On projectivity in locally presentable categories

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Abstract

We show that some fundamental results about projectivity classes, weakly coreflective subcategories and cotorsion theories can be generalized from R-modules to arbitrary locally presentable categories.

1 Introduction

Injectivity in locally presentable categories is well understood (see [AR_1]). The basic result is that a full subcategory $\mathcal{A}$ of a locally presentable category $\mathcal{K}$ is a small-injectivity class (i.e., there is a set $\mathcal{M}$ od morphisms of $\mathcal{K}$ such that $\mathcal{A}$ consists of all objects injective w.r.t. each morphism in $\mathcal{M}$) if and only if $\mathcal{A}$ is accessible and closed in $\mathcal{K}$ under products and λ-directed colimits for some regular cardinal λ. Accessibility of $\mathcal{A}$ can be replaced by $\mathcal{A}$ being also closed under λ-pure subobjects. Here, λ-pure subobjects are precisely λ-directed colimits of split subobjects. This result was re-proved for additive locally presentable categories by H. Krause [K]. Injectivity classes are closely related to weakly reflective subcategories. Every small-injectivity class of a locally presentable category $\mathcal{K}$ is weakly reflective in $\mathcal{K}$ and every weakly reflective full subcategory $\mathcal{A}$ of $\mathcal{K}$ which is closed under split subobjects is an injectivity class (i.e., it consists of all objects injective w.r.t. a class $\mathcal{M}$ of morphisms). Moreover, under set-theoretical Vopěnka's principle, injectivity classes of $\mathcal{K}$ coincide with weakly reflective full subcategories closed under split subobjects and even with full subcategories closed under products and split subobjects. Add that, in the additive setting, weakly reflective subcategories are called covariantly finite (see [AS]).

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Much less is known about projectivity in locally presentable categories. Recently, there have been found a deep result about projectivity in additive locally presentable categories saying that every accessible full subcategory $\mathcal{A}$ of an additive locally presentable category $\mathcal{K}$ which is closed under coproducts and directed colimits is weakly coreflective (see [B]). It generalizes the second method of proving the celebrated flat cover conjecture given in [BBE]. We will show that this can be extended to all locally presentable categories having enough λ-pure quotients. λ-pure quotients were introduced for general locally presentable categories in [AR_2]. The deep part of El Bashir's (and Bican's) proof is to show that additive locally presentable categories have enough λ-pure quotients.

Injectivity classes with injective hulls correspond to stably weakly reflective full subcategories. This observation goes back to Enochs [E] (see also [AR_1] Ex. 4.d); the concept of stable weak reflectivity is older (see Harris [H]). Enochs [E] proved that if a full subcategory of the category of R-modules is weakly coreflective and closed under directed colimits then it is stably weakly coreflective (see also [X]). El Bashir [B] has extended this result to Grothendieck categories. We will show that it can be proved for all locally finitely presentable categories. This is quite surprising (see [T_1] for known general results about injectivity classes with injective hulls).

In a category R-**Mod** of R-modules, injectivity and projectivity classes are often induced by cotorsion theories. This can be extended to general locally presentable categories by using weak factorization systems. Weak factorization systems originated in homotopy theory and were introduced by Beke [Be]. More about weak factorization systems can be found in [AHRT] and their relation to cotorsion theories was observed in [R]. In fact, they are also present in [EJ] where Proposition 7.2.2 shows how stable weak coreflections can be used to get a weak factorization. We show that this phenomenon does not depend on additivity.

2 Weak coreflectivity

Recall that a full subcategory $\mathcal{A}$ of a category $\mathcal{K}$ is called *weakly coreflective* if each object K in $\mathcal{K}$ has a weak coreflection, i.e., a morphism $c_K : K^* \to K$ where K^* is in $\mathcal{A}$ such that every morphism $f : A \to K$ with A in $\mathcal{A}$ factorizes (not necessarily uniquely) through c_K. Every weakly coreflective subcategory is closed under coproducts in $\mathcal{K}$.

