Weak factorization systems and topological functors

Jiří Adámek[1], Horst Herrlich, Jiří Rosický[2], and Walter Tholen[3]

Affectionately dedicated to Heinrich Kleisli who gave us the Kleisli–construction — and many other things

Abstract

Weak factorization systems, important in homotopy theory, are related to injective objects in comma–categories. Our main result is that full functors and topological functors form a weak factorization system in the category of small categories, and that this is not cofibrantly generated. We also present a weak factorization system on the category of posets which is not cofibrantly generated. No such weak factorization systems were known until recently. This answers an open problem posed by M. Hovey.

Mathematics Subject Classifications (2000): 18A22, 18A32, 18G05, 18B35.

Key words: Weak factorization systems, full functor, topological functor, cofibrantly generated, poset.

Introduction

Whereas factorization systems for morphisms in categories are one of the most studied categorical concepts, weak factorization systems have been rather neglected, although they play an important role in homotopy theory. Part of the reason may be that the basic examples, such as $(Mono, Epi)$ in the category of sets, are rather surprising for anybody accustomed to factorization systems proper. An explanation of such examples is that they are closely connected to the existence of enough injectives in comma–categories.

We analyze this relationship between weak factorization systems and injectives in comma–categories in greater detail and we provide new examples of weak factorization systems. In particular,

[1]Supported by the Grant Agency of the Czech Republic, Grant No. 201/99/0310.

[2]Supported by the Ministry of Education of the Czech Republic under the project MSM 143100009

[3]Partial financial support by NSERC is acknowledged

(a) we show that (full functors, topological functors) form a weak factorization system in the category **Cat** of small categories, and
(b) we describe a weak factorization system $(\mathcal{L}, \mathcal{R})$ in the category **Pos** of posets, where $\mathcal{L}$ is the class of embeddings (= regular monos).
These weak factorization systems are not cofibrantly generated.

1 Weak Factorization Systems

The concept of weak factorization systems plays a central role in homotopy theory, in particular in the basic definition of Quillen model categories (see 3.6 below). Formally, this notion generalizes factorization systems by weakening the unique–diagonalization property to the diagonalization property without uniqueness. However, the basic examples of weak factorization systems are fundamentally different from the basic examples of factorization systems.

1.1 Notation We denote by $\Box$ the relation *diagonalization property* on the class of all morphisms of a category: given morphisms $l\colon A \to B$ and $r\colon C \to D$ then

$$l \,\Box\, r$$

means that in every commutative square

$$\begin{tikzcd} A \arrow[r] \arrow[d, "l"'] & C \arrow[d, "r"] \\ B \arrow[r] \arrow[ur, dashed, "d"] & D \end{tikzcd}$$

there exists a diagonal $d\colon B \longrightarrow C$ rendering both triangles commutative. In this case l is also said to have the *left lifting property* with respect to r (and r to have the *right lifting property* with respect to l).

Let $\mathcal{H}$ be a class of morphisms. We denote by

$$\mathcal{H}^{\Box} \quad \text{(right box of } \mathcal{H}\text{)}$$

the class of all morphisms r with

$$h \,\Box\, r \quad \text{for all } h \in \mathcal{H}$$

and, dually, by

$$^{\Box}\mathcal{H} \quad \text{(left box of } \mathcal{H}\text{)}$$

the class of all morphisms l with

$$l \,\Box\, h \quad \text{for all } h \in \mathcal{H}.$$

1.2 Definition (see, e.g., [6] and [8]) A *weak factorization system* in a category is a pair $(\mathcal{L}, \mathcal{R})$ of morphism classes such that

1. every morphism has a factorization as an $\mathcal{L}$–morphism followed by an $\mathcal{R}$–morphism

and

2. $\mathcal{R} = \mathcal{L}^{\square}$ and $\mathcal{L} = {}^{\square}\mathcal{R}$.

1.3 Observation (see [1]) In the presence of the above condition (1), the above condition (2) is equivalent to the conjunction of the following two (this is called the "retract argument"; cf. [16] and [18]):

(2a) $\mathcal{L} \,\square\, \mathcal{R}$, i.e., $l \square r$ for all $l \in \mathcal{L}$ and $r \in \mathcal{R}$.

and

(2b) as full subcategories of the arrow–category $\mathbf{K}^{\rightarrow}$, both $\mathcal{L}$ and $\mathcal{R}$ are closed under the formation of retracts.

