CONTINUOUS CATEGORIES REVISITED

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Abstract. Generalizing the fact that Scott's continuous lattices form the equational hull of the class of all algebraic lattices, we describe an equational hull of LFP, the category of locally finitely presentable categories, over CAT. Up to a set-theoretical hypothesis this hull is formed by the category of all *precontinuous* categories, i.e., categories in which limits and filtered colimits distribute. This concept is closely related to the continuous categories of P. T. Johnstone and A. Joyal.

I. Introduction

I.A Operations on the Category LFP

Our paper has been initiated by the question whether the 2-category LFP of locally finitely presentable categories of Gabriel and Ulmer is, in any reasonable sense, equational over CAT. Recall that the morphisms considered by Gabriel and Ulmer between locally finitely presentable categories are the functors preserving limits and filtered colimits – or, equivalently, the finitary right adjoints. With natural transformations as 2-cells, this yields a 2-category LFP considered as a concrete category over CAT via the (non-full) embedding

$$U : \mathrm{LFP} \longrightarrow \mathrm{CAT}\ .$$

The question of equationality of LFP can now be formulated precisely: is the functor U pseudomonadic? The answer, as we prove below, is negative.

Since we speak about equational hull, we must first explain what operations on the 2-category LFP are and what are equations. The concept of operation follows the idea of Lawvere's dissertation where, given a category $\mathscr{C}$ concrete over **Set**, i.e., with a forgetful functor $U : \mathscr{C} \to \mathbf{Set}$, a k-ary operation ω is a natural transformation $\omega : U^k \to U$ (assigning, thus, to every object C of $\mathscr{C}$ a k-ary operation map ω_C on the set UC so that all morphisms f of $\mathscr{C}$ fulfil: Uf preserves the k-ary operations map in the usual sense). Here k can be any cardinal number, but we often want to restrict ourselves to k finite. The first idea of a generalization to LFP would be: a k-ary operation is a natural transformation $\omega : U^k \to U$. Here k can be any category, but we often want to restrict ourselves to k small. Example: the formation of limits of type k should be a k-ary operation ω. Indeed, all morphisms of LFP preserve limits, thus, they preserve ω. Alas, this is only true non-strictly, i.e., the operation "limits of type k" is not a natural transformation (= preserved by functors strictly) but a pseudonatural transformations. This leads us to the following

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Typeset by $\mathcal{A}\mathcal{M}\mathcal{S}$-TEX

Definition. By a **k-ary operation** on LFP is meant a pseudonatural transformation $\omega : U^k \to U$.

Thus, ω assigns to every locally finitely presentable category $\mathcal{A}$ an "operation functor" $\omega_{\mathcal{A}} : \mathcal{A}^k \to \mathcal{A}$ and to every finitary right adjoint $F : \mathcal{A} \to \mathcal{B}$ a natural isomorphism $\widehat{\omega}_F$ as follows:

$$\begin{tikzcd}
\mathcal{A}^k \arrow[r, "\omega^{\mathcal{A}}"] \arrow[d, "F^k"'] \arrow[dr, phantom, "\overset{\widehat{\omega}_F}{\Longrightarrow}"] & \mathcal{A} \arrow[d, "F"] \\
\mathcal{B}^k \arrow[r, "\omega^{\mathcal{B}}"'] & \mathcal{B}
\end{tikzcd}$$

with natural coherence conditions.

Examples. (1) "Limits of type k" is a k-ary operation on LFP for every small category k.

(2) "Colimits of type k" is a k-ary operation on LFP for every small filtered category k.

(3) The *trivial operation* (corresponding to variables, considered as the appropriate projection operations, in universal algebra) is ev_x, evaluation at x, for every object x of the category k:

$$ev_x^{\mathcal{A}} : \mathcal{A}^k \to \mathcal{A}\,, \quad F \longmapsto Fx \qquad \text{for all} \quad F \in \mathcal{A}^k\,.$$

One of the main results of our paper is that the above three types of operations are exhaustive: all operations of small arities on LFP are generated (in a sense made precise below) by the trivial operations, the limit-operations and the filtered-colimit-operations. This result requires a certain restriction on the set theory we work with. It holds whenever the following axiom

(R) every uniform ultrafilter is regular

is assumed – we discuss this axiom in Section V below. And it fails to hold assuming that arbitrarily large measurable cardinals exist.

We will also consider operations of large arities. However, we do not have any nice generating families for them.

I.B Equational Laws for LFP-Operations

We not only prove that limits and filtered colimits generate all operations of small arities on LFP, we also describe all equational properties of those operations. They, of course, include

(i) Equational properties of limits (in general categories).

Recall that completeness is equational over CAT in the sense that for the pseudomonad

$$Lim$$

of free limit-completion on CAT the 2-category of algebras is the 2-category of complete categories. Since locally finitely presentable categories are complete, they satisfy (i). Analogously for

(ii) Equational properties of filtered colimits (in general categories)

expressing algebras over Grothendieck's pseudomonad

$$Ind$$

of free completion w.r.t. filtered colimits on CAT. And the third obvious equational property holding in every locally finitely presentable category is

(iii) Finite limits commute with filtered colimits.

Is this all? No, there is one item missing: all locally finitely presentable categories have the following property

(iv) P roducts distribute over filtered colimits.

This means, for a given category $\mathscr{C}$ with products and filtered colimits, that for any set D_i $(i \in I)$ of filtered diagrams in $\mathscr{C}$ the canonical morphism

$$\operatorname{colim} \prod_{i\in I} D_i d_i \longrightarrow \prod_{i\in I} \operatorname{colim} D_i d_i$$

is an isomorphism. (This is less "popular" than commutation of finite limits with filtered colimits. But e.g. Grothendieck and Verdier work with the property (iv) explicitly in [GV].)

Now what we prove assuming (R) above, is that the equational properties of (i)–(iv) generate all equations between operations of small arities on LFP. Let us stress here that equations

$$\omega_1 \approx \omega_2 \qquad (\text{for } \omega_1, \omega_2 : U^k \to U)$$

are, once again, meant non-strictly: the meaning is not that the components $\omega_1^{\mathcal{A}}$ and $\omega_2^{\mathcal{A}}$ are equal for every locally finitely presentable category $\mathcal{A}$, but that a coherent collection of natural isomorphisms $i^{\mathcal{A}} : \omega_1^{\mathcal{A}} \xrightarrow{\cong} \omega_2^{\mathcal{A}}$ is given.

How is the above result proved? We introduce first an equational hull of LFP w.r.t. operations of small and large arities. This is performed by analogy to the case of algebraic lattices (a posetal shadow of LFP) which we explain in the next subsection: we find a pseudomonad $\mathbb{D}^*$ on CAT whose category of algebras is an equational hull of LFP. We then turn to the small-core $\mathbb{D}^*_{\text{small}}$ of that pseudomonad $\mathbb{D}^*$ and argue that algebras for that pseudomonad form an equational hull of LFP w.r.t. operations of small arities. By a recent result of Adámek, Koubek and Trnková [AKT] we derive the formula

$$(*) \qquad \mathbb{D}^*_{\text{small}}\mathcal{K} = \mathit{Ind}(\mathit{Lim}\,\mathcal{K}) \qquad \text{for all } \mathcal{K} \text{ small},$$

assuming the set-theoretical axiom (R). It turns out that the two pseudomonads *Lim* (free completion under limits) and *Ind* (free completion under filtered colimits) have a distributive law – here we were surprised that Grothendieck and Verdier have not come to that conclusion in [GV]: they prove that if a functor F preserves finite limits then so does $\mathit{Ind}F$, but actually, this is true (and needed) for all limits, see Lemma V.5 below. This distributive law turns $\mathit{Ind} \circ \mathit{Lim}$ into a pseudomonad biequivalent to $\mathbb{D}^*_{\text{small}}$. Thus, algebras of $\mathit{Ind} \circ \mathit{Lim}$ form an equational hull of LFP w.r.t. operations of small arities. But those algebras are just algebras of the lifting of *Ind* to the category of all *Lim*-algebras (i.e., to the category of all complete categories). We call them *precontinuous categories*. Explicitly:

Definition. A category $\mathcal{A}$ with limits and filtered colimits is called *precontinuous* provided that the functor

$$\underset{\leftarrow}{\operatorname{colim}} : \mathit{Ind}\,\mathcal{A} \longrightarrow \mathcal{A}$$

(substituting formal filtered colimits with actual filtered colimits in $\mathcal{A}$) preserves limits.

This concept is closely related to continuous categories of Johnstone and Joyal [JJ] defined as categories $\mathcal{A}$ with filtered colimits such that $\overleftarrow{\mathrm{colim}}$ is a right adjoint. The precise relationship of the concepts is discussed in Section V.

The last step we use is the following result of Adámek, Rosický and Vitale [ARV]: precontinuity is equivalent to the above conditions (iii) and (iv). Thus, categories satisfying (i)–(iv) are precisely the precontinuous categories, and assuming (R), they form an equational hull of LFP w.r.t. operations of small arities. This is analogous to the results of [ALR] characterizing the equational hull of the category of all varieties.

