Injective Hulls are not Natural

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Abstract

In a category with injective hulls and a cogenerator, the embeddings into injective hulls can never form a natural transformation, unless all objects are injective. In particular, assigning to a field its algebraic closure, to a poset or Boolean algebra its MacNeille completion, and to an R-module its injective envelope is not functorial, if one wants the respective embeddings to form a natural transformation.

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1 Introduction

Projectivity and injectivity are fundamental concepts of modern mathematics. The question whether a given category has *enough injectives* (so that every object may be embedded into an injective one) or even *injective hulls* (so that such embeddings may be chosen to be *essential*), as well as the dual questions (*enough projectives*, *projective covers*), have been investigated for many categories, particularly in commutative and homological algebra, algebraic geometry, topology, and in functional analysis.

Existence of enough injectives is often facilitated by the existence of an injective cogenerator in the given category, in which case it is easy to see that, for each object A, the embedding $\iota_A : A \to EA$ into an injective object EA may be chosen *naturally*, so that E becomes an endofunctor of $\mathcal{C}$ and $\iota : \mathrm{Id}_{\mathcal{C}} \to E$ a natural transformation. Under fairly mild additional conditions, the existence of an injective cogenerator even gives injective hulls, and the question then is again: are they natural? Although injective hulls are uniquely determined, up to isomorphism, the somewhat surprising answer that we give in this paper is: *never*, unless the situation was trivial, in the sense that all objects were injective, in which case the injective-hull functor is given by the identity functor.

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Acknowledgements. We are grateful for comments received from Peter Freyd, Jim Lambek and Bill Lawvere after the announcement of our main theorem on the "categories" network. The current formulation of the theorem grew out of a discussion on that network. The non-functoriality of injective hulls in module categories (in fact, in the category of abelian groups) was observed by Jim Lambek in his "Torsion Theories, Additive Semantics, and Rings of Quotients" (Springer Lecture Notes in Mathematics 177, 1971), pp 10, 11.

2 Preliminaries

Throughout the paper, *$\mathcal{H}$ is an arbitrary class of morphisms in a category $\mathcal{C}$.* Although classically one thinks of the morphisms in $\mathcal{H}$ as "embeddings", there is *no a-priori assumption* on $\mathcal{H}$. At first we recall some known definitions in order to fix our terminology.

Definitions 2.1. (1) An object I of $\mathcal{C}$ is *$\mathcal{H}$-injective* if the function $\mathcal{C}(h,I) : \mathcal{C}(B,I) \to \mathcal{C}(A,I)$ is surjective for every $h : A \to B$ in $\mathcal{H}$ (so that every $f : A \to I$ "extends" to some $g : B \to I$ with $g \cdot h = f$). The $\mathcal{H}$-injective objects form the full subcategory $\mathcal{H}$-**Inj** of $\mathcal{C}$.

(2) A morphism h in $\mathcal{H}$ is *$\mathcal{H}$-essential* if for every morphism g, the composite $g \cdot h$ lies in $\mathcal{H}$ only if g does. The class of all $\mathcal{H}$-essential morphism in $\mathcal{C}$ is denoted by $\mathcal{H}^*$.

(3) $\mathcal{C}$ is said to have *enough $\mathcal{H}$-injectives* if for every object A in $\mathcal{C}$ there is a morphism $\iota_A : A \to EA$ in $\mathcal{H}$ with an $\mathcal{H}$-injective object EA; if, in addition, ι_A can be chosen to be $\mathcal{H}$-essential, then $\mathcal{C}$ has *$\mathcal{H}$-injective hulls* (often also called *envelopes*). If, in either case, E extends to an endofunctor of $\mathcal{C}$ making ι a natural transformation, we shall say that $\mathcal{C}$ has *naturally enough $\mathcal{H}$-injectives* or that *$\mathcal{H}$-injective hulls are natural*, respectively.

(4) A class $\mathcal{G}$ of objects is *cogenerating* (also *coseparating*) in $\mathcal{C}$ if for any two distinct morphisms $u, v : X \to A$ in $\mathcal{C}$ there is a morphism $h : A \to G$ with $G \in \mathcal{G}$ and $h \cdot u \neq h \cdot v$; equivalently, if for every object A in $\mathcal{C}$, the source of all morphisms with domain A and codomain in $\mathcal{G}$ is monic. For $\mathcal{G}$ small, this is the same as to say that the canonial morphism

$$\iota_A : A \to \prod_{G \in \mathcal{G}} G^{\mathcal{C}(A,G)}$$

is a monomorphism, provided that the needed products exist in $\mathcal{C}$.