Moreover, in the presence of conditions (1) and (2a), condition (2b) can be replaced by the (formally weaker) condition

(2b′) (α) $s \circ f \in \mathcal{L} \Rightarrow f \in \mathcal{L}$
for any section (= split monomorphism) s, and

(β) $f \circ r \in \mathcal{R} \Rightarrow f \in \mathcal{R}$
for any retraction (= split epimorphism) r.

[To see that (2b′)(α) implies ${}^{\square}\mathcal{R} \subset \mathcal{L}$, consider $f \in {}^{\square}\mathcal{R}$. Let $f = r \circ l$ be a $(\mathcal{L}, \mathcal{R})$–factorization, and let s be a diagonal for the commutative square

$$\begin{tikzcd} \bullet \arrow[r, "l"] \arrow[d, "f"'] & \bullet \arrow[d, "r"] \\ \bullet \arrow[r, "\mathrm{id}"'] \arrow[ur, dotted, "s"] & \bullet \end{tikzcd}$$

Then $s \circ f = l \in \mathcal{L}$ and s is a section. Thus, by (2b′)(α), $f \in \mathcal{L}$.
Dually, (2b′) (β) implies $\mathcal{L}^{\square} \subset \mathcal{R}$.]

If we replace in (2) "$\square$" by "$\perp$", where $\perp$ is defined via the *unique diagonalization property* (i.e., by insisting that there exists precisely one diagonal), we arrive at the familiar notion of a factorization system in a category (cf. [2, 14.1]). Note that in this case we may replace "retracts" by "isomorphisms" in 2 (b). Factorization systems are weak factorization systems, see [2, 14.6 (3)].

However, in view of the principal examples it would be misleading to think of weak factorization systems merely as generalized factorization systems. If $(Epi, Mono)$ in Set is the "prototype of all factorization systems", then $(Mono, Epi)$ in Set is the "prototype of all weak factorization systems" (cf. [1, III.5 (3)]).

Recall that, for any class $\mathcal{H}$ of morphisms in a category $\mathbf{K}$ an object K is $\mathcal{H}$–*injective* if for each $h\colon X \longrightarrow Y$ in $\mathcal{H}$ and each morphism $f\colon X \longrightarrow K$ there exists a morphism $g\colon Y \longrightarrow K$ with $g \circ h = f$

$$\begin{tikzcd} X \arrow[rr, "h\in\mathcal{H}"] \arrow[dr, "f"'] & & Y \arrow[dl, dotted, "g"] \\ & K & \end{tikzcd}$$

Equivalently, in case $\mathbf{K}$ has a terminal object 1, K is $\mathcal{H}$–injective iff the unique morphism $K \longrightarrow 1$ belongs to $\mathcal{H}^\square$. More generally:

1.4 Remark A morphism $g\colon C \longrightarrow D$ belongs to $\mathcal{H}^\square$ iff the object (C, g) of $\mathbf{K}/D$ is $\mathcal{H}_D$–injective, where $\mathcal{H}_D$ denotes the class of all $\mathbf{K}/D$–morphisms whose underlying $\mathbf{K}$–morphism belongs to $\mathcal{H}$ (see [16, 12.4.2]).We also refer to [12] in the dual situation.

Let us denote by $\mathcal{H}$–$\mathbf{Inj}$ the full subcategory of $\mathcal{H}$–injective objects in $\mathbf{K}$. $\mathbf{K}$ is said *to have enough $\mathcal{H}$–injectives* if for every object A of $\mathbf{K}$ there is a morphism $A \longrightarrow C$ in $\mathcal{H}$ with $C \in \mathcal{H}$–$\mathbf{Inj}$.

1.5 Lemma *Let $\mathcal{H}$ be a class of $\mathbf{K}$–morphisms closed under retracts in $\mathbf{K}^\rightarrow$. Then the following conditions are equivalent:*

1. *$(\mathcal{H}, \mathcal{H}^\square)$ is a weak factorization system;*
2. *for all objects B of $\mathbf{K}$, $\mathbf{K}/B$ has enough $\mathcal{H}_B$–injectives.*

The proof follows immediately from the characterization of weak factorization systems via (1), (2a), and (2b).