I.C From Continuous Lattices to Precontinuous Categories

Recall that the fundamental concept of algebraic lattice in universal algebra is nothing else than a poset that is, as a category, locally finitely presentable. That is, a complete lattice in which every element is a (directed) join of compact (= finitely presentable) elements. It has been known since 1970's that algebraic lattices do not form a variety, and that Scott's continuous lattices are an equational hull. Recall that a lattice is continuous iff (infinite) meets distribute over directed joins – which is precisely the conditions (iv) above.

Now algebraic lattices are precisely the lattices

$$\mathit{Ind}\, L \qquad L \text{ a (meet-)semilattice}$$

where $\mathit{Ind}\, L$, a free completion under directed joins, is the well known lattice of ideals of L. Homomorphisms for algebraic lattices are the maps preserving all meets and all directed joins. Consider now the category **Slat** of semilattices and homomorphisms as a monadic category over **Pos**

$$\mathbf{Slat} = \mathbf{Pos}^{\mathbb{D}}$$

for the monad $\mathbb{D}$ of free semilattices (= semilattices of all finitely generated $\downarrow$-sets in a given poset). We obtain an adjoint situation using the two-element chain 2 as a dualizing object:

$$\begin{tikzcd} \mathbf{Slat}^{\mathrm{op}} \arrow[r, shift left, "\mathbf{Slat}(-,2)"] & \mathbf{Pos} \arrow[l, shift left, "\mathbf{Pos}(-,2)"] \end{tikzcd}$$

This adjoint situation defines a monad $\mathbb{D}^*$ over **Pos** (the filter monad) whose algebras are precisely the *continuous lattices*, i.e., complete lattices L such that the directed joins completely distribute over meets. This has been proved by A. Day [D] and O. Wyler [W]. We present a different proof here: the monad $\mathbb{D}^*$ of the above adjunction is a composite of the monad *Ind* (of free directed joins) with the monad *Meet* (of free meets) in the sense of Beck's distributive laws [B]. This implies immediately that algebras for this monad $\mathbb{D}^*$ are *Ind*-liftings to complete semilattices (= *Meet*-algebras), and this is precisely the category of continuous lattices. Observe that the comparison functor

$$K : \mathbf{Slat}^{\mathrm{op}} \longrightarrow \mathbf{Pos}^{\mathbb{D}^*}$$

is full and faithful and maps $\mathbf{Slat}^{\mathrm{op}}$ to the category of algebraic lattices. Observe further that K is a right adjoint to $\mathbf{Pos}^{\mathbb{D}^*}(-,2) : \mathbf{Pos}^{\mathbb{D}^*} \longrightarrow \mathbf{Slat}^{\mathrm{op}}$; thus, algebraic lattices are reflective in their equational hull.

We now generalize this from posets to categories by substituting

(i) the category of algebraic lattices with LFP,
(ii) **Pos** with CAT,
(iii) **Slat** with $\mathrm{CAT}_{\mathrm{lex}}$, the 2-category of lex (= finitely complete) categories, lex (= finite-limit preserving) functor and natural transformations and
(iv) *Meet* with *Lim* and the ideal lattice with Grothendieck's completion. Let us recall that for small categories $\mathcal{A}$ we have

$$Lim\mathcal{A} \cong (\mathbf{Set}^{\mathcal{A}})^{\mathrm{op}}$$

and for all categories $\mathcal{A}$ we can describe $Ind\mathcal{A}$ as the category of all *small* lex functors from $\mathcal{A}^{\mathrm{op}}$ to **Set**. Small means: a small colimit of hom-functors.

Using the category **Set** as the dualizing object, we obtain a 2-adjoint situation

$$\mathrm{CAT}_{\mathrm{lex}}^{\mathrm{op}} \underset{\mathrm{CAT}(-,\mathbf{Set})}{\overset{\mathrm{CAT}_{\mathrm{lex}}(-,\mathbf{Set})}{\rightleftarrows}} \mathrm{CAT}$$

defining a 2-monad $\mathbb{D}^*$ over CAT. A characterization of $\mathbb{D}^*$-algebras is an open problem.

For *small* categories $\mathcal{A}$ we have

$$\mathbb{D}^*\mathcal{A} = \mathrm{CAT}_{\mathrm{lex}}(\mathbf{Set}^{\mathcal{A}}, \mathbf{Set}) ,$$

and assuming (R) every lex functor from $\mathbf{Set}^{\mathcal{A}}$ to **Set** is small, as proved in [AKT]. Thus $\mathbb{D}^*\mathcal{A}$ is a free completion of $Lim\,\mathcal{A} = (\mathbf{Set}^{\mathcal{A}})^{op}$ under filtered colimits and we get $(*)$.

Set-theoretical foundations. In the present paper we are not concerned with set theory (except for the discussion of the axioms (R) and (M) restricted to section Section V below), but here we want to make clear what (very meagre) requirements are needed below. A chosen universe of small sets is assumed satisfying the Axiom of Choice (AC). In a higher universe *categories* (not assumed locally small in general) live. And when "categories" outside of this universe are considered, they are called *quasicategories*; example: CAT. Analogously with 2-categories and 2-quasicategories. But where a 2-quasicategory is biequivalent to a 2-category (such as LFP, due to Gabriel-Ulmer duality), we call them 2-categories. And we assume that every category is an ∞-*filtered* union of its small subcategories (i.e., a large union which is λ-filtered for every cardinal λ). All limits and colimits are meant to be small (unless explicitely stated large).

II. Continuous Lattices Revisited

II.1 The Monad $\mathbb{D}$ of Semilattices.

We recall shortly that the forgetful functor

$$U : \mathbf{Slat} \to \mathbf{Pos}$$

of the category **Slat** of (meet-) semilattices and their homomorphisms into the category **Pos** of poset has a left adjoint: For each poset X a free semilattice over X,

$$\eta_X^{\mathbb{D}} : X \to \mathbb{D}(X)\,,$$

is given by the poset $\mathbb{D}(X)$ of all finitely generated $\uparrow$-sets (i.e., all sets $\uparrow M$, where $M \subseteq X$ is finite), ordered by dual inclusion, and $\eta_X^{\mathbb{D}}$ is the principal-filter embedding $x \mapsto \uparrow\{x\}$.

This gives rise to the free-semilattice monad $\mathbb{D} = (\mathbb{D}, \mu^{\mathbb{D}}, \eta^{\mathbb{D}})$ over **Pos** whose category of algebras is (isomorphic to) **Slat**. Moreover, $\mathbb{D}$ is an enriched monad over the (cartesian closed) category **Pos**, i.e., the derived maps

$$\hom(X, Y) \mapsto \hom(\mathbb{D}(X), \mathbb{D}(Y))$$

are order-preserving. Consequently, every $\mathbb{D}$-algebra S yields an enriched contravariant hom-functor

$$\mathbf{Slat}(-, S) : \mathbf{Slat}^{\mathrm{op}} \longrightarrow \mathbf{Pos}\,.$$

From $S = 2$, the chain $0 < 1$, this assigns to each semilattice A the poset

$$\mathbf{Slat}(A, 2) = \text{ all filters on A}$$

ordered by inclusion; here we identify a homomorphism $h : A \to 2$ with the set $h^{-1}(1)$ which is a filter (i.e., $\uparrow$-set closed under finite meets).

II.2 The Dual Monad $\mathbb{D}^*$. Since the monad $\mathbb{D}$ is enriched over **Pos**, exponentation can be transferred from **Pos** to $\mathbb{D}$-algebras. In particular, we get the canonical functor

$$2^{(-)} : \mathbf{Pos} \to \mathbf{Slat}^{\mathrm{op}}$$

assigning to every poset X the semilattice 2^X of all $\uparrow$-sets ordered by inclusion. This functor is left adjoint to $\mathbf{Slat}(-, 2)$: for posets X and semilattices A we have the following natural bijections:

$$\begin{array}{cl}
X \longrightarrow \mathbf{Slat}(A, 2) & \text{[in Pos]} \\
\hline
X \times A \longrightarrow 2 & [\wedge\text{-preserving in the } 2^{\text{nd}} \text{ coordinate}] \\
\hline
A \longrightarrow 2^X & [\text{in } \mathbf{Slat}] \\
\hline
2^X \longrightarrow A & [\text{in } \mathbf{Slat}^{\mathrm{op}}]
\end{array}$$

This adjoint situation

$$2^{(-)} \dashv \mathbf{Slat}(-, 2)$$

leads to a new monad

$$\mathbb{D}^* = (\mathbb{D}^*, \eta^*, \mu^*)$$

called *dual monad* of $\mathbb{D}$ (w.r.t. the dualizer 2). Here $\mathbb{D}^*$ assigns to each poset X the poset

$$\mathbb{D}^* X = \mathbf{Slat}(2^X, 2) = \text{ all filters in } 2^X$$

and $\eta^*_X : X \to \mathbb{D}^* X$ is the principal-filter embedding: $\eta^*_X(x)$ is the filter of all $\uparrow$-sets containing x.

II.3 $\mathbb{D}^*$-Algebras are easy to describe when we observe that (a) the well-known monads *Ind* (of free directed joins) and *Meet* (of free meets) distribute in Beck's sense and (b) the monad $\mathbb{D}^*$ is the corresponding composite

$$\mathbb{D}^* \cong \mathit{Ind} \circ \mathit{Meet}.$$

In more detail, for each poset X we denote by

$$\eta^{\mathit{Meet}}_X : X \to \mathit{Meet}(X)$$

the principal-filter embedding into the poset $\mathit{Meet}(X)$ of all $\uparrow$-sets of X ordered by dual inclusion (i.e., dual ordering of 2^X). This is well-known to be a free completion of X under meets. And by

$$\eta^{\mathit{Ind}}_X : X \to \mathit{Ind}(X)$$

we denote the principal-ideal embedding into the poset $\mathit{Ind}(X)$ of all *ideals* (i.e., directed $\downarrow$-sets), ordered by inclusion. This is well-known to be a free completion of X under directed joins.

