PURITY IN ALGEBRA

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Abstract. There are considered pure monomorphisms and pure epimorphisms in varieties of universal algebras with applications to equationally compact (= pure injective) algebras, pure projective algebras and perfect varieties.

1. Introduction

Purity plays an important role in module theory but all its basic concepts can be extended to general universal algebras. One can proceed model-theoretically (see [19]) but our approach will be algebraic. For instance, pure monomorphisms can be characterized as directed colimits of split monomorphisms and this description works well even in all locally finitely presentable categories (see [2]). This is a class of categories containing not only varieties but also quasivarieties of many-sorted universal algebras. Having pure monomorphisms, one can introduce pure injective objects. In module theory, these objects are also called algebraically compact modules and, in universal algebra, one also speaks about equationally compact algebras. Equationally compact algebras were intensively studied during the 1970's; it is well surveyed in [22]. We will reconsider this subject and bring a new criterion for having enough pure injectives, which well covers all known occurences of this property in varieties of unary algebras.

Pure epimorphisms of modules are precisely cokernels of pure monomorphisms and they can be characterized as directed colimits of split epimorphisms. This description again works well in all locally finitely presentable categories (see [3]); there is also a model-theoretic definition of pure epimorphisms in [19]. In contrast to pure injectives, pure

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projective objects have an easy characterization as retracts of coproducts of finitely presentable objects and each locally finitely presentable category has enough pure projectives. We will call a locally finitely presentable category pure semisimple if any pure epimorphism splits, which is the same as that each object is pure projective. This concept is important in module theory where it corresponds to the fact that each module is pure injective. We will discuss pure semisimple varieties of unary algebras.

The opposite extreme to pure semisimplicity is when any strong epimorphism (monomorphism) is pure. In module theory, this means that the underlying ring R is von Neumann regular. For general varieties of algebras, we studied this situation in [10]. Then pure projectivity (injectivity) reduces to projectivity (injectivity). Flat objects K can be defined by the property that every strong epimorphism $L \to K$ is pure (see [19]). In varieties, this is equivalent to the fact that K is a directed colimit of finitely presentable projective algebras (in this way, flat algebras were introduced in [17]). Analogously, absolutely pure objects K are objects such that every strong monomorphism $L \to K$ is pure (see [19]). We will discuss these concepts in general varieties of algebras. In particular, we will mention perfect varieties, i.e., varieties where every algebra has a projective cover. This is a classical concept in modules and, for unary varieties, it was studied by Isbell [13].

2. Pure injectivity

Recall that a monomorphism $f : A \to B$ in a locally finitely presentable category $\mathcal{K}$ is *pure* provided that in each commutative square

$$\begin{array}{ccc} X & \xrightarrow{\ g\ } & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ A & \xrightarrow[\ f\ ]{} & B \end{array}$$

with X and Y finitely presentable the morphism u factorizes through g, i.e., there exists $d : Y \to A$ with $dg = u$. Pure monomorphisms in locally finitely presentable categories have the following properties:

(1) a composition of two pure monomorphisms is a pure monomorphism;
(2) if $f_2 f_1$ is a pure monomorphism then f_1 is a pure monomorphism;
(3) any pure monomorphism is a regular monomorphism (i.e., an equalizer of a parallel pair of morphisms);

(4) any split monomorphism is pure (recall that a monomorphism $f : A \to B$ splits if $hf = \mathrm{id}_A$ for some $h : B \to A$);
(5) pure monomorphisms are precisely directed colimits of split monomorphisms in the category $\mathcal{K}^{\to}$ of morphisms in $\mathcal{K}$ (recall that objects of this category are morphisms in $\mathcal{K}$ and morphisms from $f : A \to B$ to $f' : A' \to B'$ in $\mathcal{K}^{\to}$ are pairs (g, h) where $g : A \to A'$ and $h : B \to B'$ with $hf = f'g$);
(6) pure monomorphisms are stable under pushout, i.e., in a pushout

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle g}\big\downarrow & & \big\downarrow{\scriptstyle \bar{g}} \\ C & \xrightarrow[\bar{f}]{} & D \end{array}$$

$\bar{f}$ is a pure monomorphism provided that f is a pure monomorphism.

The properties (1)–(5) can be found in [2], (6) is proved in [3].

Let $(f_{ij} : K_i \to K_j)_{i<j<\lambda}$ be a smooth chain of morphisms of a category $\mathcal{K}$. This means that λ is a limit ordinal, $f_{jk} f_{ij} = f_{ik}$ for $i < j < k$ and $(f_{ij} : K_i \to K_j)_{i<j}$ is a colimit cocone for each limit ordinal $j < \lambda$. Then the component $f_0 : K_0 \to K$ of a colimit cocone $(f_i : K_i \to K)_{i<\lambda}$ is called the *transfinite composition* of $(f_{ij})_{i<j<\lambda}$.

Lemma 2.1. *Pure monomorphisms in a locally finitely presentable category $\mathcal{K}$ are closed under transfinite composition.*

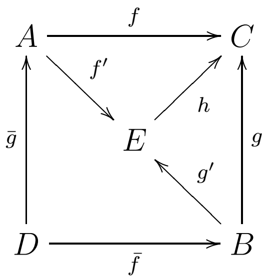
Proof. Consider a smooth chain $(f_{ij} : K_i \to K_j)_{i<j<\lambda}$ of pure monomorphisms, its transfinite composition $f : K_0 \to K$ and a commutative square

$$\begin{array}{ccc} X & \xrightarrow{g} & Y \\ {\scriptstyle u}\big\downarrow & & \big\downarrow{\scriptstyle v} \\ K_0 & \xrightarrow[f]{} & K \end{array}$$

with X and Y finitely presentable. There is $i < \lambda$ and $v' : Y \to K_i$ such that $v'g = f_{0i}u$. Since f_{0i} is pure, there is $d : Y \to K_0$ with $dg = u$. We have proved the factorization property from the definition of a pure monomorphism. This property implies that f is a monomorphism, thus a pure monomorphism (see [2], 2.29). □

The following definition mimics the concept of an effective union of subobjects due to [5] which is satisfied by both abelian categories and toposes.