The lemma simplifies if $\mathcal{H}$ is *left cancellable*, i.e., if $g \circ f \in \mathcal{H}$ implies $f \in \mathcal{H}$:

1.6 Proposition *Let $\mathbf{K}$ be a category with finite products and $\mathcal{H}$ a left cancellable class of morphisms containing all isomorphisms. Then the following conditions are equivalent:*

1. *$(\mathcal{H}, \mathcal{H}^\square)$ is a weak factorization system;*

2. **K** *has enough $\mathcal{H}$–injectives.*

Proof: Assume 2. and consider a morphism $f\colon A \longrightarrow B$. Let $h\colon A \longrightarrow C$ be an $\mathcal{H}$–morphism with $\mathcal{H}$–injective codomain C. Then

$$A \xrightarrow{f} B \;=\; A \xrightarrow{\langle h,f\rangle} C \times B \xrightarrow{\pi_2} B$$

is the desired factorization of f. In fact, $\langle h, f\rangle \in \mathcal{H}$, since $\pi_1 \circ \langle h, f\rangle = h \in \mathcal{H}$. Moreover, $\pi_2 \in \mathcal{H}^\Box$, since for any commutative square

$$\begin{tikzcd} X \arrow[r, "k"] \arrow[d, "g"'] & C \times B \arrow[d, "\pi_2"] \\ Y \arrow[r, "l"'] & B \end{tikzcd}$$

with $g \in \mathcal{H}$ a diagonal is given by $\langle t, l\rangle\colon Y \longrightarrow C \times B$ where t is obtained via $\mathcal{H}$–injectivity of C:

$$\begin{tikzcd} X \arrow[rr, "\pi_1 \circ k"] \arrow[dr, "g"'] & & C \\ & Y \arrow[ur, dotted, "t"'] & \end{tikzcd}$$

Closure of $\mathcal{H}$ under retracts in $\mathcal{K}^{\to}$ follows from left cancellability. $\diamond$

1.7 Corollary *Let* **K** *be a category with finite products. Then the following conditions are equivalent:*

1. $(Mono, Mono^\Box)$ *is a weak factorization system.*
2. **K** *has enough injectives.*

1.8 Examples (i) In an abelian category with enough injectives, $(Mono, Mono^\Box)$ is a weak factorization system. Moreover, $Mono^\Box$ consists of all epimorphisms having an injective kernel. Since this seems to be a new result, below we present a proof, using the technique of [18, 2.2.9]. The reader is also referred to the paper [11] by B. Eckmann and H. Kleisli, who introduced similar structures in order to study higher homotopy groups in a fairly general categorical context.

Proof: Any epimorphism $g\colon C \longrightarrow D$ with an injective kernel splits; choose s with $g \circ s = 1_D$. Consider a commutative square with monomorphic f:

$$
\begin{tikzcd}
A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "v"'] & D
\end{tikzcd}
$$

Consider $k = u - s \circ g \circ u$. Then $g \circ k = 0$. Thus there exists a morphism t with $k = \ker(g) \circ t$. Since the domain of $\ker(g)$ is injective and f is mono, there exists a morphism r with $t = r \circ f$. Then the morphism $d = s \circ v + \ker(g) \circ r$ is a diagonal as required.

Conversely, any $g \in Mono^\square$ is a retraction in view of the commutative square:

$$
\begin{tikzcd}
0 \arrow[r] \arrow[d] & C \arrow[d, "g"] \\
D \arrow[r, "1_D"'] & D
\end{tikzcd}
$$

Moreover, $\ker(g) \colon E \longrightarrow C$ has injective domain E, because for any monomorphism $f \colon A \longrightarrow B$ and any morphism $k \colon A \longrightarrow E$ we have a diagonal d

$$
\begin{tikzcd}
A \arrow[r, "\ker(g)\circ k"] \arrow[d, "f"'] & C \arrow[d, "g"] \\
B \arrow[r, "0"'] \arrow[ur, dotted, "d"] & D
\end{tikzcd}
$$

which induces a morphism $t \colon B \longrightarrow E$ with $d = \ker(g) \circ t$. Thus $\ker(g) \circ t \circ f = d \circ f = \ker(g) \circ k$. Hence, $t \circ f = k$. $\diamond$

(ii) Any topos has enough injectives and therefore $(Mono, Mono^\square)$ is a weak factorization system. The class $Mono^\square$ is much less transparent here. In the special case of simplicial sets, $Mono^\square$ is the class of trivial fibrations. This weak factorization system is important in homotopy theory (see [18]). In Set we have $Mono^\square = Epi$ (see [1, III 5(3)]).

(iii) Any variety $\mathbf{V}$ of algebras has enough regular projectives . Hence, $({}^\square RegEpi, RegEpi)$ is a weak factorization system. (Regular epimorphisms are the surjective homomorphisms, hence right cancellable.) If $\mathbf{V}$ is abelian, then $Epi = RegEpi$, and ${}^\square Epi$ consists of monomorphisms with projective cokernels.