II.4 Beck's Distributive Laws.

We recall here briefly the concept of distributive law of a monad $\mathbb{S}$ over monad $\mathbb{T}$ (on the same category $\mathcal{K}$) as presented by J. Beck [B]. Let $(\mathbb{S}, \eta^{\mathbb{S}}, \mu^{\mathbb{S}})$ and $(\mathbb{T}, \eta^{\mathbb{T}}, \mu^{\mathbb{T}})$ be monads on the category $\mathcal{K}$.

A distributive law of $\mathbb{S}$ over $\mathbb{T}$ is a natural transformation

$$\ell : \mathbb{S}\mathbb{T} \to \mathbb{T}\mathbb{S}$$

such that

(i) $\ell \cdot \eta^{\mathbb{S}}\mathbb{T} = \mathbb{T}\eta^{\mathbb{S}}$,

(ii) $\ell \cdot \mathbb{S}\eta^{\mathbb{T}} = \eta^{\mathbb{T}}\mathbb{S}$,

(iii) $\ell \cdot \mu^{\mathbb{S}}\mathbb{T} = \mathbb{T}\mu^{\mathbb{S}} \cdot \ell\mathbb{S} \cdot \mathbb{S}\ell$,

and

(iv) $\ell \cdot \mathbb{S}\mu^{\mathbb{T}} = \mu^{\mathbb{T}}\mathbb{S} \cdot \mathbb{T}\ell \cdot \ell\mathbb{T}$

This defines a *composite monad*

$$\mathbb{T}\mathbb{S} = (\mathbb{T}\mathbb{S}, \eta^{\mathbb{T}}\mathbb{S} \cdot \eta^{\mathbb{S}}, \mu^{\mathbb{T}\mathbb{S}})$$

where $\mu^{\mathbb{T}\mathbb{S}}$ is uniquely determined by the following properties:

(a) $\mu^{\mathbb{T}\mathbb{S}} \cdot \mathbb{S}(\eta^{\mathbb{T}} \cdot \mathbb{T}\eta^{\mathbb{S}})\mathbb{T} = \mathrm{id}_{\mathbb{T}\mathbb{S}}$,

and

(b) $\eta^{\mathbb{T}}\mathbb{S} : \mathbb{S} \to \mathbb{T}\mathbb{S}$ and $\eta^{\mathbb{S}}\mathbb{T} : \mathbb{T} \to \mathbb{T}\mathbb{S}$ are monad morphisms.

Explicitly:

$$\mu^{\mathbb{T}\mathbb{S}} = \mu^{\mathbb{T}}\mathbb{S} \cdot \mathbb{T}\mathbb{T}\mu^{\mathbb{S}} \cdot \mathbb{T}\ell\mathbb{S}\,.$$

(Conversely, every composite monad satisfying (a) and (b) gives rise to a unique distributive law.)

By a *lifting* of the monad $\mathbb{T}$ into the category of $\mathbb{S}$-algebras is meant a monad

$$\widetilde{\mathbb{T}} = (\widetilde{\mathbb{T}}, \widetilde{\eta}, \widetilde{\mu}) \quad \text{over} \quad \mathcal{K}^{\mathbb{S}}$$

such that

(α) $U^{\mathbb{S}}\widetilde{\mathbb{T}} = \mathbb{T}U^{\mathbb{S}}$,

(β) $U^{\mathbb{S}}\widetilde{\eta} = \eta U^{\mathbb{S}}$

and

(γ) $U^{\mathbb{S}}\widetilde{\mu} = \mu U^{\mathbb{S}}$.

Every distributive law ℓ defines a lifting by the rule

$$\widetilde{\mathbb{T}}(A, \alpha) = (\mathbb{T}A, \mathbb{T}\alpha \cdot \ell_A)\,,$$
$$\widetilde{\eta}(A, \alpha) = \eta_A$$

and

$$\widetilde{\mu}(A, \alpha) = \mu_A\,.$$

Conversely, every lifting gives rise to a unique distributive law.

$$\mathbb{S}\mathbb{T} \xrightarrow{\mathbb{S}\mathbb{T}\eta^{\mathbb{S}}} \mathbb{S}\mathbb{T}\mathbb{S} = \mathbb{S}\mathbb{T}U^{\mathbb{S}}F^{\mathbb{S}} = \mathbb{S}U^{\mathbb{S}}\widetilde{\mathbb{T}}F^{\mathbb{S}} =$$
$$U^{\mathbb{S}}F^{\mathbb{S}}U^{\mathbb{S}}\widetilde{\mathbb{T}}F^{\mathbb{S}} \xrightarrow{U^{\mathbb{S}}\varepsilon^{\mathbb{S}}\widetilde{\mathbb{T}}F^{\mathbb{S}}} U^{\mathbb{S}}\widetilde{\mathbb{T}}F^{\mathbb{S}} = \mathbb{T}U^{\mathbb{S}}F^{\mathbb{S}} = \mathbb{T}\mathbb{S}$$

where $F^{\mathbb{S}}$ is left adjoint to $U^{\mathbb{S}}$ and $\varepsilon^{\mathbb{S}} : F^{\mathbb{S}}U^{\mathbb{S}} \to Ind_{\mathcal{K}^{\mathbb{S}}}$ is a counit.

There is an isomorphism

$$\mathcal{K}^{\mathbb{T}\mathbb{S}} \cong (\mathcal{K}^{\mathbb{S}})^{\widetilde{\mathbb{T}}}$$

commuting with the natural forgetful functors into $\mathcal{K}$.

II.5 Distributivity of *Meet* over *Ind*.

Consider $\mathcal{K} = \mathbf{Pos}$, $\mathbb{T} = Ind$, and $\mathbb{S} = Meet$. The following lemma gives a lifting of *Ind* over complete meet-semilattices, i.e., over $\mathbf{Pos}^{\mathbb{S}}$, and therefore shows a distributive law of *Meet* over *Ind*:

Lemma.

(i) *$Ind(A)$ is a complete (semi)lattice for each poset A;*

(ii) *$Ind(f)$ is a complete-semilattice homomorphism whenever f is such;*

(iii) *η_A^{Ind} and μ_A^{Ind} are complete-semilattice homomorphisms for each complete lattice A.*

Proof. (i) $Ind(A)$ is closed under meets (=intersections) in Z^A.

(ii) $Ind(f) : Ind(A) \to Ind(B)$ assigns to every ideal Z the ideal $\downarrow f(Z)$. Given $Z = \bigcap_{t\in T} Z_t$ in $Ind(A)$ we prove $\bigcap_{t\in T} \downarrow f(Z_t) \subseteq \downarrow f(Z)$– thus, $Ind(f)$ preserves meets. For every element $z \in \bigcap_{t\in T} \downarrow f(Z_t)$ we choose $z_t \in Z_t$ with $z \leq f(z_t)$ for each $t \in T$ to conclude that

$$z \leq \bigwedge_{t \in T} f(z_t) = f(\bigwedge_{t \in T} z_t) \in f(Z)\,,$$

thus, $z \in \downarrow f(Z)$.

(iii) The principal-ideal embedding η_A^{Ind} obviously preserves meets. And μ_A^{Ind} preserves meets because it is right adjoint to $Ind(\eta_A^{Ind})$: given ideals $Z \in Ind(A)$ and $Y = \{Y_t\} \in Ind(Ind(A))$ then

$$Z \leq \mu_A^{Ind}(Y)$$

means $Z \subseteq \bigcup_{t \in T} Y_t$, and since Y is a $\downarrow$-set, this is equivalent to $\downarrow\{z\} \in Y$ for each $z \in Z$, i.e., to

$$\downarrow \eta_A^{Ind}(Z) \leq Y\,.$$

$\square$

Remark. This corollary yields a distributive law

$$\ell : Meet \circ Ind \to Ind \circ Meet;$$

since $Ind(Meet(A))$ is a complete lattice, $Ind(\eta_A^{Meet})$ extends uniquely to

$$\ell_A : Meet(Ind(A)) \to Ind(Meet(A))$$

preserving joins.

Corollary. *$Ind \circ Meet$ is a monad over* **Pos** *whose algebras are Ind-liftings to complete semilattices, i.e., complete (semi)lattices L with directed joins such that the directed-join map*

$$Ind(L) \to L$$

preserves meets. (These are precisely continuous lattices L.) And $Ind \circ Meet$-homomorphisms are the maps preserving meets and directed joins. (These are the homomorphisms of continuous lattices.)

Notation. Cont denotes the category of continuous lattices and their homomorphisms, that is, the category of $Ind \circ Meet$-algebras.