Definition 2.2. Let $\mathcal{K}$ be a locally finitely presentable category. We say that pure subobjects *have effective unions* in $\mathcal{K}$ if whenever a diagram



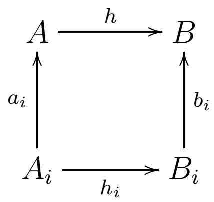
is given where f and g are pure monomorphisms, the outer square is a pullback and the inner tetragon (consisting of $f', \bar{g}, g'$ and $\bar{f}$) is a pushout then the uniquely defined morphism h is a pure monomorphism.

Definition 2.3. Let $\mathcal{K}$ be a locally finitely presentable category. An object K is called *pure injective* if for any pure monomorphism $f : A \to B$ and any morphism $g : A \to K$ there is a morphism $h : B \to K$ with $hf = g$.

We say that $\mathcal{K}$ *has enough pure injectives* if each object A of $\mathcal{K}$ has a pure monomorphism $A \to K$ into a pure injective object K.

Theorem 2.4. *Let $\mathcal{K}$ be a locally finitely presentable category having effective unions of pure subobjects. Then $\mathcal{K}$ has enough pure injectives.*

Proof. There is a regular cardinal μ such that each monomorphism $A \to B$ with B finitely presentable has A μ-presentable. Then, for every regular cardinal $\lambda \geq \mu$, each monomorphism $h : A \to B$ with B λ-presentable has A λ-presentable. In fact, we express B as a filtered colimit $(b_i : B_i \to B)_{i \in I}$ of finitely presentable objects such that $\operatorname{card} I < \lambda$ (see [16], 2.3.11) and form pullbacks



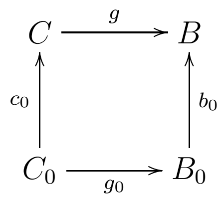
Then $(a_i : A_i \to A)_{i \in I}$ is a filtered colimit (see [2], 1.59) and A_i, $i \in I$, are μ-presentable objects. Hence A is λ-presentable (see [2], 1.16).

Let $Pure(\mathcal{K})$ be the subcategory of $\mathcal{K}$ consisting of all objects from $\mathcal{K}$ and pure monomorphisms as morphisms. By [2], 2.34 and 2.19, there is a regular cardinal ν such that $\nu \geq \mu$, $Pure(\mathcal{K})$ is ν-accessible and the embedding $Pure(\mathcal{K}) \to \mathcal{K}$ preserves ν-filtered colimits and ν-presentable objects. This implies that each object K of $\mathcal{K}$ is a ν-directed colimit $(k_i : K_i \to K)_{i \in I}$ such that K_i, $i \in I$, are ν-presentable in $\mathcal{K}$ and k_i, $i \in I$, are pure monomorphisms.

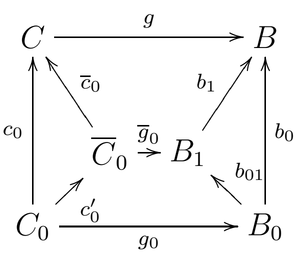
Let $\mathcal{M}$ be the set of all pure monomorphisms $A \to B$ with A and B ν-presentable. A morphism will be called $\mathcal{M}$-*cellular* if it is a transfinite composition of pushouts of morphisms from $\mathcal{M}$. According to (6) and Lemma 2.1, any $\mathcal{M}$-cellular morphism is a pure monomorphism. We will prove that, conversely, any pure monomorphism is $\mathcal{M}$-cellular. Then [1], II.10 will imply that $\mathcal{K}$ has enough pure injectives.

Let $h : A \to B$ be a pure monomorphism. Since $\mathcal{K}$ is well-powered (i.e., each object has only a set of subobjects, see [2], 0.6), we may consider the set $\mathcal{S}$ of all pure subobjects of B partially ordered by $\mathcal{M}$-cellular inclusions. Since $\mathcal{M}$-cellular morphisms are closed under transfinite composition and pure monomorphisms are closed under directed colimits in $\mathcal{K}^{\to}$, $\mathcal{S}$ is inductive, i.e., A is an $\mathcal{M}$-cellular subobject of a pure subobject $g : C \to B$ which is a maximal element of $\mathcal{S}$. It suffices to show that $C = B$, i.e., that g is an isomorphism.