(iv) Let **Pos** be the category of partially ordered sets (and isotone maps) and Emb the class of all embeddings (= regular monomorphisms) in **Pos**. Then the Emb –injectives are precisely the complete lattices; and **Pos** has enough Emb–injectives [4]. Since Emb is left cancellable, $(Emb, Emb^{\square})$ is a weak factorization system. In the next Section we will describe $Emb^{\square}$.

2 Topological Functors

2.1 Notation Cat is the category of small categories and functors.

$\mathbf{Cat_{fa}}$ is the category of small categories and faithful, amnestic[4] functors.

$Full$ (resp. $Full_{fa}$) is the class of those morphisms in **Cat** (resp. $\mathbf{Cat_{fa}}$) that are full.

Top is the class of those morphisms $G\colon \mathbf{A} \longrightarrow \mathbf{X}$ in **Cat** that are topological, (i.e., each G–structured source $(X \xrightarrow{f_i} GA_i)_{i\in I}$ has a unique G–initial[5] lift[6] $(A \xrightarrow{\bar{f}_i} A_i)_{i\in I}$ (cf. [2, Definition 21.1]).

2.2 Remark (1) Topological functors have been considered (as a natural abstraction of the forgetful functors from the categories **Top**, **Unif**, **Prox** and others to Set) independently and with slight conceptual variations since 1964 by a number of authors. For details concerning the history of the concept see [14], for a systematic treatment see [2, Sections 21–22].

(2) Some known results about topological functors which concern us here are the following:

(a) Topological functors are faithful and amnestic. See [17] or [2, 21.3 and 21.5].

(b) $Top = Full^{\square}_{fa}$ in $\mathbf{Cat_{fa}}$. See [2, 21.21] resp. [9] and [22], [21] for a partial generalization to **Cat**.

(c) Every morphism G in $\mathbf{Cat_{fa}}$ has a factorization $G = T \circ F$ with F full and T topological. See [13].

[4]A functor $G\colon \mathbf{A} \longrightarrow \mathbf{X}$ is called *amnestic* provided that an $\mathbf{A}$–isomorphism f is an identity whenever Gf is an identity.

[5]A source $(A \xrightarrow{f_i} A_i)_{i\in I}$ in $\mathbf{A}$ is called *G–initial* provided that for each source $(B \xrightarrow{g_i} A_i)_{i\in I}$ in $\mathbf{A}$ and each $\mathbf{X}$–morphism $GB \xrightarrow{h} GA$ with $Gg_i = Gf_i \circ h$ for each $i \in I$ there exists a unique $\mathbf{A}$–morphism $B \xrightarrow{\bar{h}} A$ in $\mathbf{A}$ with $G\bar{h} = h$ and $g_i = f_i \circ \bar{h}$ for each $i \in I$. Cf. [2, 10.57, 10.41 and 10.58 (1)].

[6]A source $(A \xrightarrow{\bar{f}_i} A_i)_{i\in I}$ *lifts* a G–structured source $(X \xrightarrow{f_i} GA_i)_{i\in I}$ provided that $G\bar{f}_i = f_i$ for each $i \in I$.

2.3 Theorem *$(Full, Top)$ is a weak factorization system in* **Cat***.*

Proof:

(1) Existence of $(Full, Top)$ factorizations.
Let $\mathbf{A} \xrightarrow{G} \mathbf{X}$ be a morphism in **Cat**. Then G has a (full, faithful) factorization

$$\mathbf{A} \xrightarrow{G} \mathbf{X} \quad = \quad \mathbf{A} \xrightarrow{V_1} \mathbf{B} \xrightarrow{F_1} \mathbf{X},$$

e.g., the canonical factorization of G through the quotient $\mathbf{B} = \mathbf{A}/_\sim$ of $\mathbf{A}$ obtained by the congruence relation

$$f \sim g \Leftrightarrow [\mathrm{dom}\ (f) = \mathrm{dom}\ (g), \mathrm{cod}\ (f) = \mathrm{cod}\ (g) \text{ and } G(f) = G(g)].$$

The faithful functor F_1 has a (full, faithful amnestic) factorization

$$\mathbf{B} \xrightarrow{F_1} \mathbf{X} \quad = \quad \mathbf{B} \xrightarrow{V_2} \mathbf{C} \xrightarrow{F_2} \mathbf{X},$$

see, e.g., [2, Prop. 5.33].