II.6 Corollary. *The monads $\mathbb{D}^*$ and $Ind \circ Meet$ are isomorphic.*

In fact, their Kleisli categories are concretely isomorphic since a free $\mathbb{D}^*$-algebra on any poset X is just the free $Ind \circ Meet$-algebra on X:

$$\mathbf{Slat}(\mathbf{Pos}(X, 2), 2) = \mathbf{Slat}(Meet(X), 2) = Ind(Meet(X)).$$

III. Equational Doctrines

III.1 Enriched Monads in CCC. Here we just briefly recall some concepts introduced (for the case of the category $\mathcal{C}$ of all categories in [L₂]) in the more general setting of an arbitrary cartesian closed category $\mathcal{C}$. By an *enriched monad* $\mathbb{D}$ over $\mathcal{C}$ we understand an enriched functor $\mathbb{D} : \mathcal{C} \to \mathcal{C}$ together with enriched natural transformations $\eta : 1_{\mathcal{C}} \to \mathbb{D}$ and $\mu : \mathbb{D}\mathbb{D} \to \mathbb{D}$ satisfying the usual monad axioms. The category

$$\mathcal{C}^{\mathbb{D}}$$

of (strict) algebras and homomorphisms is then also enriched over $\mathcal{C}$: the internal hom-functor

$$\mathrm{Hom}_{\mathbb{D}} : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \times \mathcal{C}^{\mathbb{D}} \to \mathcal{C}$$

is defined by means of the following equalizer for $\mathbb{D}$-algebras (A, α) and (B, β):

$$\begin{tikzcd}
\mathrm{Hom}_{\mathbb{D}}((B,\beta),(A,\alpha)) \arrow[r, "e"] & A^B \arrow[rr, "A^\beta"] \arrow[dr, "\mathbb{D}"] & & A^{(\mathbb{D}B)} \\
& & (\mathbb{D}A)^{(\mathbb{D}B)} \arrow[ur, "\alpha^{(\mathbb{D}B)}"'] &
\end{tikzcd}$$

Example. $\mathcal{C} = \mathbf{Pos}$, $\mathbb{D}$ the monad of finitely generated $\uparrow$-sets (see I.1): this is an enriched monad with
$\mathrm{Hom}_{\mathbb{D}}((B, \leq), (A, \leq))$ = all order-preserving maps in B^A ordered pointwise.

III.2 The Dual Monad $\mathbb{D}^*$.
We now fix a $\mathbb{D}$-algebra

$$\mathbb{D}S \xrightarrow{\sigma} S$$

called *dualizer*, and observe that the enriched hom-functor

$$\mathrm{Hom}_{\mathbb{D}}(-, (S, \sigma)) : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \to \mathcal{C}$$

has a left adjoint which can be constructed as follows (see [L₂]): for every object $C \in \mathcal{C}$ we denote by

$$\mathrm{eval}^*_C : C \to (\mathbb{D}S)^{\mathbb{D}(S^C)}$$

the composite of the evaluation map $C \to S^{(S^C)}$ and the $\mathbb{D}$-map $S^{(S^C)} \mapsto \mathbb{D}S^{\mathbb{D}(S^C)}$, and we consider the following adjunctions:

$$\frac{\dfrac{C \xrightarrow{\mathrm{eval}^*_C} (\mathbb{D}S)^{\mathbb{D}(S^C)}}{C \times \mathbb{D}(S^C) \longrightarrow \mathbb{D}S}}{\mathbb{D}(S^C) \xrightarrow{\varrho_C} (\mathbb{D}S)^C}$$

We now define as functor

$$S^{(-)} : \mathcal{C} \to (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}}$$

on objects by assigning to $C \in \mathcal{C}$ the $\mathbb{D}$-algebra

$$\mathbb{D}(S^C) \overset{\varrho_C}{\to} (\mathbb{D}S)^C \overset{\sigma^C}{\to} S^C$$

and to every morphism $f : C \to \bar{C}$ in $\mathcal{C}$ the $\mathbb{D}$-homomorphism

$$S^f : (S^{\bar{C}}, \sigma^{\bar{C}} \cdot \varrho_{\bar{C}}) \to (S^C, \sigma^C \cdot \varrho_C) \, .$$

Then we get an enriched adjoint situation

$$S^{(-)} \to \mathrm{Hom}_{\mathbb{D}}(-, S)$$

(the argument in this general case is analogous to that in II.2 above).

Notation. The enriched monad generated by the last adjoint situation is denoted by $\mathbb{D}^* = (\mathbb{D}^*, \eta^*, \mu^*)$ and is called the *dual monad* of $\mathbb{D}$ w.r.t. the dualizer (S, σ). Thus,

$$\mathbb{D}^* C = \mathrm{Hom}_{\mathbb{D}}(S^C, S)$$

III.3 Equational Hull of the category dual to $\mathcal{C}^{\mathbb{D}}$, the category of $\mathbb{D}$-algebras w.r.t. the "forgetful" functor $\mathrm{Hom}_{\mathbb{D}}(-, (S, \sigma)) : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \to \mathcal{C}$ is given by the comparison functor

$$K : (\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \to \mathcal{C}^{\mathbb{D}^*} \, .$$

Obviously, this is a right adjoint to an attempted inverse

$$K^* = \mathrm{Hom}_{\mathbb{D}^*}(-, S).$$

Of course, in general neither composite is the identity, but if one of them is, then the other is a closure operation or "hull". This further left adjoint was mentioned in I. for the case of posets.

Examples. (i) In the above-mentioned example we have $\mathcal{C} = \mathbf{Pos}$, $\mathbb{D} =$ the semilattice monad (thus, $(\mathcal{C}^{\mathbb{D}})^{\mathrm{op}} \cong$ the category of algebraic lattices) and K the full embedding into the category of continuous lattices.

(ii) Put $\mathcal{C} = \mathbf{Set}$ and $\mathbb{D}$ the semilattice monad, again, i.e., $\mathbb{D}X$ is the set of all finite subsets of X with $\eta^{\mathbb{D}}_X$ the singleton map and $\mu^{\mathbb{D}}_X$ the union map. Using the dualizing object 2 again, we obtain an adjoint situation

$$\begin{tikzcd} \mathbf{Slat}^{\mathrm{op}} \arrow[rr, shift left=2, "\mathbf{Slat}(-,2)"] \arrow[rr, phantom, "\top"] & & \mathbf{Set} \arrow[ll, shift left=2, "\mathbf{Set}(-,2)"] \end{tikzcd}$$

This generates as $\mathbb{D}^*$ the *filter monad* on $\mathbf{Set}$, see [D]: $\mathbb{D}^* X$ is the set of all filters on the set X (= on the semilattice 2^X) whose Eilenberg-Moore category is $\mathbf{Cont}$, and the comparison functor $K : \mathbf{Slat}^{\mathrm{op}} \to \mathbf{Cont}$ has, again, as image the category of algebraic lattices.

(iii) For $\mathcal{C} = \mathbf{Set}$ and $\mathbb{D} = \mathit{Ind}$ we obtain an adjoint situation

$$\begin{tikzcd} \mathbf{Set}^{op} \arrow[rr, shift left=2, "\mathbf{Set}(-,2)"] \arrow[rr, phantom, "\top"] & & \mathbf{Set} \arrow[ll, shift left=2, "\mathbf{Set}(-,2)"] \end{tikzcd}$$

whose dual monad $\mathbb{D}^*$ (given by $X \mapsto 2^{2^X}$) is the monad of complete atomic Boolean algebra. Here the comparison functor $K : \mathbf{Set}^{\mathrm{op}} \to \mathbf{Set}^{\mathbb{D}^*}$ is an equivalence.

III.4 Pseudomonads and Algebras. (1) Recall the concept of a *pseudomonad* on a 2-category $\mathcal{K}$, as introduced by B. Day and R. Street [DS]; we use the notation of F. Marmolejo [M$_2$]. A pseudomonad $\mathbb{D} = (\mathbb{D}, d, m, \eta, \beta, \mu)$ consists of a bifunctor

$$\mathbb{D} : \mathcal{K} \to \mathcal{K}$$

with 1-cells

$$d : \mathrm{id}_{\mathcal{K}} \to \mathbb{D} \quad \text{and} \quad m : \mathbb{D} \cdot \mathbb{D} \to \mathbb{D}$$

together with invertible 2-cells

$$\begin{tikzcd}
\mathbb{D} \arrow[r, "d\mathbb{D}"] \arrow[dr, "Id_{\mathbb{D}}"'] & \mathbb{D}\mathbb{D} \arrow[d, "m"] \arrow[dl, phantom, "\Leftarrow \beta"] \arrow[dr, phantom, "\Leftarrow \eta"] & \mathbb{D} \arrow[l, "\mathbb{D}d"'] \arrow[dl, "Id_{\mathbb{D}}"] \\
& \mathbb{D} &
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\mathbb{D}\mathbb{D} \arrow[r, "\mathbb{D}m"] \arrow[d, "m\mathbb{D}"'] \arrow[dr, phantom, "\Leftarrow \mu"] & \mathbb{D}\mathbb{D} \arrow[d, "m"] \\
\mathbb{D}\mathbb{D} \arrow[r, "m"'] & \mathbb{D},
\end{tikzcd}$$

such that two equations of the expected type must be satisfied.

