Classification theory for accessible categories

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Accessible categories = $L_{\kappa\kappa}$-elementary classes (Makkai and Paré 1989)

Abstract elementary classes (Shelah 1987) form a special case of the former (Lieberman 2011, Beke and JR 2012) and generalize $L_{\kappa\omega}$-elementary classes.

Surprisingly, I do not know any abstract elementary class which does not have an $L_{\kappa\omega}$-axiomatization (as an abstract category). With Makkai, we failed to prove that uncountable sets with injective mappings form such an example.

It seemed a long time that there is no model theory for $L_{\kappa\kappa}$-elementary classes – see the introduction to Shelah, Model theory for θ-complete utrafilters. Here, he proposes to take κ strongly compact, which makes possibly to use κ-complete ultrafilters.

Recently, Boney, Vasey and Grossberg introduced κ-abstract elementary classes.

A κ-*abstract elementary class*, or κ-*AEC*, is a subcategory $\mathcal{K}$ of the category $\mathbf{Emb}(\Sigma)$ of Σ-structure and embeddings, where Σ is a $(<\kappa)$-ary infinitary signature, such that:

1. The embedding $G : \mathcal{K} \to \mathbf{Emb}(\Sigma)$ is replete, coherent and iso-full.
2. $\mathcal{K}$ has κ-directed colimits preserved by G.
3. (Löwenheim-Skolem) There is a cardinal $\lambda = \lambda^{<\kappa} \geq |\Sigma| + \kappa$ such that for any $M \in \mathcal{K}$ and $A \subseteq |GM|$, there is $M_0 \to M$ in $\mathcal{K}$ such that $A \subseteq |GM_0|$ and $|GM_0| \leq |A|^{<\kappa} + \lambda$.

$\aleph_0$-AEC is an AEC in the usual sense.

These generalized AECs are precisely accessible categories whose morphisms are monomorphisms (Lieberman, JR 2015).

In particular, they are full subcategories of finitary structures (i.e., not only coherent and iso-full). In more detail:

Theorem 1. Any κ-AEC with Löwenheim-Skolem number λ is a λ^+-accessible category.

Theorem 2. Any κ-accessible category whose morphisms are monomorhisms is a κ-AEC with Löwenheim-Skolem number $\lambda = \max(\kappa, \nu)^{<\kappa}$ where ν is the number of morphisms among κ-presentable objects.

This indicates that κ-AECs might be too general. Boney proposed to add the existence of directed bounds. This suffices, assuming amalgamation, to embed any object to a μ-saturated object (JR 1997) where μ-saturated = injective w.r.t. morphisms between μ-presentable objects.

This also excludes well ordered sets with monomorphisms which do not admit any *EM-functor*, i.e., a faithful functor from linear orders with monomorphisms preserving directed colimits.

Any EM-functor preserves sizes where an object K of an accessible category $\mathcal{K}$ has *size* $|K|$ if $|K|^+$ is the smallest regular cardinal κ such that K is κ-presentable.

Thus the existence of an EM-functor $E : \mathbf{Lin} \to \mathcal{K}$ makes $\mathcal{K}$ *LS-accessible* in the sense that, starting from some cardinal μ, $\mathcal{K}$ has objects of all sizes $\nu \geq \mu$.

Problem 1. (Beke, JR 2012) Is any accessible category LS-accessible?

Definition 1.(Lieberman, JR 2015) We say that a pair $(\mathcal{K}, U)$ consisting of a category $\mathcal{K}$ and faithful functor $U : \mathcal{K} \to \mathbf{Set}$ is a κ-*concrete AEC*, or κ-*CAEC*, if

1. $\mathcal{K}$ is accessible with directed colimits whose morphisms are monomorphisms.
2. U is coherent and preserves monomorphims.
3. $(\mathcal{K}, U)$ is replete and iso-full.
4. U preserves κ-directed colimits.

(3) means that $\mathcal{K}$ is replete and iso-full in the canonical embedding $G : \mathcal{K} \to \mathbf{Str}(\Sigma_{\mathcal{K}})$ where $\Sigma_{\mathcal{K}}$ consists of finitary function and relation symbols *interpretable* in $\mathcal{K}$.

A *weak* κ-*CAEC* is a κ-CAEC without (3).

$\aleph_0$-CAEC = AEC.

Any metric AEC is $\aleph_1$-CAEC.

Any κ-CAEC is κ-AEC.

Let $\mathcal{K}$ be an iso-full reflective subcategory of $\mathbf{Str}(\Sigma)$ closed under limits and κ-directed colimits and $\mathcal{K}_0$ has the same objects as $\mathcal{K}$ and monomorphisms as morphisms. Then $\mathcal{K}_0$ is a κ-CAEC.

Proposition 1.(Lieberman JR 2014) Any accessible category with directed colimits whose morphisms are monomorphisms is LS-accessible.

With Lieberman, we showed that many results about AECs can be extended to weak AECs. In particular, the machinery of Galois types and the fact that saturated objects coincide with Galois saturated ones.

To our surprise, all generalizes to weak κ-CAECs.

Let Met be the category of complete metric spaces and isometric embeddings. Met is λ-accessible for any uncountable regular cardinal λ. Complete metric spaces of cardinality $\leq \aleph_0$ have presentability rank $\aleph_1$, thus size $\aleph_0$. Otherwise, size coincides with density character.

The density character of a complete metric space X, denoted $dc(X)$, is the cardinality of the smallest dense subset of X.

By replacing **Set** with Met, AECs change to mAECs, i.e., to *metric abstract elementary classes*.

They may be thought of as a kind of amalgam of AECs with the program of continuous logic, which has its origins in the work of Chang and Keisler, and has subsequently been developed by Henson, Iovino, Usvyatsov and Ben-Yaacov, among others, always with an eye toward applications of model theory to structures arising in analysis.

Any mAEC is a $\aleph_1$-CAEC.

Theorem 3. (Lieberman, JR 2015) A pair $(\mathcal{K}, U)$ consisting of a category $\mathcal{K}$ and faithful functor $U : \mathcal{K} \to \mathrm{Met}$ is a mAEC if

1. $\mathcal{K}$ is accessible with directed colimits whose morphisms are monomorphisms.
2. U is coherent and preserves monomorphims.
3. $(\mathcal{K}, U)$ is replete and iso-full.
4. U preserves $\aleph_1$-directed colimits.