The faithful amnestic functor F_2 has a (full, topological) factorization

$$\mathbf{C} \xrightarrow{F_2} \mathbf{X} \ = \ \mathbf{C} \xrightarrow{V_3} \mathbf{D} \xrightarrow{F_3} \mathbf{X},$$

see Remark 2.2 (2c) above.

Hence $\mathbf{A} \xrightarrow{G} \mathbf{X} \ = \ \mathbf{A} \xrightarrow{V_3 \circ V_2 \circ V_1} \mathbf{D} \xrightarrow{F_3} \mathbf{X}$ is a $(Full, Top)$–factorization of G.

(2) $Full \Box Top$.
Consider a commutative square

$$\begin{tikzcd} \mathbf{A} \arrow[r, "F"] \arrow[d, "V"'] & \mathbf{C} \arrow[d, "T"] \\ \mathbf{B} \arrow[r, "G"'] & \mathbf{D} \end{tikzcd}$$

in **Cat** with V full and T topological. Then a diagonal $\mathbf{B} \xrightarrow{D} \mathbf{C}$ can be obtained (precisely as in the faithful case, see, e.g., [2, Theorem 21.21]) as follows: for each $\mathbf{B}$–object B consider the V–structured source, consisting of all V–structured morphisms $f_i \colon B \longrightarrow VA_i$ $(i \in I)$ with domain B. Application of G yields a T–structured source $(GB \xrightarrow{Gf_i} T(FA_i))_{i \in I}$, which has a unique T–initial lift $(D_B \xrightarrow{\bar{f}_i} FA_i)_{i \in I}$. If $D \colon \mathbf{B} \longrightarrow \mathbf{C}$ is the unique functor with $D(B) = D_B$ for each $\mathbf{B}$–object B and $G = T \circ D$, then $F = D \circ V$. So D is a diagonal.

Hence $Full \subset {}^{\Box}Top$ and $Top \subset Full^{\Box}$.

(3) To show that $^{\square}Top \subset Full$, it suffices to verify that $\mathcal{L} = Full$ satisfies the condition (2b') (α) of 1.3. Let $\mathbf{A} \xrightarrow{F} \mathbf{B}$, $\mathbf{B} \xrightarrow{S} \mathbf{C}$ and $\mathbf{C} \xrightarrow{R} \mathbf{B}$ be small functors such that $S \circ F$ is full and $R \circ S = \mathrm{id}_{\mathbf{B}}$. If $FA \xrightarrow{f} FA'$ is $\mathbf{B}$–morphism, then there exits an $\mathbf{A}$–morphism $A \xrightarrow{g} A'$ with $S \circ F(g) = Sf$. Thus $Fg = R \circ S \circ F(g) = R \circ S(f) = f$.

(4) To show that $Full^{\square} \subset Top$, it suffices to verify that $\mathcal{R} = Top$ satisfies the condition (2b')(β) of 1.3. Let $\mathbf{A} \xrightarrow{F} \mathbf{B}$, $\mathbf{C} \xrightarrow{R} \mathbf{A}$ and $\mathbf{A} \xrightarrow{S} \mathbf{C}$ be small functors such that $R \circ F$ is topological and $R \circ S = \mathrm{id}_{\mathbf{A}}$. If $\mathfrak{S} = (B \xrightarrow{f_i} FA_i)_{i \in I}$ is an F–structured source, then the $F \circ R$–structured source

$$(B \xrightarrow{s_i} F \circ R(SA_i))_{i \in I}$$

has a unique $F \circ R$–initial lift $(C \xrightarrow{g_i} SA_i)_{i \in I}$. A straightforward computation reveals that $(RC \xrightarrow{Rg_i} A_i)_{i \in I}$ is an F–initial lift of $\mathfrak{S}$. Uniqueness follows immediately from the fact that F inherits faithfulness and amnesticity from $F \circ R$.

$\diamond$

2.4 Remark We consider the category **Pos** of posets and isotone maps as a full subcategory of **Cat**, in fact of $\mathbf{Cat_{fa}}$, as usual. It is easy to see that an isotone map $f \colon A \longrightarrow X$ between partially ordered sets is, when considered as functor,

(a) always *faithful* and *amnestic*,

(b) *full* iff it is an order–embedding,

(c) *topological* iff, for each subset B of A and each lower bound x of $f[B]$, the set of all lower bounds a of B with $f(a) \leq x$ has a largest element a_0 which satisfies $f(a_0) = x$.