(5) By an *$\mathcal{H}$-cogenerator* in $\mathcal{C}$ we mean a (small) set $\mathcal{G}$ of objects in $\mathcal{C}$ such that all small-indexed products of $\mathcal{G}$-objects exist in $\mathcal{C}$ and that the canonical morphisms ι_A of (4) lie in $\mathcal{H}$. $\mathcal{G}$ is an *$\mathcal{H}$-injective $\mathcal{H}$-cogenerator* if, in addition, all objects in $\mathcal{G}$ are $\mathcal{H}$-injective.

(6) We often omit the prefix $\mathcal{H}$ if the choice of $\mathcal{H}$ is unambiguous, particularly when $\mathcal{H}$ is the class of embeddings in the sense of [3].

Remarks 2.2. (1) The subcategory $\mathcal{H}$-**Inj** is closed under direct products and retracts in $\mathcal{C}$. To say that $\mathcal{C}$ has enough $\mathcal{H}$-injectives is equivalent to saying that $\mathcal{H}$-**Inj** is *weakly $\mathcal{H}$-reflective* in $\mathcal{C}$ (so that every object of $\mathcal{C}$ admits a weak reflection into $\mathcal{H}$-**Inj** belonging to $\mathcal{H}$; more generally see 4.1 below), whereas existence of $\mathcal{H}$-injective hulls means that $\mathcal{H}$-**Inj** is weakly $\mathcal{H}^*$-reflective in $\mathcal{C}$. In fact, in these cases $\mathcal{H}$-**Inj** is even *almost $\mathcal{H}$-reflective* (or almost $\mathcal{H}^*$-reflective) in the sense of [13], that is: weakly $\mathcal{H}$-reflective (or weakly $\mathcal{H}^*$-reflective) and closed under retracts, since the only additionally required property comes for free for $\mathcal{H}$-**Inj**.

(2) $\mathcal{H}$-injective hulls are unique, up to isomorphism. Since we work without hypotheses on $\mathcal{H}$, we include the proof of this fact for the Reader's convenience. Consider $\mathcal{H}$-essential morphisms $h : A \to I$ and $k : A \to J$ with I, J $\mathcal{H}$-injective. Then $\mathcal{H}$-injectivity of J lets k factor as $k = f \cdot h$, where $f \in \mathcal{H}$ since $h \in \mathcal{H}^*$. Now $\mathcal{H}$-injectivity of I makes 1_I factor as $1_I = g \cdot f$, hence $g \cdot k = h$. Again $k \in \mathcal{H}^*$ implies $g \in \mathcal{H}$, so that $\mathcal{H}$-injectivity of J gives l with $l \cdot g = 1_J$. Consequently, $g : J \to I$ and then f are isomorphisms. Note that in fact we proved that *any morphism connecting two $\mathcal{H}$-injective hulls of the same object must be an isomorphism.*

In order to establish naturally enough $\mathcal{H}$-injectives, one normally resorts to ι_A of 2.1(4):

Proposition 2.3. *A category with an $\mathcal{H}$-injective $\mathcal{H}$-cogenerator has naturally enough $\mathcal{H}$-injectives.*

Proof. There is a unique way of extending $EA = \prod_{G \in \mathcal{G}} G^{\mathcal{C}(A,G)}$ to a functor such that the canonical morphisms ι_A become a natural transformation. □

For examples, see Section 3.

3 Non-naturality of injective hulls

At first, we consider some typical examples which lead us to a general result.