A morphism $f : K \to L$ in $\mathcal{K}$ is called a λ-*pure quotient* (for a regular cardinal λ) provided that it is projective w.r.t. λ-presentable objects (cf. [AR_2]). Explicitly, for every λ-presentable object X, all morphisms $X \to$

L factorize through f. If $\mathcal{K}$ is locally λ-presentable then λ-pure quotients are precisely λ-directed colimits of split quotients in the category $\mathcal{K}^{\to}$ of $\mathcal{K}$-morphisms. In additive locally λ-presentable categories, λ-split quotients are precisely cokernels of λ-pure subobjects and, conversely, λ-pure subobjects are precisely kernels of λ-pure quotients.

Definition 2.1. We say that a category $\mathcal{K}$ *has enough λ-pure quotients* if for each object K there is, up to isomorphism, only a set of morphisms $f : L \to K$ such that $f = h \cdot g$, g λ-pure epimorphism implies that g is an isomorphism.

Remark 2.2. (1) Every locally presentable category has enough λ-pure subobjects in the sense that for each object K there is, up to isomorphism, only a set of morphism $f : K \to L$ such that $f = g \cdot h$, g λ-pure monomorphism implies that g is an isomorphism. It follows from the fact that there is a regular cardinal μ such that f factorizes as

$$f : K \xrightarrow{h} M \xrightarrow{g} L$$

where g is a λ-pure monomorphism and M is μ-presentable (see [AR$_1$] 2.33). Since g should be an isomorphism, L is μ-presentable. However, up to isomorphism, there is only a set of μ-presentable objects.

(2) If $\mathcal{K}$ has enough λ-pure quotients then it has enough μ-pure quotients for every regular cardinal $\mu > \lambda$. In fact, every μ-pure quotient is λ-pure.

The locally presentable category of graphs does not have enough λ-pure quotients for any λ (see [AR$_2$] Remark 11). An additive locally presentable category has enough λ-pure quotients for any regular cardinal λ (see [B] 2.1).

Proposition 2.3. *Let $\mathcal{K}$ be locally presentable category having enough λ-pure quotients for all regular cardinals λ. Let $\mathcal{A}$ be an accessible full subcategory of $\mathcal{K}$ which is closed under coproducts and directed colimits. Then $\mathcal{A}$ is weakly coreflective in $\mathcal{K}$.*

Proof. Let K be an object in $\mathcal{K}$ and take a morphism $f : A \to K$ with A in $\mathcal{A}$. Consider factorizations

$$f : A \xrightarrow{g} B \xrightarrow{h} K$$

where g is a regular epimorphism and B is in $\mathcal{A}$. Up to isomorphism, these factorization form a set which can be ordered by means of

$$(g, h) \le (g', h') \quad \text{iff } g' \text{ factorizes through } g.$$

The resulting ordered set $\mathcal{S}$ has directed joins because a directed colimit (in fact, any colimit) of regular epimorphism in $\mathcal{K}^{\to}$ is a regular epimorphism and $\mathcal{A}$ is closed under directed colimits in $\mathcal{K}$. Hence $\mathcal{S}$ contains a maximal element (g_0, h_0). Since any λ-pure epimorphism is regular (see [AR$_2$] Proposition 5) and $\mathcal{A}$ is closed under λ_0-pure quotients for some λ_0 ([AR$_2$] Proposition 14), we have

$$h_0 = t\,u\,, u\ \lambda_0\text{-pure epimorphism} \Rightarrow u \text{ isomorphism.}$$

Since $\mathcal{K}$ has enough λ-pure quotients, there is only a set of such morphisms h_0. Since $\mathcal{A}$ is closed under coproducts, it is weakly coreflective. □

Remark 2.4. Under Vopěnka's principle, every full subcategory of a locally presentable category which is closed under directed colimits is accessible. Hence we can drop the assumption that $\mathcal{A}$ is accessible.