Another characterization of "topological morphisms" in **Pos** is the following

2.5 Proposition *An isotone map $P \xrightarrow{f} X$ between posets, considered as a functor, is topological iff the following conditions are satisfied:*

(1) f is adjoint and coadjoint,

(2) f is convex, *i.e., for any p preceding q in P, every x between $f(p)$ and $f(q)$ in X can be lifted to an element between p and q in P,*

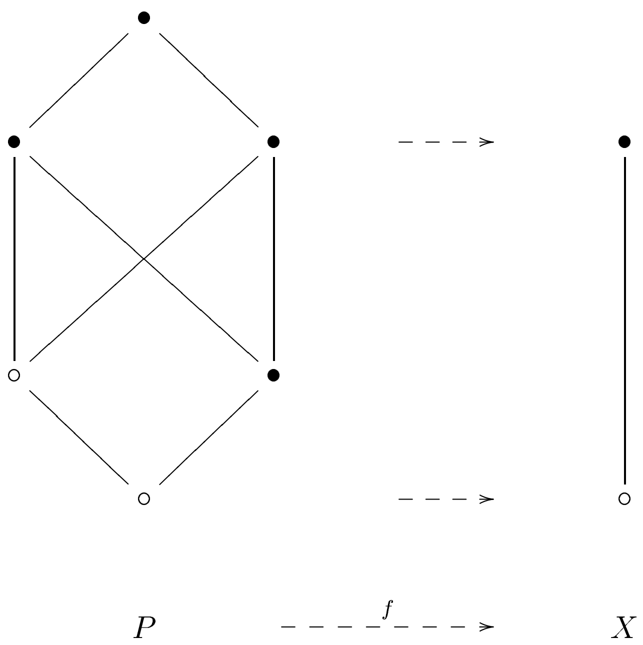
(3) f–fibres $f^{-1}(x)$ are complete lattices,

(4) the embeddings of f–fibres into P preserve all non–empty joins and meets.

Proof: First: let f be topological.
Then (1) and (3) are known to hold for topological functors in general, see [2, 21.11 and 21.12]. Moreover, (4) follows immediately from the fact that topological functors lift limits and colimits, thus meets and joins. For convexity, consider p, q and x as in condition (2). If $a \longrightarrow q$ is the initial lift of the f–structured arrow $x \longrightarrow f(q)$, then a has the desired properties.
Second: let f satisfy the conditions (1) – (4). Consider an f–structured source $S = (x \longrightarrow f(a_i))_{i \in I}$. Condition (1) implies that the fibre $f^{-1}(x)$ has a smallest element 0_x and a largest element 1_x. The set A of all lower bounds of $\{a_i \mid i \in I\}$ that belong to $f^{-1}(x)$ is non–empty, since $0_x \in A$. By (3), A has a join a in $f^{-1}(x)$ which, by (4), is a join in P. Thus $(a \longrightarrow a_i)_{i \in I}$ is a lift of S. If $I = \emptyset$, then $a = 1_x$ and $(1_x \longrightarrow a_i)_{i \in I}$ is f–initial. If $I \neq \emptyset$, consider a lower bound b of $\{a_i \mid i \in I\}$ with $f(b) \leq x$. Then convexity of f implies that for each $i \in I$ there exists some b_i in $f^{-1}(x)$ with $b \leq b_i \leq a_i$. By (3), the set $\{b_i \mid i \in I\}$ has a meet c in $f^{-1}(x)$ which, by (4) is a meet in P. Thus $b \leq c$ and $c \leq a$ (since $c \leq a_i$ for each $i \in I$), hence $b \leq a$. Consequently $(a \longrightarrow a_i)_{i \in I}$ is f–initial. ◇

2.6 Remark *It is easy to see that the four conditions of Proposition 2.5 are independent, i.e., none follows from the conjunction of the other three.*

In particular the following example demonstrates that condition (4) is not implied by the conjunction of conditions (1) – (3):



2.7 Corollary *The category* **Pos** *has a weak factorization system* (Emb, Top)*, where:*

Emb is the class of order–embeddings and
Top is the class of topological isotone maps.

Proof: Let $\mathbf{A} \xrightarrow{G} \mathbf{X}$ be a morphism in **Pos**. Then, by Theorem 2.3, G considered as a morphism in **Cat** has a $(Full, Top)$ factorization $\mathbf{A} \xrightarrow{G} \mathbf{X} = \mathbf{A} \xrightarrow{F} \mathbf{B} \xrightarrow{T} X$. Since $\mathbf{X}$ belongs to **Pos** and T is topological, thus faithful and amnestic, it follows that $\mathbf{B}$ belongs to **Pos**. Hence $G = T \circ F$ is an (Emb, Top)–factorization of G in **Pos**.