Examples 3.1. (1) For $\mathcal{C} = \mathbf{Set}$ and $\mathcal{H} = \mathrm{Mono}(\mathbf{Set})$, the injectives are precisely the non-empty sets, and every set with at least two elements is a (single-object) cogenerator. With $\mathcal{G} = \{2\}$ (where 2 is a two-element set), 2.3 gives (up to isomorphism) the embedding $A \to PPA$ (with P the contravariant powerset functor), which sends every $x \in A$ to the free ultrafilter $\{U \subseteq A \mid x \in U\}$. There are, of course, many other natural embeddings, such as $A \to A + 1$ (where 1 is a singleton set and + denotes disjoint union). The injective hull A^* of a set A may be given as $A^* = 1$ for $A = \emptyset$, and $A^* = A$ else. There is, however, no way of making $*$ a functor such that $A \hookrightarrow A^*$ becomes natural; otherwise we would have a commutative diagram (in the obvious sense)

$$\begin{tikzcd}
\emptyset \arrow[r] \arrow[d] & 1 \arrow[r, shift left, "x"] \arrow[r, shift right, "y"'] \arrow[d, "\mathrm{id}"] & 2 \arrow[d, "\mathrm{id}"] \\
\emptyset^* \arrow[r, "\mathrm{id}"] & 1^* \arrow[r, shift left, "x^*"] \arrow[r, shift right, "y^*"'] & 2^*
\end{tikzcd}$$

which, for the two distinct maps x, y into 2, would render x^*, y^* simultaneously equal and distinct.

(2) The same argumentation as in (1) carries over to much fancier contexts. Let $\mathcal{C} = \mathbf{Field}$ be the category of fields and their homomorphisms, and choose $\mathcal{H}$ to be the class of algebraic extensions. A Zorn-Lemma argument shows that $\mathcal{H}$-injective means algebraically closed, and the injective hull F^* of a field F is its algebraic closure. Again, $*$ cannot be naturally considered a functor; otherwise the two $\mathbb{R}$-automorphisms id, t of the algebraic closure $\mathbb{C}$ of the reals $\mathbb{R}$ would give a commutative diagram

$$\begin{tikzcd}
\mathbb{R} \arrow[r, hook, "j"] \arrow[d, hook, "j"'] & \mathbb{C} \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "t"'] \arrow[d, "\mathrm{id}"] & \mathbb{C} \arrow[d, "\mathrm{id}"] \\
\mathbb{R}^* \arrow[r, "j^*"] & \mathbb{C}^* \arrow[r, shift left, "\mathrm{id}"] \arrow[r, shift right, "t^*"'] & \mathbb{C}^*
\end{tikzcd}$$

which leads to a contradiction, no matter whether $j^* = \mathrm{id}$ or $j^* = t$.

By considering this diagram in the (non-full) subcategory $\mathbf{Field}_{\mathrm{alg}}$ (whose morphisms are algebraic extensions) we obtain an *example of a category with enough injectives, but not naturally so.*

(3) Turning back to sets, let $\mathcal{C}$ be the comma category $\mathbf{Set}/B$ of sets over a fixed set B (or, equivalently, the presheaf category $\mathbf{Set}^B$ of $\mathbf{Set}$-valued functors on the discrete category B), and let $\mathcal{H}$ be the set of one-to-one maps over B. The injective objects of $\mathcal{C}$ are precisely given by surjective maps onto B, and the family of maps $B + 1 \to B$ (which maps B identically) forms an injective cogenerator of $\mathbf{Set}/B$. An extension of an object $(f : X \to B)$ of $\mathbf{Set}/B$ to an injective object is therefore given by a factorization $f = h \cdot g$ with g one-to-one and h onto. A possible *natural* choice (in the sense of 2.1(3)) would be to factor f in the form

$$\begin{tikzcd}
& X + B \arrow[dr, "{(f,\mathrm{id})}"] & \\
X \arrow[ur, hook] \arrow[rr, "f"] & & B
\end{tikzcd}$$

The injective hull of f, however, is given by the fatorization

$$\begin{tikzcd}
& X + (B - \mathrm{im}\, f) \arrow[dr, "{(f,\mathrm{inclusion})}"] & \\
X \arrow[ur, hook] \arrow[rr, "f"] & & B
\end{tikzcd}$$

which, as one easily sees, fails to be natural.

Existence theorems for enough injectives and injective hulls in general comma categories and their applications to Quillen model categories are given in [2].

The negative statements above are instances of the following general fact.