A weakly coreflective subcategory $\mathcal{A}$ of a $\mathcal{K}$ is called *stably weakly coreflective* if each object K in $\mathcal{K}$ has a weak reflection $c_K : K^* \to K$ such that any morphism $f : K^* \to K^*$ with $c_K \cdot f = c_K$ is an isomorphism. A stable weak coreflection is unique up to an isomorphism.

Theorem 2.5. *Let $\mathcal{K}$ be a locally finitely presentable category and $\mathcal{A}$ a weakly coreflective full subcategory of $\mathcal{K}$ which is closed under directed colimits in $\mathcal{K}$. Then $\mathcal{A}$ is stably weakly coreflective.*

Proof. Since the comma category $\mathcal{K} \downarrow K$ is locally finitely presentable for each K in $\mathcal{K}$ and $\mathcal{A} \downarrow K$ (= the full subcategory of $\mathcal{K} \downarrow K$ consisting of $A \to K$, $A \in \mathcal{A}$) is closed under directed colimits in $\mathcal{K} \downarrow K$ provided that $\mathcal{A}$ is closed under directed colimits in $\mathcal{K}$, it suffices to prove the following result. Let $\mathcal{K}$ be a full subcategory of a locally finitely presentable category closed under directed colimits and having a weakly terminal object. Then $\mathcal{A}$ has a stably weak terminal object T (i.e., every $f : T \to T$ is an isomorphism).

Assume that $\mathcal{A}$ do not have a stably weak terminal object and consider a weakly terminal object T. Since $\mathcal{K}$ is well powered, there is a regular cardinal λ such that $T \times T$ does not have an increasing chain of subobjects of length λ. We define a chain $t_{ij} : T_i \to T_j$, $i \leq j \leq \lambda$, of weakly terminal objects of $\mathcal{A}$ by the following transfinite induction. We put $T_0 = T$ and in a limit step j we take a colimit $t_{ij} : T_i \to T_j$, $i < j$. In an isolated step we put $T_{i+1} = T_i$. Having $f : T_i \to T_i$ which is not a monomorphism, we put $t_{ii+1} = f$. If all morphisms $T_i \to T_i$ are monomorphism, there is $g : T_i \to T_i$ which is not a strong epimorphisms (otherwise, T_i would be stably weakly terminal) and we put $t_{ii+1} = g$.

Since directed colimits commute with finite limits in $\mathcal{K}$, we have a directed colimit $t_{ji} \times t_{ji} : T_j \times T_j \to T_i \times T_i$, $j < i$, for each limit ordinal $i \leq \lambda$. It implies that no $T_i \times T_i$, $i < \lambda$ has an increasing chain of subobjects of length λ. In fact, it suffices to prove it for limit ordinals $i < \lambda$. Assume that it holds for all $j < i$ and that X_k, $k < \lambda$, is an increasing chain of subobjects of $T_i \times T_i$. We get chains of subobjects X_{kj} of $T_j \times T_j$, $j < i$ by taking pullbacks

$$\begin{tikzcd} T_j \times T_j \arrow[r, "t_{ji} \times t_{ji}"] & T_i \times T_i \\ X_{kj} \arrow[u] \arrow[r, "x_{kj}"'] & X_k \arrow[u] \end{tikzcd}$$

Since all chains X_{kj}, $k < \lambda$ stabilize and $i < \lambda$, there is $k_0 < \lambda$ such that $X_{k_0 j} = X_{kj}$ for all $k_0 \leq k < \lambda$ and $j < i$. Thus it suffices to know that $x_{kj} : X_{kj} \to X_k$, $j < i$, is a directed colimit for all $k < \lambda$. But this is a general property of locally finitely presentable categories which can be proved as follows. Let $y_{kj} : X_{kj} \to Y_k$, $j < i$, be a colimit of X_{kj}, $j < i$ and $h_k : Y_k \to X_k$ the induced morphism. Since every morphism $z : Z \to X_k$ with Z finitely presentable factorizes through some x_{kj}, h_k is an isomorphism.