The rest is an easy consequence of 2.3. $\diamond$

3 Cofibrantly generated weak factorization systems

If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system in a cocomplete category $\mathbf{K}$, then $\mathcal{L}$ is cofibrantly closed in the following sense:

3.1 Definition A class $\mathcal{L}$ of morphisms in a cocomplete category $\mathbf{K}$ is called *cofibrantly closed* provided that $\mathcal{L}$ is

(i) closed under retracts in comma–categories $A \backslash \mathbf{K}$,

(ii) stable under pushouts,

(iii) closed under composition and contains all isomorphisms,

(iv) closed under transfinite compositions, i.e., given a chain of morphisms from $\mathcal{L}$, then a colimit cocone consists of morphisms from $\mathcal{L}$.

3.2 Definition A weak factorization system $(\mathcal{L}, \mathcal{R})$ in a cocomplete category $\mathbf{K}$ is said to be *cofibrantly generated* provided that there exists a set $\mathcal{H}$ of morphisms such that $\mathcal{L}$ is the smallest cofibrantly closed class containing $\mathcal{H}$.

Beke [6] calls such weak factorization systems small. Examples of cofibrantly generated factorization systems include all weak factorization–systems in a locally presentable category $\mathbf{K}$ of the form $({}^{\square}(\mathcal{C}^{\square}), \mathcal{C}^{\square})$ where $\mathcal{C}$ is any set of morphisms, see [6], or [1].

3.3 Example In a Grothendieck category, $(Mono, Mono^{\square})$ is cofibrantly generated (see [6]).

3.4 Proposition *The weak factorization system* (Emb, Top) *in* **Pos** *is not cofibrantly generated.*

Proof: Whenever $(\mathcal{L}, R)$ is a cofibrantly generated weak factorization system then $\mathcal{L}$–**Inj** is a small–injectivity class in the sense of [3, 4.1]. In fact, whenever $\mathcal{L} = \mathrm{cof}(\mathcal{C})$ for a set $\mathcal{C}$, then $\mathcal{L} =^{\square} (\mathcal{C}^{\square})$ and thus $\mathcal{L}$–**Inj** $= \mathcal{C}$–**Inj**. Any small–injectivity class in a locally presentable category is closed under λ–filtered colimits for some regular cardinal λ (see [3, 4.7]). But the full subcategory *Emb–Inj* of complete lattices is not closed in **Pos** under λ–filtered colimits for any cardinal λ. (For instance, $\lambda + 1$, i.e., the ordered set of all ordinals less than $\lambda + 1$, is not complete but a λ–filtered colimit of its complete sublattices.) $\diamond$

3.5 Corollary *The weak factorization system* $(Full, Top)$ *in* **Cat** *is not cofibrantly generated.*

Proof: Assume that $Full$ is cofibrantly generated by a set $\mathcal{C}$. Let $F\colon \mathbf{Cat} \longrightarrow \mathbf{Pos}$ be a left adjoint to the inclusion $\mathbf{Pos} \longrightarrow \mathbf{Cat}$. Since F preserves colimits, $Emb = F(Full)$ is cofibrantly generated by $F(\mathcal{C})$, a contradiction. $\diamond$

3.6 Remark Recall that a *Quillen model category* **K** is a complete and cocomplete category **K** together with three classes of morphisms, $\mathcal{F}$ ("fibrations"), $\mathcal{C}$ ("cofibrations"), and $\mathcal{W}$ ("weak equivalences") such that

(1) $\mathcal{W}$ has the 2–out–of–3 property, i.e., with any two of $f, g, g \circ f$ belonging to $\mathcal{W}$ also the third morphism belongs to $\mathcal{W}$, and $\mathcal{W}$ is closed under retracts,

and

(2) $(\mathcal{C}, \mathcal{F}_0)$ and $(\mathcal{C}_0, \mathcal{F})$ are weak factorization systems, where $\mathcal{F}_0 = \mathcal{F} \cap \mathcal{W}$ is the class of "trivial fibrations" and $\mathcal{C}_0 = \mathcal{C} \cap \mathcal{W}$ is the class of "trivial cofibrations".

It immediately follows from 1.3 that our definition is equivalent to the standard one given, e.g., in [18].

