# Injectivity and accessible categories

Jiří Rosický
Masaryk University, Department of Mathematics
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
rosicky@math.muni.cz

Since its creation by S. Eilenberg and S. MacLane [EM], category theory has brought a number of important concepts. Accessible categories are among them and we are going to show how they can help to treat injectivity in algebra, model theory and homotopy theory.

## 1 Three situation

**1.1 Injective modules.** Injective modules were introduced by R. Baer [B]. A left $R$-module $M$ is called *injective* if for each injective homomorphism $f : A \to B$ and each homomorphism $g : A \to M$ there is a homomorphism $h : B \to M$ such that $h \cdot f = g$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

The category $R$-**Mod** of left $R$-modules *has enough injectivities*, which means that for every $R$-module $A$ there is an injective homomorphism $A \to M$ with $M$ injective. This was also proved by Baer [B] using his criterion for injectivity.

**Baer's Criterion.** *A left $R$-module $M$ is injective iff for every left ideal $A$ of $R$, every homomorphism $A \to R$ can be extended to a homomorphism $R \to M$.*

One can learn about injective modules and their use in any monograph about module theory (see, e.g., [F]).

**1.2 Saturated models.** Let $T$ be a first-order theory of a countable signature $\Sigma$. Let $\mathbf{Mod}(T)$ be the category of models of the theory $T$ with elementary embeddings as morphisms. For an uncountable regular cardinal $\lambda$, a $T$-model $M$ is called $\lambda$-*saturated* if for each elementary embedding $f : A \to B$ with $\operatorname{card} A$, $\operatorname{card} B < \lambda$ and each elementary embedding $g : A \to M$ there is an elementary embedding $h : B \to M$ with $h \cdot f = g$.

$$\begin{tikzcd} A \arrow[rr, "f"] \arrow[dr, "g"'] & & B \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

We have not used the original definition of $\lambda$-saturated models (due to Morley and Vaught [MV]) but the characterization given in [S] 16.6. The category $\mathbf{Mod}(T)$ has enough $\lambda$-saturated models in the sense that each $T$-model has an elementary embedding into a $\lambda$-saturated model.

**1.3 Kan fibrations.** The category $\mathbf{SSet}$ of simplicial sets is defined as the functor category $\mathbf{Set}^{\mathbf{\Delta}^{\mathrm{op}}}$ where $\mathbf{\Delta}$ is the category of non-zero finite ordinals and order-preserving maps. The simplicial sets $\Delta^n$, $n \geq 0$ are defined as $\Delta^n = Y(n+1)$ where $Y : \mathbf{\Delta} \to \mathbf{SSet}$ is the Yoneda embedding. The simplicial subsets $\Delta^n_k \subseteq \Delta^n$, $n \geq 0$, $0 \leq k \leq n$ are obtained by excluding the identity morphism $\Delta^n \to \Delta^n$ and the morphism $\Delta^{n-1} \to \Delta^n$ given by the injective order-preserving map $n \to n+1$ whose image does not contain $k$. A morphism $p : M \to N$ of simplicial sets is called a *Kan fibration* if it has the *right lifting property* w.r.t. each embedding $i^n_k : \Delta^n_k \to \Delta^n$, $n \geq 0$, $0 \leq k \leq n$. It means that for every commutative square

$$\begin{tikzcd} \Delta^n_k \arrow[r, "g"] \arrow[d, "i^n_k"'] & M \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"'] & N \end{tikzcd}$$

there exists a diagonal

$$\begin{tikzcd} \Delta^n_k \arrow[r, "g"] \arrow[d, "i^n_k"'] & M \arrow[d, "p"] \\ \Delta^n \arrow[r, "v"'] \arrow[ur, dashed, "h"] & N \end{tikzcd}$$

making both triangles commutative.