We will show that ordinals $i < \lambda$ such that $t_{i\lambda}$ is a monomorphism are cofinal in λ. Let $j < i < \lambda$ and consider kernel pairs

$$\begin{tikzcd} S_j \arrow[r, shift left, "t'_{j\lambda}"] \arrow[r, shift right, "t''_{j\lambda}"'] & T_j \arrow[r, "t_{j\lambda}"] & T_\lambda \end{tikzcd}$$

$$\begin{tikzcd} S_{ji} \arrow[r, shift left, "t'_{ji}"] \arrow[r, shift right, "t''_{ji}"'] & T_j \arrow[r, "t_{ji}"] & T_i \end{tikzcd}$$

Since directed colimits commute with finite limits in $\mathcal{K}$, we get that S_j is a directed union of S_{ji}, $j < i < \lambda$. Since $T_j \times T_j$ does not have an increasing chain of subobjects of length λ, there is $i < j < \lambda$ such that $S_j = S_{ji}$. Now, consider $j < \lambda$ and define a sequence of ordinals $i_0 \leq i_1 \leq \cdots \leq i_n \leq \cdots$, $n < \omega$, by putting $i_0 = j$ and $S_{i_n} = S_{i_n i_{n+1}}$. Let $i_\omega = \sup\limits_{n<\omega} i_n$. Since

$$S_{i_\omega} \cong \bigcup_{n<\omega} S_{i_n} = \bigcup_{n<\omega} S_{i_n i_{n+1}} \cong T_{i_\omega} \,,$$

$t_{i_\omega \lambda}$ is a monomorphism.

If $t_{i\lambda}$ is a monomorphism then $t_{i\,i+1}$ is a monomorphism too and, following the construction, it is not a strong epimorphism. Hence T_λ contains an increasing chain of subobjects of length λ. Since T is weakly terminal, there is a morphism $h : T_\lambda \to T$. If $t_{i\lambda}$ is a monomorphism then, following the construction, all morphisms $f : T_i \to T_i$ are monomorphisms. Hence $h \cdot t_{i\lambda}$ is a monomorphism because we have a monomorphism

$$T_i \xrightarrow{ht_{i\lambda}} T \xrightarrow{t_{0i}} T_i \,.$$

Consequently, h is a monomorphism and we get an increasing chain of subobjects of T of length λ, which is a contradiction. $\square$

3 Weak factorization systems

Let $\mathcal{K}$ be a category and $f : A \to B$, $g : C \to D$ morphisms such that in each commutative square

$$\begin{tikzcd} A \arrow[r, "u"] \arrow[d, "f"'] & C \arrow[d, "g"] \\ B \arrow[r, "v"'] & D \end{tikzcd}$$

there is a diagonal $d : B \to C$ with $d \cdot f = u$ and $g \cdot d = v$. One says that g has the *right lifting property* w.r.t. f and that f has a *left lifting property* w.r.t. g. For a class $\mathcal{H}$ of morphisms of $\mathcal{K}$ we put

$$\mathcal{H}^\square = \{g \mid g \text{ has the right lifting property w.r.t. each } f \in \mathcal{H}\} \text{ and}$$

$$^\square\mathcal{H} = \{f \mid f \text{ has the left lifting property w.r.t. each } g \in \mathcal{H}\}.$$

A *weak factorization system* $(\mathcal{L}, \mathcal{R})$ in $\mathcal{K}$ consists of two classes $\mathcal{L}$ and $\mathcal{R}$ of morphisms of $\mathcal{K}$ satisfying

(WF1) $\mathcal{R} = \mathcal{L}^\square$ and $\mathcal{L} = {}^\square\mathcal{R}$ and

(WF2) any morphism h of $\mathcal{K}$ has a factorization $h = g \cdot f$ with $f \in \mathcal{L}$ and $g \in \mathcal{R}$.