Theorem 3.2. *Assume that every isomorphism is in $\mathcal{H}^*$ and that every morphism h in $\mathcal{H}^*$ is an extremal monomorphism of $\mathcal{C}$ (so that $h = f \cdot e$ with an epimorphism e forces e to be iso). Then $\mathcal{C}$ cannot have natural $\mathcal{H}$-injective hulls, unless every object in $\mathcal{C}$ is $\mathcal{H}$-injective.*

Proof. Suppose we had natural $\mathcal{H}$-injective hulls, given by $\iota_A : A \to A^*$ for each object A in $\mathcal{C}$. We first show that with $\iota = \iota_A$, $\iota^* : A^* \to A^{**}$ is an isomorphism. Certainly, ι_{A^*} is an isomorphism since A^* is $\mathcal{H}$-injective; in fact, we may assume $\iota_{A^*} = 1_{A^*}$ since every isomorphism is in $\mathcal{H}^*$. By naturality, $\iota = \iota^* \cdot \iota$, so that by Remark 2.2(2) ι^* is an isomorphism. Now, by virtue of the following simple lemma, we may conclude that ι must be an epimorphism and, already being an extremal monomorphism, in fact an isomorphism. Hence, with A^* also A is $\mathcal{H}$-injective. □

Lemma 3.3. *For any endofunctor $*$ of $\mathcal{C}$ pointed by ι, if ι is pointwise monic and ι^* pointwise epic, then ι is pointwise epic.*

Proof. For an object A in $\mathcal{C}$ we write $\iota = \iota_A$, and with $a, b : A^* \to B$ we assume $a \cdot \iota = b \cdot \iota$. Then $a^* \cdot \iota^* = b^* \cdot \iota^*$, hence $a^* = b^*$ by hypothesis on ι^*. Now

$$\iota_B \cdot a = a^* \cdot \iota_{A^*} = b^* \cdot \iota_{A^*} = \iota_B \cdot b$$

implies $a = b$ by hypothesis on ι.

$$\begin{tikzcd}
A \arrow[r, "\iota"] \arrow[d, "\iota"'] & A^* \arrow[r, shift left, "a"] \arrow[r, shift right, "b"'] \arrow[d, "\iota_{A^*}"'] & B \arrow[d, "\iota_B"] \\
A^* \arrow[r, "\iota^*"] & A^{**} \arrow[r, shift left, "a^*"] \arrow[r, shift right, "b^*"'] & B^*
\end{tikzcd}$$

□

Examples 3.4. (1) The group $\mathbb{Q}/\mathbb{Z}$ is an injective cogenerator of the category **Ab** of abelian groups. For any ring R, the character module $R^+ = \mathrm{Hom}_{\mathbb{Z}}(R, \mathbb{Q}/\mathbb{Z})$ takes on this role in the category $\mathbf{Mod}_R$ of right R-modules. Hence, there are naturally enough injectives in these categories, a classical result due to [5]. More generally, every complete abelian category with a generator and enough injectives has an injective cogenerator and therefore naturally enough injectives; moreover, the hypothesis on enough injectives is redundant if in each subobject lattice directed joins distribute over finite meets, i. e., in any Grothendieck category with a generator (see, for example, [20]). Such categories have even injective hulls, a result that for $\mathbf{Mod}_R$ goes back to [9]; but unless every object is injective, they cannot be natural. Unital rings R for which every unital right R-module is injective have been characterized as the right Artinian non-zero rings R which contain no non-zero nilpotent ideal (see [10], p. 56).

(2) All varieties of algebras have naturally enough projectives, with the algebra with one free generator serving as a projective generator. In fact, more generally, let $\mathcal{C}$ be a regular epi-reflective subcategory of a monadic category over **Set** (i. e., an essentially algebraic category in the sense of [1]), then the object representing the underlying-set functor is a projective generator. Prop. 2.3 and Thm. 3.2 applied to $\mathcal{C}^{\mathrm{op}}$ with $\mathcal{H} = \mathrm{RegEpi}(\mathcal{C}) = \{\text{onto homomorphisms}\}$ show that there are naturally enough projectives, but that projective covers,

whenever they exist and are not trivial, fail to be natural. Non-trivial projective covers seem to be fairly rare though. For example, an abelian group has a projective cover in **Ab** only if it is projective, i.e. if it is free. Bass [8] characterized so-called right-perfect rings (those R for which every right R-module has a projective cover) as the rings satisfying the descending chain condition on principal left ideals (see also [11], [12]).

