-category is called *strongly homotopy locally λ-presentable* if it has homotopy weighted colimits and a small full subcategory $\mathcal{A}$ consisting of homotopy λ-presentable objects such that every object of $\mathcal{K}$ is a homotopy λ-filtered colimit of objects from $\mathcal{A}$.

Proposition 2. Every strongly homotopy locally λ-presentable $\mathcal{V}$-category is homotopy locally λ-presentable.

We do not know whether the both concepts coincide, which is true for a trivial model category $\mathcal{V}$.

Theorem 6. $\operatorname{Int}\mathcal{M}$ is strongly homotopy locally presentable for every combinatorial model $\mathcal{V}$-category $\mathcal{M}$. If $\mathcal{M}$ is tractable then each small homotopy orthogonality class in $\operatorname{Int}\mathcal{M}$ is strongly homotopy locally presentable.

A *weak equivalence* $F : \mathcal{K} \to \mathcal{L}$ is a local weak equivalence which is *homotopically surjective* in the sense that each $L \in \mathcal{L}$ is homotopy equivalent to some FK.

A local weak equivalence $F : \mathcal{K} \to \mathcal{L}$ is called a *strong weak equivalence* if there is a local weak equivalence $G : \mathcal{L} \to \mathcal{K}$ such that GFK is homotopy equivalent to K and FGL is homotopy equivalent to L for each $K \in \mathcal{K}$ and $L \in \mathcal{L}$.

A strong weak equivalence is a weak equivalence. For trivial model structure, the both concepts coincide with equivalences of categories.

Proposition 3. If $\mathcal{L}$ is homotopy locally λ-presentable and $F : \mathcal{K} \to \mathcal{L}$ is a weak equivalence then $\mathcal{K}$ is homotopy locally λ-presentable.

Proposition 4. If $\mathcal{K}$ is strongly homotopy locally λ-presentable and $\mathcal{K} \to \mathcal{L}$ is a strong weak equivalence then $\mathcal{L}$ is strongly homotopy locally λ-presentable.

Theorem 7. Assuming Vopěnka's principle, the following conditions are equivalent to any fibrant $\mathcal{V}$-category:

(1) $\mathcal{K}$ is homotopy cocomplete and homotopy bounded.

(2) $\mathcal{K}$ is homotopy locally presentable.

(3) There is a weak equivalence $\mathcal{K} \to \operatorname{Int} \mathcal{M}$ for some combinatorial model $\mathcal{V}$-category.

(4) There is a weak equivalence $\mathcal{K} \to \operatorname{Int} \mathcal{M}$ where $\mathcal{M}$ is a left Bousfield localization of a $\mathcal{V}$-presheaf category w.r.t. a set of morphisms.

For a trivial model category $\mathcal{V}$, we do not need VP for the equivalence of (2), (3) and (4) but it is needed for the equivalence of (1) and (2).

Problem 1. Is VP needed for (2)⇒(3)?