(2) Recall further the concept of an *algebra* for a pseudomonad $\mathbb{D}$: it is a quadruple

$$(X, x, \psi, \chi)$$

consisting of
  an object (0-cell) X,
  a morphism (1-cell) $x : \mathbb{D}X \to X$
and
  invertible 2-cells

$$\begin{tikzcd}
X \arrow[r, "dX"] \arrow[dr, "Id_X"'] & \mathbb{D}X \arrow[d, "x"] \arrow[dl, phantom, "\Leftarrow \psi"] \\
& X
\end{tikzcd}
\qquad
\begin{tikzcd}
\mathbb{D}\mathbb{D}X \arrow[r, "\mathbb{D}x"] \arrow[d, "mX"'] \arrow[dr, phantom, "\Leftarrow \chi"] & \mathbb{D}X \arrow[d, "x"] \\
\mathbb{D}X \arrow[r, "x"'] & X,
\end{tikzcd}$$

This data must satisfy two expected equations.

We denote by

$$\mathcal{K}^{\mathbb{D}}$$

the 2-category of $\mathbb{D}$-algebras. Its 1-cells, called *homomorphisms*, from (X, x, ψ, χ) to (Z, z, ξ, θ) are pairs (h, ϱ) where

$$h : X \to Z$$

is a 1-cell of $\mathcal{K}$ and

$$\begin{tikzcd}
\mathbb{D}X \arrow[r, "\mathbb{D}h"] \arrow[d, "x"'] \arrow[dr, phantom, "\Leftarrow \rho"] & \mathbb{D}Z \arrow[d, "z"] \\
X \arrow[r, "h"'] & Z
\end{tikzcd}$$

is an invertible 2-cell, such that two expected equations are satisfied.

The 2-cells of $\mathcal{K}^{\mathbb{D}}$ are defined as follows:

Given homomorphisms (h,ρ), $(h',\rho') : (X,x,\psi,\chi) \longrightarrow (Z,z,\zeta,\theta)$, a 2-cell

$$\xi : (h,\rho) \longrightarrow (h',\rho')$$

is a 2-cell $\xi : h \longrightarrow h'$ such that $(\xi \circ x)\cdot\rho = \rho'\cdot(z\circ \mathbb{D}\xi)$. Vertical composition is the obvious one.

Horizontal composition: for $(h,\rho) : (\psi,\chi) \longrightarrow (\zeta,\theta)$ and $(k,\pi) : (\zeta,\theta) \longrightarrow (\tau,\sigma)$ we define $(k,\pi)\circ(h,\rho) = (k\circ h, (k\circ\rho)\cdot(\pi\circ\mathbb{D}h))$.

IV. Gabriel-Ulmer Duality Revisited

IV.1 The Pseudomonad $\mathbb{D}$ of Lex Categories. Generalizing the situation of $U : \mathbf{Slat} \to \mathbf{Pos}$ in II.1, we consider here the forgetful functor (non-full inclusion)

$$U : \mathrm{CAT}_{\mathrm{lex}} \to \mathrm{CAT}$$

where CAT is the 2-quasicategory of all categories (and functors, and natural transformations), and $\mathrm{CAT}_{\mathrm{lex}}$ is the sub-2-quasicategory of all categories with finite limits (and functors preserving finite limits, and natural transformations).

This functor U has a left adjoint: for every category $\mathcal{X}$ we consider a free completion of $\mathcal{X}$ under finite limits. We obtain a 2-adjoint situation and denote by $\mathbb{D} = (\mathbb{D}, \eta^{\mathbb{D}}, \mu^{\mathbb{D}})$ the pseudomonad on CAT it generates.

The category $\mathrm{CAT}^{\mathbb{D}}$ of $\mathbb{D}$-algebras (see III.4) is biequivalent to $\mathrm{CAT}_{\mathrm{lex}}$ – this easily follows from the fact that $\mathbb{D}$ is a (typical example of a) co-KZ-doctrine, see [K], [M_1].

IV.2 The Dual Pseudomonad $\mathbb{D}^*$. We use the category **Set** of (small) sets as the dualizing objects. We thus consider the enriched hom-functor

$$\mathrm{CAT}_{\mathrm{lex}}(-,\mathbf{Set}) : \mathrm{CAT}_{\mathrm{lex}}^{\mathrm{op}} \to \mathrm{CAT}$$

assigning to every lex category $\mathcal{A}$ the category

$$\mathrm{LEX}\,\mathcal{A} = CAT_{\mathrm{lex}}(\mathcal{A},\mathbf{Set})$$

of all lex functors in $\mathbf{Set}^{\mathcal{A}}$. Since $\mathbb{D}$ is an enriched monad over CAT, exponentation is transferred from CAT to $\mathrm{CAT}^{\mathbb{D}}$, and we obtain the canonical 2-functor

$$\mathbf{Set}^{(-)} = \mathrm{CAT}(-,\mathbf{Set}) : \mathrm{CAT} \to \mathrm{CAT}_{\mathrm{lex}}^{\mathrm{op}}\ .$$

Analogously to II.2, the 2-functors above form a 2-adjoint situation: for a category $\mathcal{X}$ and a lex category $\mathcal{A}$ we have the following natural bijections:

$$\mathcal{X} \xrightarrow{\ F\ } \mathrm{LEX}\ \mathcal{A} \qquad [\text{in}\,\mathrm{CAT}]$$

$$\mathcal{X}\times\mathcal{A} \longrightarrow \mathbf{Set} \qquad [\text{lex in the } 2^{\text{nd}} \text{ coordinate}]$$

$$\mathcal{A} \xrightarrow{\ F'\ } \mathbf{Set}^{\mathcal{X}} \qquad [\text{in}\,\mathrm{CAT}_{\mathrm{lex}}]$$

$$\mathbf{Set}^{\mathcal{X}} \longrightarrow \mathcal{A} \qquad [\text{in } \mathrm{CAT}_{\mathrm{lex}}^{\mathrm{op}}]$$

This adjoint situation

$$\mathbf{Set}^{(-)} \dashv \mathrm{LEX}(-)$$

generates the dual pseudomonad

$$\mathbb{D}^* = (\mathbb{D}^*, \eta^*, \mu^*)$$

over CAT. Thus, $\mathbb{D}^*$ is the 2-functor defined on objects (categories) $\mathcal{X}$ by

$$\mathbb{D}^* \mathcal{X} = \mathrm{LEX}(\mathbf{Set}^{\mathcal{X}})$$

on 1-cells (functors) $F : \mathcal{X} \to \mathcal{Y}$ by

$$\mathbb{D}^* F : H \longmapsto H \cdot \mathbf{Set}^F \text{ for each lex } H : \mathbf{Set}^{\mathcal{X}} \to \mathbf{Set}\,,$$

and on 2-cells (natural transformations) $f : F \to F'$ by

$$(\mathbb{D}^* f)_H = H * \mathbf{Set}^f\,.$$

The unit

$$\eta^*_{\mathcal{X}} : \mathcal{X} \to \mathrm{LEX}(\mathbf{Set}^{\mathcal{X}})$$

is given by $X \mapsto ev_X$, where

$$ev_X = \mathbf{Set}^{\mathcal{X}}(\mathcal{X}(X,-),-)$$

is the evaluation-at-X functor. And the co-unit

$$\mu^*_{\mathcal{X}} : \mathrm{LEX}(\mathbf{Set}^{\mathbb{D}^* \mathcal{X}}) \to \mathrm{LEX}(\mathbf{Set}^{\mathcal{X}})$$

is defined by means of the evaluation functor

$$ev : \mathbf{Set}^{\mathcal{X}} \to \mathbf{Set}^{\mathrm{LEX}(\mathbf{Set}^{\mathcal{X}})}$$

by

$$\mu^*_{\mathcal{X}}(G) = G \cdot ev \text{ for each lex } G : \mathbf{Set}^{\mathbb{D}^* \mathcal{X}} \to \mathbf{Set}\,.$$

Notation. For every functor

$$F : \mathcal{X} \to \mathrm{LEX}\,\mathcal{A} \qquad (\mathcal{X} \in \mathrm{CAT},\ \mathcal{A} \in \mathrm{CAT}_{\mathrm{lex}})$$

we denote by

$$\widehat{F} : \mathbb{D}^* \mathcal{X} \to \mathrm{LEX}\,\mathcal{A}$$

the functor which to every $H \in \mathbb{D}^* \mathcal{X} = \mathrm{LEX}(\mathbf{Set}^{\mathcal{X}})$ assigns the composite of the lex functor $F' : \mathcal{A} \to \mathbf{Set}^{\mathcal{X}}$ (see IV.2) with H:

$$\widehat{F} : H \to H \cdot F' \text{ for all } H \in \mathbb{D}^* \mathcal{X}\,.$$

Observation. (1) For each functor $F : \mathcal{X} \to Y$, the functor

$$\mathbb{D}^* F : H \longmapsto H \cdot \mathbf{Set}^F$$

preserves limits and filtered colimits (even those large ones that exist in $\mathbb{D}^* \mathcal{X}$). In fact, $\mathbb{D}^* F$ preserves all object-wise (large or small) limits, and $\mathbb{D}^* \mathcal{X}$ is closed under all existing limits in $\mathbf{Set}^{\mathbf{Set}^{\mathcal{X}}}$; analogously with filtered colimits.