(3) Being dually equivalent to **Ab**, the category of compact abelian groups has naturally enough injectives, but fails to have injective hulls ([6]). Similarly, the category **CompHaus** of compact Hausdorff spaces has naturally enough injectives (the closed unit interval beings a cogenerator), but the only space with a non-trivial essential extension is the empty space. Dually, the category **CC*Alg** of commutative C^*-algebras has naturally enough projectives, but only the trivial algebra has a non-trivial projective cover. The projective objects in **CompHaus** are extremally disconnected compact Hausdorff spaces (i.e., the retracts of Čech-Stone-compactifications of discrete spaces), and the coessential quotient morphisms are the irreducible quotient morphisms (i.e., those continuous surjections $X \to Y$ that map no proper closed subset of X onto Y). Projective covers (normally called resolutions) exist in **CompHaus** but are not natural, and the same is true for its full subcategory of Boolean spaces (i.e., zero-dimensional compact Hausdorff spaces). The dual statements are therefore valid for **CC*Alg**, and for the category of Boolean algebras, where injective means complete (see also (5) below).

(4) In the category $\mathbf{Top}_0$ of T0-spaces with its injective (Sierpiński) cogenerator, there are naturally enough injectives. Although each object has a unique maximal essential extension, these are generally not injective ([7]); in fact, a T0-space has an injective hull if and only if its lattice of open sets is completely distributive ([15]). For extensions of these facts to **Top**, see [25], [15]. In $\mathbf{Top}_1$, $\mathbf{Top}_2 = \mathbf{Haus}$ and $\mathbf{Top}_{3\frac{1}{2}} = \mathbf{Tych}$, the only injective objects have exactly one point (see [14]). The situation changes drastically if we restrict the class of embeddings to $\mathcal{H} = \{\text{epimorphic embeddings}\}$; then $\mathcal{H}$-injective hulls exist in $\mathbf{Top}_0$ and are given by the sober reflection (see [22]) and, hence, are natural.

(5) In the category **Pos** of partially ordered sets, with $\mathcal{H} = \{\text{order embeddings}\}$, the two-element chain is an injective cogenerator, and the injective objects are precisely the complete lattices. Injective hulls are given by MacNeille completion but are not natural. Similarly, in the category **Met** of metric spaces, with non-expanding maps and $\mathcal{H} = \{\text{isometric embeddings}\}$, injective hulls exist and are given by the hyperconvex envelope (see [16]) but are not natural.

We refer to [17], [3], [14] as sources for further examples and references to such.

Remark 3.5. In generalization of results known for abelian categories (see 3.4(1)), for our general category $\mathcal{C}$ and without recourse to an $\mathcal{H}$-injective $\mathcal{H}$-cogenerator, sufficient conditions for the existence of enough $\mathcal{H}$-injectives and for the existence of $\mathcal{H}$-injective hulls were developed in [6] and extendend in

[24]. Here is a set of sufficient conditions which is often applicable when $\mathcal{H}$ is a class of "embeddings" (for details see [24]):

1. $\mathcal{H}$ is part of a proper orthogonal $(\mathcal{E}, \mathcal{H})$-factorization system for morphisms in $\mathcal{C}$, such that $\mathcal{C}$ is $\mathcal{E}$-cowellpowered and $\mathcal{H}$-wellpowered;
2. for every object A in $\mathcal{C}$, every small well-ordered diagram in the full subcategory $\mathcal{H}_A$ of the comma category $A \backslash \mathcal{C}$ given by the morphisms in $\mathcal{H}$ with domain A admits a cocone in $\mathcal{H}_A$; this is referred to as the "$\mathcal{H}$-chain condition" in [24].

Then the following conditions are equivalent:

i. *$\mathcal{C}$ has $\mathcal{H}$-injective hulls,*

ii. *$\mathcal{C}$ has enough $\mathcal{H}$-injectives,*

iii. *$\mathcal{C}$ is $\mathcal{H}^*$-cowellpowered and has the $\mathcal{H}$-transferability property* (so that for all morphisms f, h with common domain and $h \in \mathcal{H}$ there are f', h' with $h' \cdot f = f' \cdot h$ and $h' \in \mathcal{H}$).