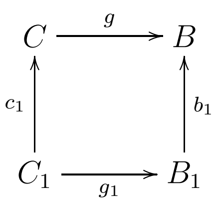
Assume that g is not an isomorphism. Then there is a pure monomorphism $b_0 : B_0 \to B$ with B_0 ν-presentable which does not factorize through g. Form a pullback



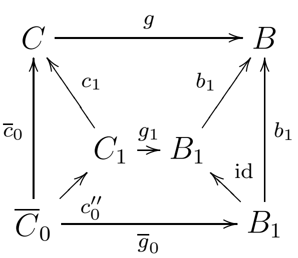
Since g_0 is a monomorphism, C_0 is ν-presentable. Since C_0 is ν-presentable and C is a ν-directed union of its ν-presentable pure subobjects, c_0 factorizes through one of them. Since the same argument applies to $b_0 : B_0 \to B$, there is a filling



with $\overline{c}_0$ and b_1 pure monomorphism and $\overline{C}_0$, B_1 ν-presentable. Clearly, $\overline{g}_0$ is a pure monomorphism. Form a pullback

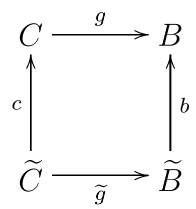


and take the induced morphism $c_0'' : \overline{C}_0 \to C_1$, i.e., $g_1 c_0'' = \overline{g}_0$ and $c_1 c_0'' = \overline{c}_0$. Then C_1 is ν-presentable. Denote $c_{01} = c_0'' c_0'$. We have $c_1 c_{01} = c_0$ and $g_1 c_{01} = b_{01} g_0$. Thus we have got another filling



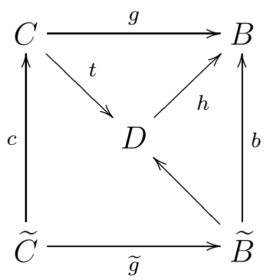
By continuing this procedure, we get ν-presentable objects B_n, C_n monomorphisms $g_n : C_n \to B_n$, $c_{n,n+1} : C_n \to C_{n+1}$, $c_n : C_n \to C$, $c_n' : C_n \to \overline{C}_n$ and pure monomorphisms $\overline{g}_n : \overline{C}_n \to B_{n+1}$, $b_{n,n+1} : B_n \to B_{n+1}$, $b_n : B_n \to B$, $c_n'' : \overline{C}_n \to C_{n+1}$, $\overline{c}_n : \overline{C}_n \to C$ such that $gc_n = b_n g_n$, $c_{n+1} c_{n,n+1} = c_n$, $b_{n+1} b_{n,n+1} = b_n$, $g_{n+1} c_n'' = \overline{g}_n$, $c_{n,n+1} = c_n'' c_n'$, $c_n = \overline{c}_n c_n'$ and $\overline{c}_n = c_{n+1} c_n''$.

Let $(\widetilde{c}_n : C_n \to \widetilde{C})_{n=0}^{\infty}$ and $(\widetilde{b}_n : B_n \to \widetilde{B})_{n=0}^{\infty}$ be colimit cocones. We get the induced morphisms



satisfying $c\widetilde{c}_n = c_n$, $b\widetilde{b}_n = b_n$ and $\widetilde{g}\widetilde{c}_n = \widetilde{b}_n g_n$ for $n = 0, 1, \dots$. Moreover, the square above is a pullback (as a colimit of a chain of pullbacks). At the same time, $\widetilde{g} = \mathit{colim}\, \overline{g}_n$ where $\overline{g}_n : \overline{C}_n \to B_{n+1}$ are pure monomorphism. Thus $\widetilde{g}$ is a pure monomorphism. The objects $\widetilde{C}$ and $\widetilde{B}$ are ν-presentable and $b = \mathit{colim}\, b_n$ is a pure monomorphism.

Form the effective union of g and b:



Then h is a pure monomorphism and t is $\mathcal{M}$-cellular because $\widetilde{g} \in \mathcal{M}$. This contradicts the maximality of C. □

Remark 2.5. In the proof of Theorem 2.4, we have established that pure injectives form a small injectivity class $\mathcal{M}$-*Inj* consisting of objects injective w.r.t. a set $\mathcal{M}$ of morphisms. In the literature concerning pure injective (=equationally compact) universal algebras, this smallness property corresponds to equational λ-compactness for a cardinal λ.

The property of having effective unions modifies an approach to the existence of enough injective objects going back to Grothendieck (see [7]).

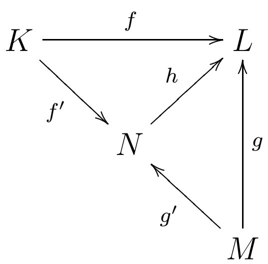
In what follows, $\mathbf{Set}^{\mathcal{C}}$ will denote the category of all functors $\mathcal{C} \to \mathbf{Set}$ (morphisms are natural transformations).

Corollary 2.6. *Let $\mathcal{C}$ be a small category such that*

(a) *whenever $f : C \to D_1$ and $g : C \to D_2$ are in $\mathcal{C}$ then either $f = hg$ or $g = hf$ for some h;*
(b) *if there is a monomorphism $K \to L$ or an epimorphism $L \to K$ in $\mathbf{Set}^{\mathcal{C}}$ with L finitely presentable then K is finitely presentable too.*

Then $\mathbf{Set}^{\mathcal{C}}$ has enough pure injectives.

Proof. Since $\mathbf{Set}^{\mathcal{C}}$ is locally finitely presentable (see [2]), by Theorem 2.4 it suffices to show that $\mathbf{Set}^{\mathcal{C}}$ has effective unions of pure subobjects. Consider pure monomorphisms $f : K \to L$ and $g : M \to L$ and form their effective union



Since $\mathbf{Set}^{\mathcal{C}}$ has effective unions of subobjects (see [5]), h is a monomorphism. Consider a commutative square

$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ {\scriptstyle u}\downarrow & & \downarrow{\scriptstyle v} \\ N & \xrightarrow[h]{} & L \end{array}$$

with X and Y finitely presentable. Without any loss of generality we can assume that v is a monomorphism and the square is a pullback (and thus u and t are monomorphism as well). Indeed, take the (epi, mono)-factorization $v = v_2 v_1$ of v, the pullback of h and v_2 and the induced morphism u' such that $u''u' = u$ and $t'u' = v_1 t$:

$$\begin{array}{ccc} X & \xrightarrow{t} & Y \\ {\scriptstyle u'}\downarrow & & \downarrow{\scriptstyle v_1} \\ X' & \xrightarrow{t'} & Y' \\ {\scriptstyle u''}\downarrow & & \downarrow{\scriptstyle v_2} \\ N & \xrightarrow[h]{} & L \end{array}$$