If $N = \Delta^0$ then the unique morphism $p : M \to \Delta^0$ ($\Delta^0$ is a terminal object in $\mathbf{SSet}$) is a Kan fibration iff for each $i^n_k$, $n \geq 0$, $0 \leq k \leq n$ and for each morphism $g : \Delta^n_k \to M$ there is a morphism $h : \Delta^n \to M$ with $h \cdot i^n_k = g$

$$\begin{tikzcd} \Delta^n_k \arrow[rr, "i^n_k"] \arrow[dr, "g"'] & & \Delta^n \arrow[dl, dashed, "h"] \\ & M & \end{tikzcd}$$

Such simplicial sets $M$ are called *Kan complexes*. **SSet** has enough Kan complexes in the sense that each simplicial set $A$ has an embedding $f : A \to B$ into a Kan complex. Moreover, this embedding $f$ is an *anodyne extension*, which is defined by having the *left lifting property* w.r.t. each Kan fibration $p$. It means that for every commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & M \arrow[d, "p"] \\ B \arrow[r, "v"'] & N \end{tikzcd}$$

there exists a diagonal $h$ making both triangles commutative. Of course, every embedding $\Delta^n_k \to \Delta^n$ is an anodyne extension. The just explained property of having enough Kan complexes can be equivalently formulated in the way that each morphism $A \to \Delta^0$ has a factorization

$$A \xrightarrow{\ f\ } B \xrightarrow{\ p\ } \Delta^0$$

where $f$ is an anodyne extension and $p$ a Kan fibrations. More generally, every morphism $A \to N$ of simplicial sets has a factorization

$$A \xrightarrow{\ f\ } B \xrightarrow{\ p\ } N$$

where $f$ is an anodyne extension and $p$ a Kan fibration (see, e.g. [GJ]). Kan fibrations were introduced D. M. Kan [K].

# 2 Accessible categories

An object $K$ of a category $\mathcal{K}$ is called $\lambda$-*presentable*, where $\lambda$ is a regular cardinal, provided that its hom-functor $\hom(K, -)$ preserves $\lambda$-directed colimits. A category $\mathcal{K}$ is called it $\lambda$-accessible provided that

(1) $\mathcal{K}$ has $\lambda$-directed colimits,

(2) $\mathcal{K}$ has a set $\mathcal{A}$ of $\lambda$-presentable objects such that every object is a $\lambda$-directed colimit of objects of $\mathcal{A}$.

A category is called *accessible* if it is $\lambda$-accessible for some regular cardinal $\lambda$. Accessible categories were introduced by C. Lair [L] and their theory was created by M. Makkai and R. Paré [MP]. We will use the monograph [AR]. The first steps towards the theory of accessible categories were made by M. Artin, A. Grothendieck and J. L. Verdier [AGV] and especially by P. Gabriel and F. Ulmer [GU].

**2.1 Examples.** (1) The category $R$-**Mod** is $\aleph_0$-accessible for every ring $R$. It has all colimits and $\aleph_0$-presentable objects are finitely presentable $R$-modules in the usual module-theoretic sense. Every $R$-module is a directed colimit of finitely presentable modules. The same argument applies to every variety of universal algebras.

(2) The category $\mathbf{Mod}(T)$ is $\aleph_1$-accessible for every first-order theory $T$ of a countable signature. It has directed colimits (see [AR] 5.39) and $\aleph_1$-presentable objects are $T$-models having countably many elements. Every $T$-model is an $\aleph_1$-directed colimit of countable $T$-models. This can be found in [AR] 5.42 but it is an immediate consequence of the downward Löwenheim-Skolem theorem.

(3) The category **SSet** is $\aleph_0$-accessible. It has all colimits and $\aleph_0$-presentable objects are finite colimits of simplicial sets $\Delta^n$, $n \geq 0$. Every simplicial set is a directed colimit of finite colimits of $\Delta^n$, $n \geq 0$. The same argument applies to every functor category $\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$ where $\mathcal{X}$ is a small category.

(4) Let $N$ be a simplicial set and consider the comma-category $\mathbf{SSet} \downarrow N$. Objects of this category are morphisms $p : A \to N$ of simplicial sets. Morphisms $(A, p) \to (B, q)$ are morphisms $f : A \to B$ of simplicial sets with $q \cdot f = p$

$$\begin{tikzcd} A \arrow[d, "f"'] \arrow[dr, "p"] & \\ B \arrow[r, "q"'] & N \end{tikzcd}$$

Then $\mathbf{SSet} \downarrow N$ is an $\aleph_0$-accessible category. It has all colimits and $\aleph_0$-presentable objects are $f : A \to N$ with $A$ $\aleph_0$-presentable in **SSet**. Every object in $\mathbf{SSet} \downarrow N$ is a directed colimit of $\aleph_0$-presentable objects (see [AR] 1.57).

Let $\mathcal{H}$ be a class of morphisms in a category $\mathscr{C}$. An object $M$ in $\mathscr{C}$ is called *$\mathcal{H}$-injective* if for each morphism $f : A \to B$ in $\mathcal{H}$ and each morphism $g : A \to M$ there is a morphism $h : B \to M$ such that $h \cdot f = g$.