Furthermore, it is shown in [24] that in the presence of a "well-behaved" separating set in $\mathcal{C}$ and for $\mathcal{H} = \mathrm{Mono}(\mathcal{C})$, conditions i–iii are also equivalent to:

iv. *$\mathcal{C}$ has a coseparating set of injective objects.*

4 Extensions of the naturality theme

Problems 4.1. The functoriality/naturality problem may be considered more generally, as follows: for a functor $U : \mathcal{A} \to \mathcal{C}$ and for every object X in $\mathcal{C}$, we are given an object FX in $\mathcal{A}$ and a morphism $\eta_X : X \to UFX$ (normally in a class $\mathcal{H}$ of morphisms in $\mathcal{C}$). Then the following questions arise:

I (*Natural functoriality of F*) Can F be made a functor $F : \mathcal{C} \to \mathcal{A}$ such that $\eta : \mathrm{Id}_{\mathcal{C}} \to UF$ becomes a natural transformation?

II (*Unnatural functoriality of F*) If the answer to I is negative, can F be made a functor (no condition on η)?

In Sections 2 and 3 we dealt with the situation when $U : \mathcal{H}\text{-}\mathbf{Inj} \hookrightarrow \mathcal{C}$ is the inclusion functor. It seems natural to consider more generally, in lieu of $\mathcal{H}$-**Inj**, any weakly $\mathcal{H}$-reflective subcategory $\mathcal{A}$ of $\mathcal{C}$, or even an arbitrary weakly right adjoint functor $U : \mathcal{A} \to \mathcal{C}$, so that for every object X in $\mathcal{C}$ the given morphism η_X in $\mathcal{H}$ is weakly universal, that is: any morphism $f : X \to UA$ with $A \in \mathcal{A}$ factors as $f = Ug \cdot \eta_X$ for some $g : FX \to A$ in $\mathcal{A}$.

We limit ourselves here to pointing to three interesting examples treated in the literature.

Examples 4.2. (1) Suppose that an embedding of a set X into an injective (= non-empty, see 3.1(1)) set FX has been given, with $FX = X$ for each $X \neq \emptyset$. As we saw, Question I then has a negative answer, but so does II: it is shown in [21] (and it follows also from a result in [18]) that there is no functor $F : \mathbf{Set} \to \mathbf{Inj}$ with $FX = X$ for each non-empty set X. There is in fact no functor $F : \mathbf{Set} \to \mathbf{Inj}$ with $|Fn| = n$ for $n = 1, 2, 3$. However, there exists a functor $F : \mathbf{Set} \to \mathbf{Inj}$ with $|F1| = 1$ and $|F2| = 2$: just put

$$FX = 1 + \{P \subseteq X \mid |P| = 2\},$$

and for $f : X \to Y$ let Ff map P to $f(P)$ if $|f(P)| = 2$; everything else gets mapped to the only point of 1.

(2) Let $\mathbf{ACP}_p$ be the category of algebraically-closed fields of characteristic p (p prime or 0), let $\mathbf{Set}_{\mathrm{mono}}$ be the category of sets and their monomorphisms, and let $U : \mathbf{ACP}_p \to \mathbf{Set}_{\mathrm{mono}}$ be the forgetful functor. For each set X, let $\eta_X : X \to FX$ be the embedding of X into an algebraically-closed field FX of characteristic p with transcendency basis X. It is shown in [4] that the answer to problem I is negative (which also implies the negative result of 3.1(2)), but that II has a positive answer.

(3) In [23] Problem I is discussed in conjunction with the Lefschets-Nöbeling-Pontryagin Theorem which says that every n-dimensional compact metric space can be homeomorphically embedded into a $(2n+1)$-dimensional separable metrizable group, namely into $\mathbb{R}^{2n+1}$. Hence let $U : \mathbf{HausGrp}_{<\infty} \to \mathbf{Top}$ be the forgetful functor of finite-dimensional Hausdorff groups into the category of topological spaces. The question then is whether for a suitable full subcategory $\mathcal{C}$ of $\mathbf{Top}$, there are a functor $F : \mathcal{C} \to \mathbf{HausGrp}_{<\infty}$ and a natural transformation $\eta : J \to UF$ (where $J : \mathcal{C} \hookrightarrow \mathbf{Top}$ is the inclusion functor) such that every η_X is an embedding. Shakhmatov shows that

- for $\mathcal{C}$ the category of 0-dimensional compact metrizable spaces, the answer is positive, while
- for $\mathcal{C}$ the category of 1-dimensonal compact metrizable spaces the answer is negative.

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