According to (b), Y' is finitely presentable and, since t' is a monomorphism, X' is finitely presentable too. Therefore, having $d' : Y' \to N$ with $d't' = u''$, we get $d = d'v_1$ satisfying

$$dt = d'v_1 t = d't'u' = u''u' = u\,.$$

Form pullbacks

$$\begin{array}{ccc} K & \xrightarrow{f} & L \\ {\scriptstyle u_1}\uparrow & & \uparrow{\scriptstyle v} \\ X_1 & \xrightarrow[t_1]{} & Y \end{array} \qquad \begin{array}{ccc} M & \xrightarrow{g} & L \\ {\scriptstyle u_2}\uparrow & & \uparrow{\scriptstyle v} \\ X_2 & \xrightarrow[t_2]{} & Y \end{array}$$

Since t_1 and t_2 are monomorphisms, X_1 and X_2 are finitely presentable and, since f and g are pure monomorphisms, we get $d_1 : Y \to K$ and $d_2 : Y \to M$ with $d_1 t_1 = u_1$ and $d_2 t_2 = u_2$. For $C \in \mathcal{C}$, let $Z(C)$ be the set

$$\{y \in Y(C) |\ L(s)(v_C(y)) \in f_D K(D) - g_D M(D) \text{ for some } s : C \to D\}$$

Let $d_C : Y(C) \to N(C)$ be defined as follows

$$d_C(y) = \begin{cases} f'_C d_{1C}(y) & \text{for } \ y \in Z(C) \\ g'_C d_{2C}(y) & \text{otherwise}. \end{cases}$$

Then $d_C t_C = u_C$ because if $u_C(x) \in f'_C K(C) - g'_C M(C)$ then $t_C(x) \in Z(C)$ (take $s = \mathrm{id}_C$) and thus $d_C t_C(x) = u_C(x)$ because

$$\begin{aligned} h_C d_C t_C(x) &= h f'_C d_{1C} t_{1C}(x) = h f'_C u_{1C}(x) \\ &= f_C u_{1C}(x) = v_C t_{1C}(x) = h_C u_C(x)\,. \end{aligned}$$

If $u_C(x) \in g'_C M(C)$ then $t_C(x) \notin Z(C)$ (because $L(s) v_C t_C(x) = L(s) h_C u_C(x) \in g_D M(D)$ for each $s : C \to D$) and thus $d_C t_C(x) = u_C(x)$ because

$$h_C d_C t_C(x) = h_C g'_C d_{2C} t_{2C}(x) = g_C u_{2C}(x) = v_C b_{2C}(x) = h_C v_C(x)\,.$$

We have to prove that $d : Y \to N$ is a natural transformation (i.e., a morphism in $\mathbf{Set}^{\mathcal{C}}$). Consider $y \in Y(C)$ and $p : C \to E$. We have the following possibilities.

(i) $y \in Z(C)$. Let $s : C \to D$ be from the definition of $Z(C)$.

(i$_1$) $Y(p)(y) \in Z(E)$. Then

$$d_E Y(p)(y) = f'_E d_{1E} Y(p)(y) = N(p) f'_C d_{1C}(y) = N(p) d_C(y)\,.$$

(i$_2$) $Y(p)(y) \notin Z(E)$. Then s does not factorize through p because if $s = qp$ then

$$L(q) v_C Y(p)(y) = L(qp) v_C(y) = L(s) v_C(y) \in f_D K(D) - g_D M(D)$$

and thus $Y(p)(y) \in Z(E)$ (due to q). By (a), p factorizes through s, i.e., $p = qs$. Hence

$$v_E Y(p)(y) = L(p) v_C(y) = L(qs) v_C(y) \in f_E K(E)\,.$$

Since $Y(p)(y) \notin Z(E)$, we have

$$v_E Y(p)(y) \in f_E K(E) \cap g_E M(E).$$

Hence $Y(p)(y) = t_E(x)$ for some $x \in X_1(E) \cap X_2(E)$. Therefore

$$\begin{aligned} d_E Y(p)(y) &= g'_E d_{2E} Y(p)(y) = g'_E d_{2E} t_{2E}(x) = g'_E u_{2E}(x) \\ &= f'_E u_{1E}(x) = f'_E d_{1E} t_{1E}(x) = f'_E d_{1E} Y(p)(y) \\ &= d N(p) f'_C d_{1C}(y) = N(p) d_E(y)\,. \end{aligned}$$

(ii) $y \notin Z(C)$. Then $Y(p)(y) \notin Z(E)$ because $s : E \to D$ making $Y(p)(y) \in Z(E)$ would give $sp : C \to D$ making $y \in Z(C)$. Thus

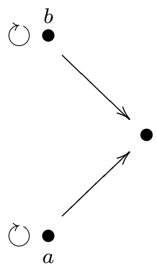
$$d_E Y(p)(y) = g'_E d_{2E} Y(p)(y) = N(p) g'_E d_{2C}(y) = N(p) d_C(y)\,.$$

□

Example 2.7. (1) We will give an example of a small category $\mathcal{C}$ such that $\mathbf{Set}^{\mathcal{C}}$ does not have enough pure injectives.