If we denote by ${}_A\mathcal{R}$ the class of all morphisms in the comma-category $A \downarrow \mathcal{K}$ such that the underlying $\mathcal{K}$-morphism belongs to $\mathcal{R}$ then a morphism $f : A \to B$ belongs to ${}^\square\mathcal{R}$ iff the object (A, f) is projective w.r.t. ${}_A\mathcal{R}$ in $A \downarrow \mathcal{K}$. Therefore a pair $(\mathcal{L}, \mathcal{R})$ satisfying (WF1) is a weak factorization system iff for every (A, h) in $A \downarrow \mathcal{K}$ there is an ${}_A\mathcal{R}$-morphism $g : f \to h$ from an ${}_A\mathcal{R}$-projective object (A, f). Consequently, the full subcategory ${}^\triangle({}_A\mathcal{R})$ of $A \downarrow \mathcal{K}$ consisting of all ${}_A\mathcal{R}$-projective objects is weakly coreflective.

Moreover, this full subcategory coincides with the full subcategory of $A \downarrow \mathcal{K}$ consisting of morphisms belonging to $\mathcal{L}$. Therefore, if $\mathcal{K}$ has an initial object 0 and a terminal object 1, then a weak factorization system $(\mathcal{L}, \mathcal{R})$ yields a weakly coreflective full subcategory ${}^{\triangle}\mathcal{R}$ of $\mathcal{K}$ and, dually, a weakly reflective full subcategory $\mathcal{L}^{\triangle}$ of $\mathcal{K}$. Weak ${}^{\triangle}\mathcal{R}$-coreflections are given by factorization

$$0 \to K^* \xrightarrow{c_K} K\,.$$

The pairs $({}^{\triangle}\mathcal{R}, \mathcal{L}^{\triangle})$ given by $(\mathcal{L}, \mathcal{R})$ satisfying (WF1) can be viewed as a generalization of cotorsion theories to non-additive setting.

Recall that a pair $(\mathcal{F}, \mathcal{C})$ of classes of R-modules is called a *cotorsion theory* if

$$\begin{aligned} \mathcal{C} &= \{C \mid \operatorname{Ext}(F, C) = 0 \quad \text{for all} \quad F \in \mathcal{F}\} \quad \text{and} \\ \mathcal{F} &= \{F \mid \operatorname{Ext}(F, C) = 0 \quad \text{for all} \quad C \in \mathcal{C}\}\,. \end{aligned}$$

We call a monomorphism $f : A \to B$ in R-**Mod** an *$\mathcal{F}$-monomorphism* if coker $f : B \to F$ has $F \in \mathcal{F}$. The class of all $\mathcal{F}$-monomorphisms is denoted *$\mathcal{F}$-Mono*. Analogously a *$\mathcal{C}$-epimorphism* is an epimorphism $g : A \to B$ such that ker $g : C \to A$ has $C \in \mathcal{C}$ and *$\mathcal{C}$-Epi* denotes the class of all $\mathcal{C}$-epimorphisms. Then the definition of a cotorsion theory can be rewritten as

$$\mathcal{C} = (\mathcal{F}\text{-}\mathit{Mono})^{\triangle} \quad \text{and} \quad \mathcal{F} = {}^{\triangle}(\mathcal{C}\text{-}\mathit{Epi})\,.$$

(see [EJ] 7.2, Ex. 2 or [R] 4.3). Morphisms from $(\mathcal{F}\text{-}\mathit{Mono})^{\square}$ are called $\mathcal{F}$-fibrations and morphisms from ${}^{\square}(\mathcal{C}\text{-}\mathit{Epi})$ $\mathcal{C}$-cofibrations.

Proposition 3.1. *Let $(\mathcal{F}, \mathcal{C})$ be a cotorsion theory in R-**Mod**. Then ($\mathcal{F}$-Mono, $\mathcal{C}$-Epi) satisfies* (WF1).

