Birkhoff's Variety Theorem in Many Sorts

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The classical result of Garrett Birkhoff [3] that equational classes of Σ-algebras are precisely those closed under products, subalgebras and regular quotients (or homomorphic images) does not fully generalize to S-sorted signatures: it holds for finitely sorted algebras, but in case of infinitely many sorts one has to add closedness under directed unions. Or one has to leave the realm of finitary logic and start working in the logic $\mathscr{L}_{\omega,\lambda}$ allowing quantifications of less than λ variables where $\lambda > \text{card}\, S$. However, the "naive" generalization without directed unions appears rather persistently in books and papers. (See e.g. page 141 of [1], page 105 of [4], page 107 of [5] and page 248 of [6]). We first show a trivial counter-example to the "naive" generalization, and then prove the two generalizing theorems.

The failure of the "naive" generalization is caused by the role empty sorts of algebras are playing. If one would define a many-sorted algebra as a collection of non-empty sets with finitary many-sorted operations, no such problem with Birkhoff's variety theorem would appear. But such a definition would not work in applications. For example when considering a graph (with multiple edges) as an algebra of two sorts (vertex, edge), we certainly do not want to exclude graphs without edges. (Thus the situation reminds somewhat of the problem that the sorts cause in equational logic, empty sorts make equational logic more delicate, but we do not deal with this issue in our note.)

Recall that an S-sorted signature is a set Σ together with an arity function assigning to every symbol $\sigma \in \Sigma$ a pair $(s_1, \dots, s_n, s) \in S^* \times S$. By a Σ-algebra is then meant an S-sorted set $A = (A_s)_{s \in S}$ together with functions $\sigma^A \colon A_{s_1} \times \cdots \times A_{s_n} \to A_s$ for every σ of arity $(s_1 \dots s_n, s)$; if $n = 0$ then σ^A is an element of A_s. The category Σ-Alg of Σ-algebras has as morphisms

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the S-sorted functions preserving the operations in the expected sense. For every many-sorted set X (of variables) we denote by $F_\Sigma X$ the free Σ-algebra on X. Recall that X is called finite provided that $\coprod\limits_{s\in S} X_s$ is a finite set. For an infinite cardinal λ we call X λ-*presentable* provided that card $\coprod\limits_{s\in S} X_s < \lambda$.

Definition. An *equation* (in the finitary logic) is a formula $\forall\, X : t = t'$ where X is a finite S-sorted set and t, t' are elements of F_Σ X of the same sort. A Σ-algebra A *satisfies* the equation provided that for every S-sorted function $f : X \to A$ the homomorphism $\bar{f} : F_\Sigma X \to A$ extending f merges t and t'. An *equational class* is a full subcategory of $Alg\,\Sigma$ specified by a set of equations.

Example of a class of Σ-algebras which is closed under products, subalgebras and regular quotients in $Alg\,\Sigma$, but is not equational.

Put $S = \mathbb{N}$ and $\Sigma = \emptyset$. Thus $Alg\,\Sigma = \mathbf{Set}^{\mathbb{N}}$ is the category of sequences of sets. Let $\mathcal{A}$ be the full subcategory of all $(A_n)_{n\in\mathbb{N}}$ such that either $A_n = \emptyset$ for some $n \in \mathbb{N}$, or card $A_n = 1$ for every $n \in \mathbb{N}$. This subcategory is clearly closed under products, subalgebras and regular quotients.

However, $\mathcal{A}$ is not equational because then it would clearly be closed under directed unions. This is not the case: every object A of $\mathbf{Set}^{\mathbb{N}}$ is a directed union of objects A_i with almost all coordinates empty (and $A_i \in \mathcal{A}$).

The Birkhoff Variety Theorem. *For every set S of sorts and every S-sorted signature Σ the equational classes of Σ-algebras are precisely the full subcategories of Σ-Alg closed under products, subalgebras, regular quotients and directed unions.*

Proof. The verification of necessity is easy and standard. The only fact one needs to know is that regular epimorphisms are precisely the homomorphisms that are surjective in every sort (see e.g. [2], Corollary 3.5).