Consider the category $\mathcal{C}$ having two objects C and D and two non-identity morphisms $f, g : C \to D$. Then $\mathbf{Set}^{\mathcal{C}}$ is the category of oriented multigraphs because C is the object of edges, D the object of vertices and f (respectively, g) gives the source (respectively, target) vertex of an edge.

$\mathbf{Set}^{\mathcal{C}}$ satisfies the condition (b) from 2.6 but not (a). It neither has effective unions of pure monomorphisms nor enough pure injectives. In fact, in the graph



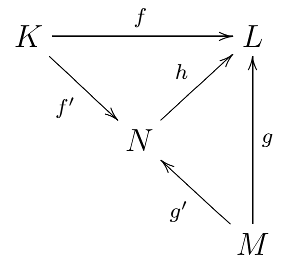
both $\{a\}$ and $\{b\}$ are pure (because split) subobjects but $\{a, b\}$ is not.

Let K be a complete oriented graph without loops having countably many vertices. Then a monomorphism $K \to L$ is pure iff L does not have loops. Thus there is no pure monomorphism $f : K \to L$ with L pure injective (f should be injective w.r.t. $g : K \to M$ where M is a complete oriented graph without loops having more vertices than L, which implies that L has a loop). Hence $\mathbf{Set}^{\mathcal{C}}$ does not have enough pure injectives.

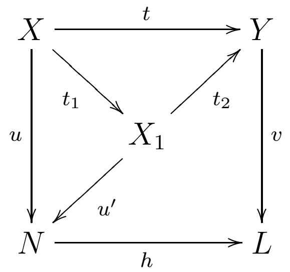
(2) Let $(\mathbb{N}, +)$ be the monoid of natural numbers considered as a category with a single object. Then $\mathbf{Set}^{\mathbb{N}}$ is the variety of algebras with one unary operation. By 2.6, $\mathbf{Set}^{\mathbb{N}}$ has enough pure injectives. This result was proved by Wenzel [21]; see also [20].

Corollary 2.8. *Let $\mathcal{C}$ be a small category such that each morphism of $\mathcal{C}$ is an isomorphism. Then $\mathbf{Set}^{\mathcal{C}}$ has enough pure injectives.*

Proof. Again, we will prove that $\mathbf{Set}^{\mathcal{C}}$ has effective unions of pure subobjects. Consider pure monomorphism $f : K \to L$ and $g : M \to L$, form their effective union



and consider a commutative square



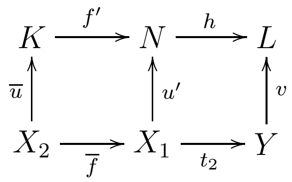
with X and Y finitely presentable. Inside, we have the (epi, mono)-factorization $t = t_2 t_1$ and the induced morphism u' (recall from the proof of 2.6 that h is a monomorphism).

At first, we will prove that any monomorphism $m : U \to V$ in $\mathbf{Set}^{\mathcal{C}}$ is a coproduct injection. Consider $s : C \to D$ in $\mathcal{C}$ and $x \in V(C)$ such that $V(s)(x) \in m_D U(D)$. Then $V(s)(x) = m_D(y)$ for some $y \in U(D)$ and thus

$$x = V(s^{-1}s)(x) = V(s^{-1})m_D(y) = m_C U(s^{-1})(y) \in m_C U(C)\,.$$

We have proved that $m(U)$ is a connected component of V, i.e., m is a coproduct injection.

Thus, we can assume that $Y = X_1 \coprod Y_1$ and it suffices to find $u'' : Y_1 \to N$; then $(u', u'') : Y \to N$ satisfies $(u', u'')t_1 = u$. Consider



where the left square is a pullback. Since $\overline{f}$ is a monomorphism, it is a coproduct injection and we can assume that $X_1 = X_2 \coprod X_3$, i.e., $Y = X_2 \coprod X_3 \coprod Y_1$. It is easy to see that X_2 is finitely presentable and, since $f = hf'$ is a pure monomorphism, there is $X_3 \coprod Y_1 \to K$. This yields $Y_1 \to N$. □

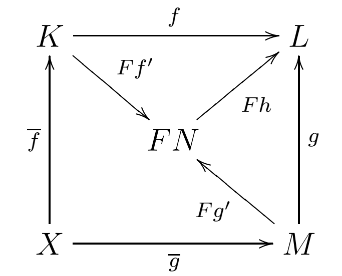
Corollary 2.9 (Banaschewski [4]). *Let G be a group. Then the category G-**Set** of left G-sets has enough pure injectives.*

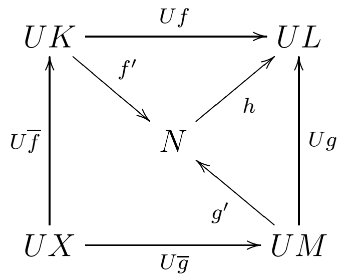
Proof. G can be considered as a category with a single object whose set of endomorphisms is G. Hence the result follows from 2.8. □

Let $\mathcal{K}$ be a locally finitely presentable category and $\mathcal{C}$ be the set of all representatives of finitely presentable objects in $\mathcal{K}$. Then $\mathcal{K}$ is (equivalent to) a full reflective subcategory of $\mathbf{Set}^{\mathcal{C}^{\mathrm{op}}}$ closed under directed colimits. So, we can apply the following result.