Proof. Following [R] 4.4, $(\mathcal{F}\text{-}\mathit{Mono})^{\square} = \mathcal{C}\text{-}\mathit{Epi}$ because $\mathcal{F}$ contains all projective R-modules. Hence $\mathcal{F}\text{-}\mathit{Mono} \subseteq {}^{\square}(\mathcal{C}\text{-}\mathit{Epi})$. Consider $f : A \to B$ in ${}^{\square}(\mathcal{C}\text{-}\mathit{Epi})$ and let $e : A \to E$ be an embedding of A into an injective R-module E. Since $\mathcal{C}$ contains all injective R-modules, we get a diagonal t in the square

$$\begin{tikzcd} A \arrow[r, "e"] \arrow[d, "f"'] & E \arrow[d] \\ B \arrow[r] \arrow[ur, dashed, "t"] & 0 \end{tikzcd}$$

Thus f is a monomorphism. We have to show that coker $f : B \to F$ has $F \in \mathcal{F}$, i.e., that every $\mathcal{C}$-epimorphism $g : X \to F$ splits. But this follows from the existence of a diagonal in the square

$$\begin{tikzcd} A \arrow[r, "0"] \arrow[d, "f"'] & X \arrow[d, "g"] \\ B \arrow[r, "\operatorname{coker} f"'] \arrow[ur, dashed] & F \end{tikzcd}$$

□

Remark 3.2. A cotorsion theory $(\mathcal{F}, \mathcal{C})$ is said to be have *enough injectives* if any morphism $M \to 0$ satisfies (WF2). Dually, $(\mathcal{F}, \mathcal{C})$ has *enough projectives* if any morphism $0 \to M$ satisfies (WF2). Cotorsion theories having enough projectives and enough injectives are also called *complete*. The basic result is that every cotorsion theory cogenerated by a set is complete (see [ET], or [E] 7.4.1). The proof of Theorem 4.5 in [R] show that if $(\mathcal{F}, \mathcal{C})$ is cogenerated by a set then $(\mathcal{F}\text{-}\mathit{Mono}, \mathcal{C}\text{-}\mathit{Epi})$ satisfies (WF2) for all morphisms, i.e., that it is a weak factorization system. Moreover, this weak factorization system is cofibrantly generated.

Definition 3.3. A weak factorization system $(\mathcal{L}, \mathcal{R})$ will be called a *stable weak factorization system* if (WF2) is strengthened to

(WF2$_s$) any morphism h of $\mathcal{K}$ has a factorization $h = g\,f$ such that $f \in \mathcal{L}$, $g \in \mathcal{R}$ and any t with $tf = f$ and $gt = g$ is an isomorphism.

In a stable weak factorization system, full subcategories ${}^{\triangle}({}_A\mathcal{R})$ are stably weakly coreflective, and dually, full subcategories $(\mathcal{L}_B)^{\triangle}$ are stably weakly reflective.

Remark 3.4. Stable weak factorization system were introduced by Tholen [T$_2$] using essential morphism instead of stable weak reflections. They were called essential weak factorization systems. The use of stability immediately yields that every left essential weak factorization system is right essential.

Proposition 3.5. *Let $\mathcal{K}$ be a category with pushouts and $\mathcal{L}$, $\mathcal{R}$ classes of morphisms satisfying* (WF1) *and such that full subcategories ${}^{\triangle}({}_A\mathcal{R})$ are stably weakly coreflective for each A in $\mathcal{K}$. Then $(\mathcal{L}, \mathcal{R})$ is a stable weak factorization system.*

Proof. Consider $h : A \to B$ a take a stable weak coreflection $c_h : h^* \to h$ of h to ${}^{\triangle}({}_A\mathcal{R})$:

$$\begin{tikzcd} A \arrow[rr, "h"] \arrow[dr, "h^*"'] & & B \\ & B^* \arrow[ur, "c_h"'] & \end{tikzcd}$$