**2.2 Examples.** (1) Injective $R$-modules are $\mathcal{H}$-injective objects in $R$-**Mod** for $\mathcal{H}$ consisting of all monomorphisms.

(2) $\lambda$-saturated models are $\mathcal{H}$-injective objects in $\mathbf{Mod}(T)$ for $\mathcal{H}$ consisting of morphisms $f : A \to B$ with $\operatorname{card} A$, $\operatorname{card} B < \lambda$. We recall that these objects are precisely $\lambda$-presentable objects.

(3) Kan complexes are $\mathcal{H}$-injective objects in **SSet** for $\mathcal{H}$ consisting of anodyne extensions. In fact, we defined them as being injective w.r.t. embeddings $\Delta^n_k \to \Delta^n$, $n \geq 0$, $0 \leq k \leq n$ but it immediatelly follows from the definition that they are injective w.r.t. every anodyne extension.

(4) Let $N$ be a simplicial set and consider the comma-category $\mathbf{SSet} \downarrow N$. Kan fibrations $p : M \to N$ are $\mathcal{H}$-injective objects for $\mathcal{H}$ consisting of morphisms $(A, a) \to (B, b)$ carried by anodyne extensions $f : A \to B$. In fact the defining

property of a Kan fibration exactly means that

$$\begin{tikzcd}
(A,pu) \arrow[rr, "f"] \arrow[dr, "u"'] & & (B,v) \arrow[dl, dashed, "h"] \\
& (M,p) &
\end{tikzcd}$$

An accessible category does not need to have all colimits (see, for example, 2.1 (2)). We say that a diagram $D : \mathcal{D} \to \mathcal{K}$ has a *bound* in a category $\mathcal{K}$ if there is a compatible cocone $(Dd \xrightarrow{C_d} C)_{d \in \mathcal{D}^{\mathrm{obj}}}$ in $\mathcal{K}$. We say that $\mathcal{K}$ has *directed bounds* if every directed diagram has a bound in $\mathcal{K}$ and that $\mathcal{K}$ has *pushout bounds* if every diagram

$$\begin{tikzcd}
A \arrow[r] \arrow[d] & B \\
C &
\end{tikzcd}$$

has a bound in $\mathcal{K}$.

**2.3 Theorem.** *Let $\mathcal{K}$ be an accessible category with directed and pushout bounds and $\mathcal{H}$ a set of morphisms in $\mathcal{K}$. Then every object $K$ in $\mathcal{K}$ has a morphism $K \to M$ into an $\mathcal{H}$-injective object $L$.*

*Proof.* Following [AR] 2.14 and 2.2 (3), there is a regular cardinal $\lambda$ such that $\mathcal{K}$ is $\lambda$-accessible and every morphism in $\mathcal{H}$ has a $\lambda$-presentable domain. Consider an object $K$ in $\mathcal{K}$. Let $\mathcal{X}_K$ be the set of all spans

$$\begin{tikzcd}
K & \\
C \arrow[u, "u"] \arrow[r, "g"'] & D
\end{tikzcd}$$

with $g \in \mathcal{H}$. We will index these spans by ordinals $i < \mu_K = \operatorname{card} \mathcal{X}_K$. We define a chain $k_{ij} : K_i \to K_j$, $i \le j \le \mu_K$ by the following transfinite induction:

First step: $K_0 = K$.

Isolated step: $K_{i+1}$ is given by a pushout bound

$$\begin{tikzcd}
K_i \arrow[r, "k_{i,i+1}"] & K_{i+1} \\
C_i \arrow[u, "k_{0i} \cdot u_i"] \arrow[r, "g_i"'] & D_i \arrow[u]
\end{tikzcd}$$

where $k_{0,i+1} = k_{i,i+1} \cdot k_{0i}$.

Limit step: $K_i$ is a bound of the chain

$$K_0 \xrightarrow{k_{01}} K_1 \xrightarrow{k_{12}} \dots K_j \xrightarrow{k_{j,j+1}} \dots$$

where $j < i$ and $k_{0i} : K_0 \to K_i$ is given by this bound.