Quillen model categories provide a framework for abstract homotopy theory (see [20], [5], [16] or [18]). We have mentioned a part of a Quillen structure in simplicial sets in our Example 1.8 (ii). A Quillen model category is called *cofibrantly generated* provided that both weak factorization systems $(\mathcal{C}, \mathcal{F}_0)$ and $(\mathcal{C}_0, \mathcal{F})$ are cofibrantly generated. M. Hovey asks in [19] for an example of a Quillen model category that can be proven to be not cofibrantly generated.

3.7 Example If $(\mathcal{L}, \mathcal{R})$ is a weak factorization system in a category **K** with finite limits and colimits, then $\mathcal{F} = \mathcal{R}$, $\mathcal{C} = \mathcal{L}$ and $\mathcal{W} = \mathbf{Mor K}$ is a Quillen model category. Hence 3.4 and 3.5 provide examples of Quillen model categories which are not cofibrantly generated.

M. Hovey has informed us that recently there have been found other examples of non–cofibrantly generated Quillen model categories.

References

[1] J. Adámek, H. Herrlich, J. Rosický, and W. Tholen: On a Generalized Small–Object Argument for the Injective Subcategory Problem. Preprint May 2000.

[2] J. Adámek, H. Herrlich, and G. Strecker: Abstract and Concrete Categories. Wiley 1990.

[3] J. Adámek and J. Rosický: Locally presentable and accessible categories. Cambridge University Press 1994.

[4] B. Banaschewski and G. Bruns: Categorical characterization of the Mac Neille completion. Archiv Math. **18** (1967) 369–377.

[5] H. J. Baues: Algebraic Homotopy. Cambridge Univ. Press, Cambridge 1989.

[6] T. Beke: Sheafifiable homotopy model categories. Math. Proc. Cambr. Phil. Soc. **129** (2000) 447–475.

[7] F. Borceux: Handbook of Categorical Algebra. Vol. II. Cambridge University Press 1995.

[8] A.K. Bousfield: Constructions of factorization systems in categories. J. Pure Appl. Alg. **9** (1977) 207–220.

[9] G.C.L. Brümmer and R.-E. Hoffmann: An external characterization of topological functors. Springer Lecture Notes Math. **540** (1976) 136–151.

[10] G. Bruns and H. Lakser: Injective hulls of semilattices. Canad. Math. Bull. **13** (1970) 115–118.

[11] B. Eckmann and H. Kleisli: Algebraic homotopy groups and Frobenius algebras. Ill. J. Math. **6** (1962) 533 - 552.

[12] K.A. Hardie and J.J.C. Vermeulen: A projective homotopy theory for non–additive categories. Rendiconti dell' Istituto di Matematica dell' Univ. di Trieste **25** (1993) 263–276.

[13] H. Herrlich: Initial completions. Math. Z. **150** (1976) 101–110.

[14] H. Herrlich: Categorical Topology 1971–1981. In: Gener. Topol. and its Rel. to Mod. Analysis and Algebra V, ed. J. Novák, Heldermann 1983, 279–383.

[15] H. Herrlich and G.E. Strecker: Cartesian closed topological hulls as injective hulls. Quaest. Math. **9** (1986) 263–280.

[16] P. Hirschhorn: Localization of model categories. Preprint, 1998.

[17] R.-E. Hoffmann: Die kategorielle Auffassung der Initial– und Finaltopologie. Thesis, Univ. Bochum 1972.

[18] M. Hovey: Model Categories. AMS Math. Surveys and Monographs **63** (1998).

[19] M. Hovey: Problems, http://www.math.wesleyan.edu.

[20] D. Quillen: Homotopical Algebra. Springer Lecture Notes Math. **43** (1967).

[21] W. Tholen and M.B. Wischnewsky: Semi–topological functors II: external characterizations. J. Pure Appl. Algebra **15** (1979) 75–92.

[22] H. Wolff: On the external characterization of topological functors: Manuscripta Math. **22** (1977) 63–76.

Institut für Theoretische Informatik
Technische Universität Braunschweig
38032 Brausnchweig, Germany
j.adamek@tu-bs.de

Fachbereich 3
Universität Bremen
28334 Bremen, Germany
herrlich@math.uni-bremen.de

Department of Mathematics
Masaryk University
66295 Brno, Czech Republic
rosicky@math.muni.cz

Department of Mathematics and Statistics
York University
Toronto M3Y 1P3, Canada
tholen@pascal.math.yorku.ca