We have to prove that $c_h \in \mathcal{R}$. Let

$$\begin{tikzcd} X \arrow[r, "u"] \arrow[d, "l"'] & B^* \arrow[d, "c_h"] \\ Y \arrow[r, "v"'] & B \end{tikzcd} \qquad (*)$$

be a commutative square with $l \in \mathcal{L}$. Fill it with a pushout of l and u

$$\begin{tikzcd} X \arrow[rr, "u"] \arrow[dd, "l"'] & & B^* \arrow[dl, "\bar{l}"'] \arrow[dd, "c_h"] \\ & P \arrow[dr, "t"] & \\ Y \arrow[ur, "\bar{u}"] \arrow[rr, "v"'] & & B \end{tikzcd}$$

and take the induced morphism t. Since $\mathcal{L}$ is closed under taking pushout and composition (se [Hi] 8.2.9 and 8.2.5), we have $\bar{l} \in \mathcal{L}$ and $\bar{l}h^* \in \mathcal{L}$ ($h^* \in \mathcal{L}$ as an ${}_A\mathcal{R}$-projective object). Hence $\bar{l}h^* \in {}^{\triangle}({}_A\mathcal{R})$ and since c_h is a stable weak coreflection, we get w making the following diagram commutative

$$\begin{tikzcd} & B^* \arrow[dr, "c_h"] & \\ A \arrow[ur, "h^*"] \arrow[dr, "\bar{l}h^*"'] & & B \\ & P \arrow[uu, "w"] \arrow[ur, "t"'] & \end{tikzcd}$$

Thus $w\bar{l}$ is an isomorphism and $g = (w\bar{l})^{-1}w\bar{u}$ satisfies

$$g\,l = (\bar{w}l)^{-1}w\bar{u}l = (\bar{w}l)^{-1}w\bar{l}u = u$$

and

$$c_h g = c_h(\bar{w}l)^{-1}w\bar{u} = c_h w\bar{u} = t\bar{u} = v$$

because $c_h w\bar{l} = t\bar{l} = c_h$. Hence g is a diagonal in $(*)$. □

Corollary 3.6. *Let $\mathcal{K}$ be a locally finitely presentable category and $\mathcal{L}$, $\mathcal{R}$ classes of morphisms satisfying (WF1) and such that ${}^{\triangle}({}_A\mathcal{R})$ is weakly coreflective and closed under directed colimits in $A \downarrow \mathcal{K}$ for each A in $\mathcal{K}$. Then $(\mathcal{L}, \mathcal{R})$ is a stable weak factorization system.*

It follows from 2.5 and 3.5.

Theorem 3.7. *Let $(\mathcal{F},\mathcal{C})$ be a cotorsion theory in R-**Mod** such that $\mathcal{F}$ is closed under directed colimits. Then $(\mathcal{F}\text{-}Mono, \mathcal{C}\text{-}Epi)$ is a stable weak factorization system.*

Proof. Following 3.1, $(\mathcal{F}\text{-}Mono,\ \mathcal{C}\text{-}Epi)$ satisfies (WF1). Since $\mathcal{F}$ is closed under directed colimits in R-**Mod**, $(\mathcal{C}\text{-}Epi)^{\triangle}$ is closed under directed colimits in $K \downarrow R$-**Mod** (because it consists of $\mathcal{F}$-monomorphisms $K \to X$). We will show that $({}_K\mathcal{C}\text{-}Epi)^{\triangle}$ is weakly coreflective in $K \downarrow R$-**Mod**. Then the result follows from 3.6.

Since $\mathcal{F}$ is closed under coproducts, it is weakly coreflective (see [B]). Let $f : K \to X$ be an $\mathcal{F}$-monomorphism and consider the diagram

$$\begin{tikzcd} K \arrow[r, "f"] \arrow[dr, "\ker v"'] & X \arrow[r, "\operatorname{coker} f"] & Y \\ & Z \arrow[u, "u"] \arrow[r, "v"'] & Y^* \arrow[u, "c_Y"'] \end{tikzcd}$$

where c_Y is a weak $\mathcal{F}$-coreflection of Y and the square is a pullback. Then $u\cdot\ker v = f$ and thus $(Z, \ker v)$ is a weak $({}_K\mathcal{C}\text{-}Epi)^{\triangle}$-coreflection of (X, f). $\square$

An analogous result was proved in [R] under stronger assumptions about $\mathcal{F}$ (cf. Theorem 4.5).

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