(2) Also $\eta^*_{\mathcal{X}}$ and $\mu^*_{\mathcal{X}}$ preserve limits and filtered colimits (including all existing large ones). For $\eta^*_{\mathcal{X}}$ this follows from the formula $\eta^*_{\mathcal{X}} X = \mathbf{Set}^{\mathcal{X}}(\mathcal{X}(X, -), -)$ and the fact that $\mathrm{LEX}\,\mathbf{Set}^{\mathcal{X}}$ is closed under limits and filtered colimits in $\mathbf{Set}^{\mathbf{Set}^{\mathcal{X}}}$. For $\mu^*_{\mathcal{X}} : G \mapsto G \cdot ev$ this is also evident.

(3) Finally, each $\widehat{F} : \mathbb{D}^* \mathcal{X} \to \mathrm{LEX}\,\mathcal{A}$ preserves (all existing) limits and filtered colimits.

IV.3 Open Problem. Characterize the category $\mathrm{CAT}^{\mathbb{D}^*}$ of $\mathbb{D}^*$-algebras.

Remark. The answer to the above problem could be called a full equational hull of LFP. Here "full" refers to the possibility of forming k-ary operations where k is an arbitrary (possibly large) category.

In what follows we will aim at describing an equational hull of LFP w.r.t. small-ary operations.

IV.4 Comparison Functor

$$K : \mathrm{CAT}^{\mathrm{op}}_{\mathrm{lex}} \to \mathrm{CAT}^{\mathbb{D}^*} .$$

This functor assigns to every lex category $\mathcal{A}$ the $\mathbb{D}^*$-algebra whose underlying category is $\mathrm{LEX}\,\mathcal{A}$, and whose operation-map is the functor

$$\widehat{Id} : \mathbb{D}^* (\mathrm{LEX}\,\mathcal{A}) \to \mathrm{LEX}\,\mathcal{A}$$

(see Notation IV.2), thus,

$$K\mathcal{A} = (\mathrm{LEX}\,\mathcal{A}, \widehat{Id}) .$$

For every lex functor $F : \mathcal{A} \to \mathcal{B}$ we have

$$K(F) : (\mathrm{LEX}\,\mathcal{B}, \widehat{Id}) \to (\mathrm{LEX}\,\mathcal{A}, \widehat{Id})$$

defined by

$$K(F) : H \longmapsto H \cdot F \qquad \text{for all lex } H : \mathcal{B} \to \mathbf{Set}$$

For reasons analogous to the above (IV.2), $K(F)$ preserves limits and filtered colimits for each F.

IV.5 Gabriel-Ulmer Duality. The image of the comparison functor

$$K : \mathrm{CAT}^{\mathrm{op}}_{\mathrm{lex}} \to \mathrm{CAT}^{\mathbb{D}^*}$$

restricted to small categories is precisely LFP. That is, denote by

$$\mathbf{Cat}_{\text{lex}}$$

the 2-category of all small lex categories (and functors preserving finite limits and natural transformations). Then $K[\mathbf{Cat}_{\text{lex}}^{\text{op}}]$ is biequivalent to LFP. In fact:

(a) for every small lex category $\mathcal{A}$ the algebra $K\mathcal{A} = (\mathrm{LEX}\,\mathcal{A}, \widehat{Id})$ is determined by the (locally finitely presentable) category $\mathrm{LEX}\,\mathcal{A}$. And conversely, every object of LFP is isomorphic to one of the form $\mathrm{LEX}\,\mathcal{A}$, $\mathcal{A}$ small and lex;

(b) for every lex functor $F : \mathcal{A} \to \mathcal{B}$ the homomorphism $K(F) : \mathrm{LEX}\,\mathcal{B} \to \mathrm{LEX}\,\mathcal{A}$ preserves limits and filtered colimits (see IV.4). The main ingredient of the Gabriel-Ulmer duality is, then, the following converse: every homomorphism from $(\mathrm{LEX}\,\mathcal{D}, \widehat{Id})$ has the form $K(F)$ for some lex functor $F : \mathcal{A} \to \mathcal{B}$.

Lemma. *Let $\mathcal{A}$ and $\mathcal{B}$ be small lex categories. For every functor $H : \mathrm{LEX}\,\mathcal{B} \to \mathrm{LEX}\,\mathcal{A}$ equivalent are:*

(i) $H : (\mathrm{LEX}\,\mathcal{B}, \widehat{Id}) \to (\mathrm{LEX}\,\mathcal{A}, \widehat{Id})$ *is a $\mathbb{D}^*$-homomorphism;*

(ii) *H preserves limits and filtered colimits,*

and

(iii) *$H \cong K(F)$ for some lex functor $F : \mathcal{A} \to \mathcal{B}$.*

Proof. i→ii. Let $J : \mathcal{J} \to \mathrm{LEX}\,\mathcal{B}$ be a diagram. Since $\widehat{Id}$ preserves limits, we have

$$\lim J \cong \widehat{Id}(\lim \eta_{\mathrm{LEX}\,\mathcal{B}} \cdot J).$$

Hence

$$\begin{aligned} H(\lim J) &\cong H(\widehat{Id}(\lim \eta_{\mathrm{LEX}\,\mathcal{B}} \cdot J)) \\ &\cong \widehat{Id} \cdot \mathbb{D}^*(H)(\lim \eta_{\mathrm{LEX}\,\mathcal{B}} \cdot J) \\ &\cong \lim(\widehat{Id} \cdot \mathbb{D}^*(H) \cdot \eta_{LEX\mathcal{B}} \cdot J) \\ &\cong \lim(\widehat{Id} \cdot \eta_{\mathrm{LEX}\,\mathcal{A}} \cdot H) \cong \lim(H \cdot J) \end{aligned}$$

because $\mathbb{D}^*(H)$ preserves limits. Therefore H preserves limits. Analogously we prove that H preserves filtered colimits.

ii→iii. By the Adjoint Functor Theorem 1.66 in [AR], H has a left adjoint $G : \mathrm{LEX}\,\mathcal{A} \to \mathrm{LEX}\,\mathcal{B}$. Since H preserves filtered colimits, the functor G preserves finite presentability of objects. Since $\mathcal{A}$ is lex, finitely presentable objects of $\mathrm{LEX}\,\mathcal{A}$ are precisely the representable functors, analogously for $\mathcal{B}$. Consequently, there is a functor $F : \mathcal{A} \to \mathcal{B}$ with $G(\mathcal{A}(A, -)) \cong \mathcal{B}(FA, -)$ for all $A \in \mathcal{A}$. The functor $K(F)$ is easily seen to be a right adjoint to G, thus, $K(F) \cong H$.

iii→i. trivial. □

IV.6 Corollary. *The quasicategory*

$$\mathrm{CAT}^{\mathbb{D}^*}$$

of $\mathbb{D}^*$-algebras is an equational hull of LFP.

That is, given a pseudomonad $\mathbb{T}$ over CAT and a functor

$$\mathrm{E} : \mathrm{LFP} \to \mathrm{CAT}^{\mathbb{T}}$$

such that the functor $U_{\mathbb{T}} \cdot \mathrm{E} : \mathrm{LFP} \to \mathrm{CAT}$ is naturally isomorphic to the forgetful functor $U : \mathrm{LFP} \to \mathrm{CAT}$, then there exists a pseudofunctor

$$\mathrm{E}^* : \mathrm{CAT}^{\mathbb{D}^*} \to \mathrm{CAT}^{\mathbb{T}}$$

with

$$\mathrm{E} \cong \mathrm{KE}^* \quad \text{and} \quad U_{\mathbb{D}^*} \cong U_{\mathbb{T}} \cdot \mathrm{E}^*$$

$$\begin{tikzcd}
\mathrm{LFP} \arrow[rr, "K"] \arrow[dr, "\mathrm{E}"] \arrow[dd, "U"'] & & \mathrm{CAT}^{\mathbb{D}^*} \arrow[dl, "\mathrm{E}^*"'] \arrow[dd, "U_{\mathbb{D}^*}"] \\
& \mathrm{CAT}^{\mathbb{T}} \arrow[dl, "U_{\mathbb{T}}"] \arrow[dr, "U_{\mathbb{T}}"'] & \\
\mathrm{CAT} \arrow[rr, "Id"'] & & \mathrm{CAT}
\end{tikzcd}$$

Proof. It is sufficient to show that

(a) every $\mathbb{D}^*$-algebra is an absolute pseudocolimit of limits of locally finitely presentable categories

and

(b) the forgetful functors $U_{\mathbb{T}}$ (in particular, $U_{\mathbb{D}^*}$) create absolute pseudocolimits and limits.

In fact, we than define E^* using $E^*(K(\mathcal{A})) = \mathcal{A}$ and extending this to $\mathrm{CAT}^{\mathbb{D}^*}$ by means of absolute pseudocolimits and limits.

Concerning (a), every $\mathbb{D}^*$-algebra is an absolute pseudocolimit of free $\mathbb{D}^*$-algebras, as proved in [LV], and every free $\mathbb{D}^*$-algebra

$$(\mathbb{D}^*)^2\mathcal{A} \xrightarrow{\mu_{\mathcal{A}}} \mathbb{D}^*\mathcal{A} = \mathrm{LEX}(\mathbf{Set}^{\mathcal{A}})$$

is a limit of the (large) ∞-cofiltered diagram of all $\mathrm{LEX}(\mathscr{C})$ where $\mathscr{C}$ is a small subcategory of $\mathbf{Set}^{\mathcal{A}}$ closed under finite limits (where for $\mathscr{C} \subseteq \mathscr{C}'$ we have an embedding $I : \mathscr{C} \to \mathscr{C}'$ giving a connecting morphism $\mathrm{LEX}(I) : \mathrm{LEX}(\mathscr{C}') \to \mathrm{LEX}(\mathscr{C})$). The limit cone is then, analogously, given by LEX-images of the embeddings $\mathscr{C} \hookrightarrow \mathbf{Set}^{\mathcal{A}}$. Each $\mathrm{LEX}(\mathscr{C})$ is a locally finitely presentable category.