Proposition 2.10. *Let $\mathcal{K}$ be a full reflective subcategory of a locally finitely presentable category $\mathcal{L}$ closed under directed colimits. If $\mathcal{L}$ has effective unions of pure subobjects then $\mathcal{K}$ has effective unions of pure subobjects.*

Proof. Let $U : \mathcal{K} \to \mathcal{L}$ be the inclusion, $F : \mathcal{L} \to \mathcal{K}$ its left adjoint and consider pure monomorphisms $f : K \to L$ and $g : M \to L$ in $\mathcal{K}$ and their effective unions in $\mathcal{K}$ and $\mathcal{L}$





We have used the fact that U preserves pullbacks and V preserves pushouts. Since h is a pure monomorphism, it is a directed colimit of split monomorphisms and thus Fh is a directed colimit of split monomorphisms. Since $\mathcal{K}$ is locally finitely presentable (see [2], Corolalry 2.48), Fh is a pure monomorphism. □

3. Pure semisimplicity

Recall that a morphism $h : A \to B$ in a locally finitely presentable category $\mathcal{K}$ is a *pure epimorphism* if any morphism $f : C \to B$ with

C finitely presentable factorizes through h, i.e., $f = hg$ for some $g : C \to A$ (see [3]). Any split epimorphism $f : A \to B$ is pure; recall that being split means the existence of $s : B \to A$ with $fs = \mathrm{id}_B$. By [3], pure epimorphisms in a locally finitely presentable category are precisely directed colimits of split epimorphisms in $\mathcal{K}^{\to}$. This explains why the concept of a pure epimorphism is not dual to that of a pure monomorphism.

An object K of $\mathcal{K}$ is called *pure projective* if for any pure epimorphism $h : A \to B$ and for any morphism $f : K \to B$ there is $g : K \to A$ with $hg = f$. Therefore any finitely presentable object is pure projective.

Lemma 3.1. *Let $\mathcal{K}$ be a locally finitely presentable category. Then an object K is pure projective iff it is a retract of a coproduct of finitely presentable objects.*

Proof. The sufficiency is evident because pure projective objects are clearly closed under coproducts and retracts. Under a retract of K we mean an object L having a split monomorphism $L \to K$. Conversely, let K be pure projective and express it as a directed colimit

$$(k_i : K_i \to K)_{i \in I}$$

of finitely presentable objects K_i. Consider the coproduct

$$(u_i : K_i \to \coprod_{i \in I} K_i)_{i \in I}$$

and the induced morphism $t : \coprod_{i \in I} K_i \to K$ satisfying $tu_i = k_i$ for each $i \in I$. Then t is a pure epimorphism because each $f : C \to K$ with C finitely presentable factorizes through some k_i and thus through t. Since K is pure projective, t splits, i.e., $ts = \mathrm{id}$ for some s and thus K is a retract of $\coprod_{i \in I} K_i$. □

Remark 3.2. In the proof (which goes back to Warfield), we have shown that every locally finitely presentable category has enough pure projectives, i.e., that each object K admits a pure epimorphism $t : L \to K$ with a pure projective domain ($L = \coprod_{i \in I} K_i$ in the proof).

Definition 3.3. A locally finitely presentable category $\mathcal{K}$ will be called *pure semisimple* if each pure epimorphism in $\mathcal{K}$ splits, i.e., if every object is pure projective.

Proposition 3.4. *A locally finitely presentable category $\mathcal{K}$ is pure semisimple iff every object is a retract of a coproduct of finitely presentable objects.*

Proof. By Lemma 3.1, it suffices to show that $\mathcal{K}$ is pure semisimple iff each its object is pure projective. Since each pure epimorphism with a pure projective codomain splits, $\mathcal{K}$ is pure semisimple provided that it has all objects pure projective. The converse follows from Remark 3.2 and Lemma 3.1. □

An object K of a locally finitely presentable category $\mathcal{K}$ is called *abstractly finite* (see [15]) if any morphism $f : K \to \coprod\limits_{i \in I} K_i$ factorizes through a finite subcoproduct, i.e., there are $i_1, \dots, i_n \in I$ with $f = gu$ where $g : K \to \coprod\limits_{j=1}^{n} K_{i_j}$ and $u : \coprod\limits_{j=1}^{n} K_{i_j} \to \coprod\limits_{i \in I} K_i$ is the induced morphism. Each *indecomposable* object K, i.e., an object such that $\hom(K, -) : \mathcal{K} \to \mathbf{Set}$ preserves coproducts, is abstractly finite.

Proposition 3.5. *Let $\mathcal{K}$ be a pure semisimple locally finitely presentable category. Then an object of $\mathcal{K}$ is finitely presentable iff it is abstractly finite.*

Proof. Every finitely presentable object is abstractly finite. Let K be an abstractly finite object and, by 3.4, we express it as a retract of a coproduct of finitely presentable objects $\coprod\limits_{i \in I} K_i$

$$K \underset{r}{\overset{f}{\rightleftarrows}} \coprod K_i$$

$rf = \mathrm{id}_K$. Since f factorizes through a finite subcoproduct $\coprod\limits_{j=1}^{n} K_{i_j}$, K is a retract of $\coprod\limits_{j=1}^{n} K_{i_j}$ and thus it is finitely presentable. □

Coproducts are *universal* in a category $\mathcal{K}$ (see [8]) if, given a coproduct $(u_i : L_i \to \coprod\limits_{i \in I} L_i)_{i \in I}$ and a morphism $f : K \to \coprod\limits_{i \in I} L_i$, the pullbacks

$$\begin{array}{ccc} K_i & \xrightarrow{\ v_i\ } & K \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ L_i & \xrightarrow[\ u_i\ ]{} & \coprod L_i \end{array}$$

make K a coproduct $(v_i : K_i \to K)_{i \in I}$. Coproducts are universal in each category $\mathbf{Set}^{\mathcal{C}}$. In varieties which are not unary, coproducts are rarely universal.