The object $K_{\mu_K}$ will be denoted by $K^*$ and the morphism $K_{0\mu_k} : K \to K^*$ by $t_K$. Following the construction, each span $(u_i, g_i) \in \mathcal{X}_K$ has a pushout bound

$$\begin{tikzcd} K \arrow[r, "t_K"] & K^* \\ C_i \arrow[u, "u_i"] \arrow[r, "g_i"'] & D_i \arrow[u] \end{tikzcd}$$

We define a chain $m_{ij} : M_i \to M_j$, $i \le j \le \lambda$ by the following transfinite induction:

First step: $M_0 = K$.

Isolated step: $m_{i,i+1} : M_i \to M_{i+1}$ is $t_{M_i} : M_i \to M_i^*$.

Limit step: $M_i$ is a directed bound of the chain

$$M_0 \xrightarrow{m_{01}} M_1 \xrightarrow{m_{12}} \dots M_j \xrightarrow{m_{j,j+1}} \dots \tag{1}$$

for $j < i < \lambda$ and $M_\lambda$ is a colimit of (1) for $i = \lambda$.

We will show that $m_{0\lambda} : K \to M_\lambda$ is a desired morphism of $K$ into an $\mathcal{H}$-injective object. Consider a span

$$\begin{tikzcd} M_\lambda & \\ C \arrow[u, "u"] \arrow[r, "g"'] & D \end{tikzcd}$$

Since the object $C$ is $\lambda$-presentable and $M_\lambda$ is a directed colimit of $M_i$, $i < \lambda$, there is a factorization

$$\begin{tikzcd} M_\lambda & \\ C \arrow[u, "u"] \arrow[r, "u'"'] & M_i \arrow[ul, "m_{i\lambda}"'] \end{tikzcd}$$

of $u$ through $M_i$ for some $i < \lambda$. Since the span

$$\begin{tikzcd} M_i & \\ C \arrow[u, "u'"] \arrow[r, "g"'] & D \end{tikzcd}$$

is in the set $\mathcal{X}_{M_i}$, it has a pushout bound

$$\begin{tikzcd} M_i \arrow[r, "m_{i,i+1}"] & M_{i+1} \\ C \arrow[u, "u'"] \arrow[r, "g"'] & D \arrow[u, "v"'] \end{tikzcd}$$

We have

$$u = m_{i\lambda} \cdot u' = m_{i+1,\lambda} \cdot m_{i,i+1} \cdot u' = m_{i+1,\lambda} \cdot v \cdot g \,.$$

Hence $u$ factorizes through $g$, which proves that $M_\lambda$ is $\mathcal{H}$-injective. $\square$

**2.4 Examples.** (1) The category $R$-**Mod** is $\aleph_0$-accessible and has all colimits. Let $\mathcal{H}$ be the set of all embeddings $A \to R$ where $A$ is a left ideal in $R$. Following Baer's Criterion $\mathcal{H}$-injective modules are precisely injective modules. Following Theorem 2.3 every $R$-module has a homomorphism into an injective R-module.

To prove that $R$-**Mod** has enough injectives, we have to replace the category $R$-**Mod** by the category $R$-$\mathbf{Mod}_0$ of $R$-modules and injective homomorphisms taken as morphisms. Following [AR] 2.3 (6), $R$-$\mathbf{Mod}_0$ is an accessible category. It has directed colimits (by [AR] 1.62) and pushouts because monomorphisms in $R$-**Mod** are stable under pushouts. Hence, by applying Theorem 2.3, to the category $R$-$\mathbf{Mod}_0$, we get that $R$-**Mod** has enough injectives.

(2) Let $T$ be a first-order theory of a countable signature and $\lambda$ an uncountable regular cardinal. The category $\mathbf{Mod}(T)$ has pushout bounds (see [H], p. 288). Hence Theorem 2.3 together with Example 2.1 (2) implies that every $T$-model has an elementary embedding into a $\lambda$-saturated $T$-model. Of course, we take for $\mathcal{H}$ the set of all elementary embedding $A \to B$ with $\operatorname{card} A$, $\operatorname{card} B < \lambda$.

(3) The category **SSet** is $\aleph_0$-accessible and has all colimits. Let $\mathcal{H}$ consist of embeddings $\Delta_k^n \to \Delta^n$, $n \geq 0$, $0 \leq k \leq n$. Following Theorem 2.3, every simplicial set $A$ has a morphism $m : A \to M$ into a Kan complex $M$.