Concerning (b), this has been proved by [LV]. □

IV.7 Corollary. LFP *is not pseudomonadic over* CAT.

This follows from the case of continuous lattices and the fact that LFP contains all algebraic lattices (which, being posets, are strict $\mathbb{D}^*$-algebras). In fact, let L be a continuous lattice which is not algebraic. Being an $Ind \circ Meet$-algebra, L is a split coequalizer of homomorphisms of algebraic lattices (recall that $Ind(-)$ is always algebraic). This gives, then, a split coequalizer of two strict homomorphisms of strict $\mathbb{D}^*$-algebras such that the codomain of the coequalizer (necessarily, a strict $\mathbb{D}^*$-algebra) L lies outside of LFP: in fact, lattices which are in LFP are *precisely* the algebraic ones.

IV.8 Remark. As stated in IV.3, we do not know how to describe the equational hull of LFP, i.e., the category $\mathrm{CAT}^{\mathbb{D}^*}$ – we do not even know whether this is a "property" (i.e., a subcategory of CAT) or a "structure".

It is natural to restrict our attention to operations of small arities, more precisely, to work instead of the pseudomonad $\mathbb{D}^*$ with its "small core" $\mathbb{D}^*_{\mathrm{small}}$ defined by on objects by forming the ∞-filtered colimit

$$\mathbb{D}^*_{\mathrm{small}}\mathcal{A} = \operatorname{colim} \mathbb{D}^* \mathcal{B}$$

of the free algebras generated by all small subcategories $\mathcal{B}$ of $\mathcal{A}$. Can we, then, describe the "small-equational hull"

$$\mathrm{CAT}^{\mathbb{D}^*_{\mathrm{small}}}$$

of LFP?

V. Small-Equational Hull of LFP

V.1 Remark. We have seen in Section IV that the pseudomonad $\mathbb{D}^*$ whose algebras form an equational hull of the category LFP is given by

$$\mathbb{D}^* \mathcal{K} = \mathrm{LEX}(\mathbf{Set}^{\mathcal{K}})\,.$$

In the present section we describe algebras of the pseudomonad

$$\mathbb{D}^*_{\mathrm{small}}$$

which is the small core of $\mathbb{D}^*$; see IV.8. We prove that, under a certain set-theoretical axiom (R), we have the formula

$$\mathbb{D}^*_{\mathrm{small}} k = \mathit{Ind}(\mathit{Lim}\, k) \quad \text{for all } k \text{ small},$$

analogous to the formula for algebraic lattices in II.6 above. And we start this section by observing a distributive law of the pseudomonad *Lim* on *Ind*, from which it follows that $\mathit{Ind} \circ \mathit{Lim}$ is a pseudomonad. The above formula yields a biequivalence

$$\mathbb{D}^*_{\mathrm{small}} \approx \mathit{Ind} \circ \mathit{Lim}\,.$$

Therefore, an equational hull of LFP w.r.t. small-ary operations is the category of algebras of $\mathit{Ind} \circ \mathit{Lim}$. We call these algebras precontinuous categories.

V.2 Pseudomonad *Ind*. The role that *Ind*, the free completion under directed joins, played for continuous lattices in II.3 above is now played by Grothendieck's completion *Ind*, i.e., free completion under filtered colimits. For every category $\mathcal{X}$ we describe $\eta^{\mathit{Ind}}_{\mathcal{X}} : \mathcal{X} \to \mathit{Ind}\,\mathcal{X}$ as the codomain restriction of the Yoneda embedding $\mathcal{X} \to \mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$ into the full subcategory $\mathit{Ind}\,\mathcal{X}$ of all flat functors (i.e., filtered colimits of hom-functors) in $\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$. This leads to a pseudomonad

$$(\mathit{Ind},\ \eta^{\mathit{Ind}},\ \mu^{\mathit{Ind}})$$

over CAT.

An alternative description of $\mathit{Ind}\,\mathcal{X}$ can be presented for all categories with finite coproducts:

V.3 Notation. For every category $\mathscr{L}$ with finite products denote by

$$\mathrm{Lex}\mathscr{L}$$

the full subcategory of $\mathbf{Set}^{\mathscr{L}}$ formed by all lex functors which are *small*, i.e., small colimits of hom-functors.

V.4 Remark. If $\mathcal{X}$ is a category with finite coproducts, then

$$Ind\,\mathcal{X} = \mathrm{Lex}\mathcal{X}^{\mathrm{op}}\,.$$

More precisely, the Yoneda embedding

$$\mathcal{X} \longrightarrow \mathrm{Lex}\mathcal{X}^{\mathrm{op}}$$

is a free completion of $\mathcal{X}$ under filtered colimits.

Now we prove that *Ind* distributes over the pseudomonad of free completion under limits:

V.5 Lemma.

(i) *$Ind\,\mathcal{X}$ is complete for every complete category $\mathcal{X}$.*

(ii) *$Ind\,F$ preserves limits whenever $F : \mathcal{X} \to \mathcal{Y}$ preserves limits ($\mathcal{X}$ and $\mathcal{Y}$ complete)*

(iii) *$\eta^{Ind}_{\mathcal{X}}$ and $\mu^{Ind}_{\mathcal{X}}$ preserve limits for each complete category $\mathcal{X}$.*

Proof. (i) See [AGV].

(ii) We use the following description of $Ind\,\mathcal{K}$ (see e.g. [JJ]): objects are all filtered diagrams in $\mathcal{K}$. Morphisms from $D : \mathcal{D} \to \mathcal{K}$ to $D' : \mathcal{D}' \to \mathcal{K}$ are compatible families of equivalence classes $[f_d]$ ($d \in \mathrm{obj}\mathcal{D}$) of morphisms $f_d : Dd \to D'd'$ in $\mathcal{K}$ under smallest equivalence $\sim$ with $f_d = D'\delta \cdot f_d$ for every $\delta \in \mathrm{mor}\mathcal{D}'$ with domain d'; compatibility means $[f_d] = [f_{\bar{d}} \cdot D\bar{\delta}]$ for all $\bar{\delta} : d \to \bar{d}$ in $\mathcal{D}$. The embedding $\eta^{Ind}_{\mathcal{K}}$ sends an object X to the corresponding single-morphism diagram (X). For finite limits (ii) is also proved in [AGV]. Thus, it remains to prove that $Ind\,F$ preserves products. The functor $Ind\,F$ takes a diagram $X = (X_i)_{i\in I}$, an object of $Ind\,\mathcal{X}$, to the diagram $FX = (FX_i)_{i\in I}$, an object of $Ind\,\mathcal{Y}$. Products of objects X^t, $t \in T$, of $Ind\,\mathcal{X}$ can be described as follows: let $I = \prod\limits_{t\in T} I^t$ be the usual product category of the filtered categories I^t which are the domains of X^t, then

$$\prod_{t\in T} X^t = (X_i)_{i\in I}$$

where for each $i = (i^t) \in I$ the object X_i is a product of $X^t_{i^t}$, $t \in T$, and the equivalence classes of the projection $\pi^t_i : X_i \to X^t_{i^t}$ form then the projection morphisms of the product ΠX^t in $Ind\,\mathcal{X}$. The claim

$$Ind\,F(\prod_{t\in T} X^t) = \prod_{t\in T} Ind\,F(X^t)$$

now easily follows from the fact that F preserves products, thus, from $X_i = \prod\limits_{t\in T} X^t_{i^t}$ we conclude $FX_i = \prod\limits_{t\in T} FX^t_{i^t}$.

(iii) The proof that $\eta^{Ind}_{\mathcal{X}}$ preserves limits can be found in [AGV]; it is based on the description of *Ind* via flat functors above: $Ind\,\mathcal{X}$ is obviously closed under limits in $\mathbf{Set}^{\mathcal{X}^{\mathrm{op}}}$. Finally, $\mu^{Ind}_{\mathcal{X}}$ preserves limits because *Ind* is a KZ-monad (and therefore $\mu^{Ind}_{\mathcal{X}}$ has $Ind(\eta^{Ind}_{\mathcal{X}})$ as a right adjoint), see [M$_1$]). $\square$

V.6 Pseudomonad Lim. We next consider a free completion of categories under limits. This yields a pseudomonad

$$(Lim, \eta^{Lim}, \mu^{Lim})$$

on CAT.