Proposition 3.6. *Let $\mathcal{K}$ be a locally finitely presentable category with universal coproducts. Then $\mathcal{K}$ is pure semisimple iff each of its objects is a coproduct of finitely presentable objects.*

Proof. By 3.4, it suffices to show that coproducts of finitely presentable objects are closed under retracts. Let

$$K \underset{r}{\overset{f}{\rightleftarrows}} \coprod A_i$$

where $rf = \mathrm{id}_K$ and A_i, $i \in I$, are finitely presentable. Forming pullbacks along coproduct injections

$$\begin{array}{ccc} K_i & \xrightarrow{\;v_i\;} & K \\ {\scriptstyle f_i}\downarrow & & \downarrow{\scriptstyle f} \\ A_i & \xrightarrow[\;u_i\;]{} & \coprod A_i \end{array}$$

we get that $K = \coprod\limits_{i \in I} K_i$. Since we have pullbacks

$$\begin{array}{ccc} A_i & \xrightarrow{\;u_i\;} & \coprod A_i \\ {\scriptstyle r_i}\downarrow & & \downarrow{\scriptstyle r} \\ K_i & \xrightarrow[\;v_i\;]{} & \coprod K_i \end{array}$$

we have $r_i f_i = \mathrm{id}_{K_i}$ and thus K_i, $i \in I$, are finitely presentable as retracts of A_i, $i \in I$. □

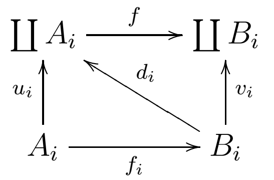
Proposition 3.7. *Let $\mathcal{C}$ be a small category. Then $\mathbf{Set}^{\mathcal{C}}$ is pure semisimple iff each indecomposable object is finitely presentable.*

Proof. Necessity follows from 3.6. Since every object in $\mathbf{Set}^{\mathcal{C}}$ is a coproduct of indecomposable objects (see, e.g., [9], 6.1.5 and the forth example after it), we get the sufficiency. □

Remark 3.8. Hence $\mathbf{Set}^{\mathcal{C}}$ is pure semisimple iff every object is a coproduct of indecomposable finitely presentable objects. The same is true (but much harder to prove) in any category R-**Mod** of left R-modules over a ring R (cf. [11]).

Corollary 3.9. *If $\mathbf{Set}^{\mathcal{C}}$ is pure semisimple then monomorphisms split in $\mathbf{Set}^{\mathcal{C}}$.*

Proof. By 3.7, every morphism f in $\mathbf{Set}^{\mathcal{C}}$ is of the form $\coprod\limits_{i\in I} f_i : \coprod\limits_{i\in I} A_i \to \coprod\limits_{i\in I} B_i$ where $f_i : A_i \to B_i$ and A_i, B_i, $i \in I$ are indecomposable finitely presentable. If f is pure, we have d_i with $d_i f_i = u_i$ for each $i \in I$.



This induces the morphism $d : \coprod B_i \to \coprod A_i$ satisfying $dv_i = d_i$, $i \in I$. We have $dfu_i = dv_i f_i = d_i f_i = u_i$ and thus $df = \mathrm{id}$. □

Theorem 3.10. *Let G be a group. Then the category G-**Set** of left G-sets is pure semisimple iff every subgroup of G is finitely generated.*

Proof. Let G be a group such that G-**Set** is pure semisimple. Let H be a subgroup of G. Recall that ϱ is a left congruence on G if

$$s\varrho t \Rightarrow (us)\varrho(ut)$$

for every s, t, $u \in G$. By [12] (see also [14], 4.39), the left congruence ϱ_H on G generated by $H \times H$ satisfies

$$H = \{x | (x, 1) \in \varrho_H\}.$$

Since G/ϱ_H is a cyclic, i.e., indecomposable G-set, it is finitely presentable (see 3.7). Consequently, ϱ is finitely generated (cf. [14], I.5.30) and thus H is finitely generated.

Conversely, let subgroup of G is finitely generated. Then every left congruence on G be finitely generated and thus every cyclic G-set is finitely presentable. Consider a G-set A and a, $b \in A$. If the 1-generated sub-G-sets Aa and Ab intersect then $Aa = Ab$. Indeed, if $c \in Aa \cap Ab$ then $c = xa = yb$ and thus $a = x^{-1}xa = x'yb \in Ab$. Hence $Aa \subseteq Ab$ and, conversely $Ab \subseteq Aa$. Therefore A is a coproduct of cyclic G-sets and, by 3.6, G-**Set** is pure semisimple. □

Corollary 3.11. *Let G be a commutative group. Then G-**Set** is pure semisimple iff G is finitely generated.*

Remarks 3.12. (1) A ring R has *finite representation type* if it is pure semisimple (i.e., every left R-module is a coproduct of indecomposable finitely presentable left R-modules) and there are, up to isomorphism, finitely many indecomposable left R-modules. The famous pure semisimplicity conjecture asks whether every pure semisimple ring has finite representation type. This is true for commutative rings (see [11]).

Since any subgroup of $(\mathbb{Z}, +)$ is cyclic, $\mathbb{Z}$-**Set** is pure semisimple (using 3.11). But $\mathbb{Z}$ is not of "finite representation type" because the quotient left $\mathbb{Z}$-sets $\mathbb{Z}_n$, $n \in \mathbb{Z}$ yield infinitely many non-isomorphic indecomposable left $\mathbb{Z}$-sets.