To prove the sufficiency, let $\mathcal{A}$ be a full subcategory of Σ-*Alg* closed as above. Then $\mathcal{A}$ is a full reflective subcategory such that for every Σ-algebra A the reflection $r_A : A \to R(A)$ is a regular epimorphism. We prove that $\mathcal{A}$ is presented by all equations $\forall\, X : t = t'$ where X is a finite S-sorted set and t, $t' \in (F_\Sigma X)_s$ are merged by $(r_{F_\Sigma X})_s$. Every algebra of $\mathcal{A}$ clearly satisfies all these equations. Conversely, if A in Σ-*Alg* satisfies them, we prove $A \in \mathcal{A}$. We can assume that A is finitely generated (since $\mathcal{A}$ is closed under directed unions and A is a directed union of finitely generated subalgebras). That is there exists a finite S-sorted subset $i\colon X \hookrightarrow A$ such that the corresponding homomorphism $\bar{i}\colon F_\Sigma X \to A$ is a regular epimorphism. Observe that $\bar{i}$ factorizes through $r_{F_\Sigma X}$: this follows from the fact that whenever $r_{F_\Sigma X}$ merges two elements t, t' (of the same sort s), then so does $\bar{i}$. Indeed, A satisfies $\forall\, X\colon t = t'$ which implies $\bar{i}_s(t) = \bar{i}_s(t')$. Consequently, we have an S-sorted function $e\colon R(F_\Sigma X) \to A$ with $\bar{i} = e \cdot r_{F_\Sigma X}$. Since $r_{F_\Sigma X}$ is a regular epimorphism, the fact that $\bar{i}$ is a homomorphism implies that e is one, too. And since $\bar{i}$ is a regular epimorphism, so is e. This proves $A \in \mathcal{A}$. ☐

Definition. Let λ be the least infinite cardinal larger than card S. By a *λ-ary equation* is meant a formula $\forall\ X\colon\ t = t'$ where X is a λ-presentable S-sorted set and t, t' are elements of $F_\Sigma X$ of the same sort. Satisfaction and λ-ary equational classes are defined analogously to the finitary case above.

Theorem. *For every set S of sorts and every S-sorted signature Σ the λ-ary equational classes of Σ-algebras are precisely those closed under products, subalgebras, and regular quotients.*

Proof. Given a full subcategory $\mathcal{A}$ of Σ-*Alg* closed under products, subalgebras, and regular quotients, we prove that $\mathcal{A}$ is closed under λ-directed unions. This is sufficient: the proof is as above, using that every algebra A is a λ-directed union of λ-generated subalgebras. Let A be a Σ-algebra and A_i $(i \in I)$ a λ-directed collection of subalgebras with

$$A = \bigcup_{i\in I} A_i \quad \text{and} \quad A_i \in \mathcal{A} \quad \text{for all } i \in I.$$

For every sort $s \in S$ with $A_s \neq \emptyset$ choose $i_0 \in I$ so that $(A_{i_0})_s \neq \emptyset$. The choice of i_0 can be made independently of s (since I is λ-directed and card $S < \lambda$). The algebra $\prod\limits_{i \geq i_0} A_i$ lies in $\mathcal{A}$. Consider the S-sorted subset B of this product consisting in sort s of precisely those tuples $(x_i)_{i\geq i_0}$ (where $x_i \in (A_i)_s$) which are eventually constant. That is, there exists $j \in I$ with $x_i = x_j$ for all $i \geq j$. It is easy to see that B carries a subalgebra of $\prod\limits_{i\geq i_0} A_i$. We have $B \in \mathcal{A}$. And this proves $A \in \mathcal{A}$ because we have a regular quotient $e\colon B \to A$ assigning to every eventually constant tuple the constant value. In fact, since for every sort s with $A_s \neq \emptyset$ we have $B_s \neq \emptyset$, due to our choice of i_0, it is easy to see that e_s is surjective. ☐

Corollary. *In case of finitely many sorts the Birkhoff variety theorem holds without the requirement of closure under directed unions.*

Remark. For more on equational classes of many-sorted algebras the reader may consult [2].

References

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