Since **SSet** is cocomplete, we can use colimits instead of bounds in the proof of Theorem 2.3. Hence $m$ belongs to the closure of $\mathcal{H}$ under pushouts, compositions and colimits of chains. Every morphism of this closure belongs to $\Box(\mathcal{H}^\Box)$ where the box on the right (left) means the use of the right (left) lifting property. Hence $m$ is an anodyne extension.

More generally, by applying Theorem 2.3 to the category **SSet** $\downarrow N$ (for $\mathcal{H}$ consisting of morphism carried by embeddings $\Delta^n_k \to \Delta^n$, $n \geq 0$, $0 \leq k \leq n$), we get that each morphism $A \to N$ has a factorization

$$A \xrightarrow{\ f\ } B \xrightarrow{\ p\ } N$$

where $f$ is an anodyne extension and $p$ a Kan fibration.

The last example gives the essence of essence of the *small object argument* already present in [GZ]. This argument is commonly used in homotopy theory (see [Ho]) but the theory of accessible categories has started to be used in homotopy theory only recently (see T. Beke [B]). Our Theorem 2.3 is a very general formulation of the small object argument. The point is that every object of an accessible category is presentable (= small), which makes possible to stop the construction of an $\mathcal{H}$-injective object $M$ for $K$. The next example shows that it is necessary to assume that $\mathcal{H}$ is a set.

**2.5 Example.** Let **Gr** be the category of groups and $\mathcal{H}$ the class of all injective homomorphisms. Every group $K$ is a subgroup of a simple group $L \neq K$ (see [Sc]). If $K$ is $\mathcal{H}$-injective, the embedding $f : K \to L$ splits, i.e., there exists $g : L \to K$ with $g \cdot f = \mathrm{id}_K$; by applying $\mathcal{H}$-injectivity to

$$\begin{tikzcd}
K \arrow[rr, "\mathrm{id}_K"] \arrow[dr, "f"'] & & K \\
& L \arrow[ur, dashed, "g"'] &
\end{tikzcd}$$

Since $L$ is simple and $L \not\cong K$, the homomorphism $g$ has to be constant, i.e., $K = \{1\}$. Therefore the trivial group $\{1\}$ is the only injective (= $\mathcal{H}$-injective) group. Hence the category of groups does not have enough injectives. On the other hand, the category $\mathbf{Gr}_0$ of groups and injective homomorphisms is accessible (following the same reasons as the category $R$-$\mathbf{Mod}_0$) and the only obstacle to apply Theorem 2.3 is that $\mathcal{H}$ is not a set.

# References


[AR] J. Adámek and J. Rosický, Locally presentable and accessible categories, Cambridge University Press 1994.

[AGV] M. Artin, A. Grothendieck and J. L. Verdier, Théorie des topos et cohomologie étale des schémas, Lect. Notes in Math. 269, Springer-Verlag 1972.

[B] R. Baer, Abelian groups that are direct summands of every containing abelian group, *Bull. Amer. Math. Soc.* 46 (1940), 800-806.

[EM] S. Eilenberg and S. MacLane, General theory of natural equivalences, *Trans. Amer. Math. Soc.* 58 (1945), 231-294.

[F] C. Faith, Algebra: Rings, Modules and Categories I, Springer-Verlag 1973.

[GU] P. Gabriel and F. Ulmer, Lokal Präsentierbare Kategorien, Lect. Notes in Math. 221, Springer-Verlag 1971.

[GZ] P. Gabriel and M. Zisman, Calculus of fractions and homotopy theory, Ergebnisse in Math. and Grenzgebiete 35, Springer-Verlag 1967.

[GJ] P. G. Goerss and J. F. Jardine, Simplicial homotopy theory, Birkhäuser 1999.

[H] W. Hodges, Model theory, Cambridge University Press 1993.

[Ho] M. Hovey, Model categories, Amer. Math. Soc. 1999.

[K] D. M. Kan, On c.s.s. complexes, *Amer. J. Math.* 79 (1957), 449-476.

[L] C. Lair, Catégories modélables et catégories esquissables, *Diagrammes* 6 (1981), 1-20.

[MP] M. Makkai and R. Paré, Accessible categories: the foundations of categorical model theory, *Contemp. Math.*, 104, Amer. Math. Soc. 1989.

[MV] M. Morley and R. Vaught, Homogenenous universal models, *Math. Scand.* 11 (1962), 37-57.

[S] G. E. Sacks, Saturated model theory, Benjamin 1972.

[Sc] W. R. Scott, Group theory, Prentice Hall 1964.