(2) By [4], pure monomorphisms split in G-**Set** for a group G iff all subgroups of G are finitely generated. Hence G-**Set** is pure semisimple iff all its objects are pure injective. The same property has the category R-**Mod** of left modules over a ring R because pure epimorphisms are precisely cokernels of pure monomorphisms.

(3) The category $\mathbf{Set}^{\to}$ is not pure semisimple although all of its objects are pure injective. An example of a pure epimorphism in $\mathbf{Set}^{\to}$ which does not split is $(p, p') : f \to g$ (cf. (5) before 2.1) where $p = f : \coprod_{n\in\mathbb{N}} n \to \mathbb{N}$ is induced by the inclusions $n \to \mathbb{N}$ on each summand $(n = \{0, 1, \dots, n-1\})$, $g : \mathbb{N} \to 1$.

(4) If $\mathcal{K}$ be a full reflective subcategory of a locally finitely presentable category $\mathcal{L}$ closed under directed colimits and $\mathcal{L}$ is pure semisimple then $\mathcal{K}$ is pure semisimple as well.

Indeed, $\mathcal{K}$ is locally finitely presentable ([2], 2.41) and the inclusion preserves pure epimorphisms.

4. Perfectness

An object K of a locally finitely presentable category $\mathcal{K}$ will be called *projective* if for every strong epimorphism (cf. [8], 4.3.5) $h : A \to B$ and every morphism $f : K \to B$ there is a morphism $g : K \to A$ with $hg = f$. If $\mathcal{K}$ is a variety then we get usual projective algebras. Since every pure epimorphism in a locally finitely presentable category $\mathcal{K}$ is regular (see [3], Proposition 4) and thus strong, every projective object is pure projective.

An object K of a locally finitely presentable category $\mathcal{K}$ will be called *flat* if every strong epimorphism $L \to K$ is pure (see [19]). We will show that, in varieties, this definition coincides with that from [17].

Lemma 4.1. *An object of a variety is flat iff it is a directed colimit of (finitely presentable) projective objects.*

Proof. Let $(k_i : P_i \to K)_{i\in I}$ be a directed colimit of projectives and consider a strong epimorphism $h : L \to K$. Let $f : X \to K$ be a homomorphism with X finitely presentable. There is $i \in I$ and $f' : X \to P_i$ such that $k_i f' = f$. Moreover, we have $f'' : P_i \to L$ with $hf'' = k_i$. Hence $h(f''f') = f$, which proves that h is a pure epimorphism.

Conversely, let K be flat. Since every variety has enough projectives, there is a strong epimorphism $p : P \to K$ with P projective. Since K is flat, p is a pure epimorphism. Consider $f : X \to K$ where X is finitely presentable. Then $f = pg$ for some $g : X \to P$ and, since every projective object is a directed colimit of finitely presentable projectives, there is a factorization

$$\begin{array}{ccc} P & \xrightarrow{\;p\;} & K \\ {\scriptstyle h}\uparrow & & \uparrow{\scriptstyle f} \\ Q & \xleftarrow[\;u\;]{} & X \end{array}$$

where Q is a finitely presentable projective. Consequently, K is a directed colimit of finitely presentable projectives. □

Definition 4.2. A locally finitely presentable category will be called *perfect* if every object K has a projective cover, i.e., a strong epimorphism $p : P \to K$ with P projective such that each monomorphism $f : P \to P$ with $pf = p$ is an isomorphism.

Remark 4.3. Our definition is equivalent to the standard one saying that each object K admits a strong epimorphism $p : P \to K$ with P projective such that no restriction of p to a proper subobject of P is a strong epimorphism.

In fact, let $p : P \to K$ be a projective cover and consider a monomorphism $u : Q \to P$ such that pu is a strong epimorphism. Since P is projective, there is $v : P \to Q$ such that $puv = p$. Thus uv is an isomorphism and hence u is an isomorphism.

Conversely, consider a strong epimorphism $p : P \to K$ with P projective such that the restriction of p to any proper subobject is not a strong epimorphism. Let $f : P \to P$ satisfy $pf = p$. By [2], 1.61, there is a factorization $f = hg$ where g is a strong epimorphism and h a monomorphism. Since $phg = pf = p$ is a strong epimorphism, h is an isomorphism. Since P is projective, g splits, i.e., $gt = \mathrm{id}$ for some t. Since $phgt = ph$ is a strong epimorphism, t is an isomorphism and thus g is an isomorphism. We have proved that f is an isomorphism.

Proposition 4.4. *Let $\mathcal{K}$ be a locally finitely presentable category having enough projectives and such that each flat object is projective. Then $\mathcal{K}$ is perfect.*

Proof. It follows from [18], 2.5. □

Remark 4.5. For a ring R the following conditions are equivalent (see [6]):

(i) R-**Mod** is perfect;
(ii) every flat left R-module is projective;
(iii) R satisfies the descending chain condition on principal right ideals.

Isbell [13] proved that, for any small category $\mathcal{C}$, $\mathbf{Set}^{\mathcal{C}}$ is perfect iff every flat object is projective. We do not know any example of a perfect locally finitely presentable category where flat objects are not projective.

Proposition 4.6. *In a pure semisimple variety, every flat object is projective.*

Proof. Let K be flat and $(k_i : K_i \to K)_{i\in I}$ be a corresponding directed colimit of finitely presentable projectives (see 4.1). Analogously as in the proof of 3.1, the induced morphism $t : \coprod_{i\in I} K_i \to K$ is a pure epimorphism. Since $\mathcal{K}$ is pure semisimple, t splits and thus K is projective. □

Remark 4.7. In fact, we have proved that, in any variety, an object which is flat and pure projective is projective.

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