Corollary. *$Ind \circ Lim$ is a pseudomonad over CAT whose quasicategory of algebras is equivalent to the quasicategory of algebras of $\widetilde{Ind}$. The latter is the lifting of Ind to the quasicategory of complete categories.*

Proof. In fact, given pseudomonads $\mathbb{S}$ and $\mathbb{T}$ on a 2-category $\mathcal{K}$, then a lifting $\widetilde{\mathbb{S}}$ of $\mathbb{S}$ to the category $\mathcal{K}^{\mathbb{T}}$ of algebras for $\mathbb{T}$ is equivalent to providing a pseudomonad structure $\mathbb{S} \circ \mathbb{T}$ of the corresponding composite endofunctor of $\mathcal{K}$. And the category of algebras for $\mathbb{S} \circ \mathbb{T}$ is biequivalent to that of $\widetilde{\mathbb{S}}$ (see [M_2] and [M_3]). Thus the result follows from Lemma V.5 which precisely yields a lifting of *Ind* to CAT^{Lim}. □

V.7 Remark. We are prepared to describe a small equational hull of LFP. As seen in Section IV, for all small categories k we have (due to $Lim\, k = (\mathbf{Set}^k)^{\mathrm{op}}$)

$$\mathbb{D}^*_{\mathrm{small}}\, k = \mathbb{D}^*\, k = \mathrm{LEX}(\mathbf{Set}^k) = \mathrm{LEX}(\lim k)^{\mathrm{op}}\,.$$

If LEX could be substituted by Lex (i.e., if every lex functor from $\mathbf{Set}^k$ to $\mathbf{Set}$, k a small category, were small), we would have the described formula

$$\mathbb{D}^*_{\mathrm{small}}\, k = \mathrm{Lex}(Lim\, k)^{\mathrm{op}} = Ind\,(Lim\, k)\,.$$

The problem of

$$\mathrm{LEX}(\mathbf{Set}^k) \stackrel{?}{=} \mathrm{Lex}(\mathbf{Set}^k)$$

has been proved to depend on set theory in the recent paper [AKT], inspired by an earlier version of the present work.

Recall that an ultrafilter U on a set of cardinality k is called *uniform* if its members all have cardinality k. And U is called *regular* provided a subset V of U of cardinality k exists such that every member of U intersects only finitely many of members of V. The following statement

(R) every uniform ultrafilter is regular

is consistent with set theory, as proved by H.-D. Donder, see [Do]. Now the following has been proved in [AKT]:

V.8 Theorem. *For every small category $\mathcal{K}$, (R) implies*

$$\mathrm{LEX}(\mathbf{Set}^{\mathcal{K}}) = Lex(\mathbf{Set}^{\mathcal{K}}).$$

V.9 Corollary. *Assuming (R), the pseudomonads $\mathbb{D}^*_{\mathrm{small}}$ and $Ind \circ Lim$ are biequivalent.*

Proof. Since both pseudomonads $\mathbb{D}^*_{\mathrm{small}}$ and $Ind \circ Lim$ preserve ∞-filtered colimits, it suffices to show that they are biequivalent on small categories. Following Theorem V.8, they are biequivalent as bifunctors. It is easy to see that they have the same unit and multiplication. □

V.10 Corollary. *Assuming (R), the category of algebras of the pseudomonad $Ind \circ Lim$ is a small equational hull of* LFP.

That is, given a pseudomonad $\mathbb{T}$ over CAT preserving ∞-filtered colimits and a functor

$$E : \mathrm{LFP} \to \mathrm{CAT}^{\mathbb{T}}$$

as in IV.6, then there exists a pseudofunctor

$$E^* : \mathrm{CAT}^{\mathbb{D}^*_{\mathrm{small}}} \to \mathrm{CAT}^{\mathbb{T}}$$

with the properties from IV.6. The proof is analogous to that in IV.6.

Now, we are going to spell out the property of being an algebra of $Ind \circ Lim$ explicitly.

V.11 Definition. A category $\mathcal{A}$ is called *precontinuous* provided that it has limits and filtered colimits, and they distribute in the following sense: the functor

$$\underset{\leftarrow}{colim} : Ind\,\mathcal{A} \to \mathcal{A}$$

(substituting formal filtered colimits with actual colimits in $\mathcal{A}$) preserves limits.

V.12 Examples. (i) Every complete, continuous category in the sense of [JJ] is precontinuous. We recall that a category $\mathcal{A}$ is called *continuous* if it has filtered colimits and $\underset{\leftarrow}{colim} : Ind\,\mathcal{A} \to \mathcal{A}$ is a right adjoint.

(ii) Every locally finitely presentable category is precontinuous (in fact, continuous).

(iii) For every complete category $\mathcal{A}$ the category $Ind\,\mathcal{A}$ is precontinuous because it is complete by V.5 and continuous by [JJ].

(iv) The following category is precontinuous but not continuous: we start with a large class X and form the ordered class P of all pairs

$$(V, i) \text{ with } V \subseteq X \text{ and } i \leq \omega$$

(i.e., V is a subclass of X and i is a natural number or ω), subject to the condition

$$i = \omega \Rightarrow V \text{ is small,}$$

and ordered as follows:

$$(V, i) \leq (W, j) \text{ iff } W \subseteq V \text{ and } \ i \leq j\,.$$

We claim that the ordered class $P_\perp$ obtained by adding the least element $\perp$ to P is precontinuous but not continuous.

(a) $P_\perp$ is precontinuous. In fact, P is a subposet of the ordered class $Q = (\exp X)^{\mathrm{op}} \times (\omega + 1)$ which is easily seen to be a continuous category. The following maps

$$f : Q \to P_\perp\,,\ f(V, i) = (V, i) \text{ if } (v, i) \in P, = \perp \ \text{ else}$$

and

$$g : P_\perp \to Q\,,\ g(V,i) = (V,i) \text{ and } g(\perp) = (X,\omega)$$

preserve small meets, thus, $Ind\, g$ does (see Lemma V.1). And $\underleftarrow{colim} : Ind\, P_\perp \to P_\perp$ is the composite of $Ind\, g$, $\underleftarrow{colim} : Ind\, Q \to Q$, and f.

(b) $P_\perp$ is not continuous because $\underleftarrow{colim} : Ind\, P_\perp \to P_\perp$ does not preserve large meets. In fact, for each $v \in X$ we have an ideal $I_v = \{x\} \times \omega \cup \{\perp\}$ in $P_\perp$ and $\cap_{v\in X} I_v = \{\perp\}$. However, $colim\, I_v = \vee I_v = (\{v\}, \omega)$ and the meet of all the latter elements is (V, ω).

V.13 Theorem [ARV]. *A category with limits and filtered colimits is precontinuous iff*

(i) *finite limits commute with filtered colimits*

and

(ii) *products distribute over filtered colimits (see Introduction).*

V.14 Lemma. *A category with a dense set of finitely presentable objects is precontinuous iff it is locally finitely presentable.*

Proof. Let $\mathcal{A}$ be such a dense set in a precontinuous category $\mathcal{K}$. The functor $E : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ assigning to every object $\mathcal{K}$ the domain-restriction of $\hom(-, K)$ to $\mathcal{A}^{\mathrm{op}}$ is fully faithful (since $\mathcal{A}$ is dense) and preserves limits and filtered colimits (since objects of $\mathcal{A}$ are finitely presentable). Consequently, it has a left adjoint (see Theorem 2.48 in [AR]). Thus, $\mathcal{K}$ is equivalent to a full, reflective subcategory of a locally finitely presentable category $\mathcal{L} = \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$, and being closed in $\mathcal{L}$ under filtered colimits, is itself locally finitely presentable. ☐

V.15 Proposition. *For every small category $\mathcal{K}$, the embedding $\mathcal{K} \hookrightarrow Ind(Lim\mathcal{K})$ which is a codomain restriction of $\eta_{\mathcal{K}}^{\mathbb{D}^*}$ is a free precontinuous category on $\mathcal{K}$. That is,*

(a) *$Ind(Lim\mathcal{K})$ is precontinuous*

and

(b) *every functor $F : \mathcal{K} \to \mathcal{L}$, where $\mathcal{L}$ is precontinuous, has an essentially unique extension to a functor $F^* : Ind(Lim\mathcal{K}) \to \mathcal{L}$ preserving limits and filtered colimits.*

Proof. (a) follows from Example V.12 (iii).

To prove (b), denote by $F' : Lim\mathcal{K} \to \mathcal{L}$ the essentially unique limit-preserving extension of F, and by $F^* : Ind(Lim\,\mathcal{K}) \to \mathcal{L}$ the essentially unique extension of F' preserving filtered colimits. It is sufficient to prove that F^* preserves limits. By Lemma V.5, $Ind\,F'$ preserves limits, and since the functor $\underleftarrow{colim}\mathcal{L} : Ind\,\mathcal{L} \to \mathcal{L}$ preserves filtered colimits, we have $F^* \cong \underleftarrow{colim} \cdot Ind\,F'$. Since $\mathcal{L}$ is precontinuous, $\underleftarrow{colim}$ preserves limits, therefore F^* preserves limits. ☐

V.16 Notation. We denote by

$$\mathrm{PCONT}$$

the 2-quasicategory of all
(0-cells) precontinuous categories
(1-cells) functors preserving limits and filtered colimits
(2-cells) natural transformations.

V.17 Corollary. *Assuming (R), the 2-quasicategory* PCONT *of precontinuous categories is a small equational hull of* LFP*.*

In other words, as claimed in the Introduction, the operations "limits of type k" and "filtered colimits of type k" generate all operations of small arity on LFP. And the only equational laws, besides the general laws of limits in categories and filtered colimits in categories, are (by V.13) the commutation of finite limits with filtered colimits, and the distributivity of products over filtered colimits.

V.18 Remark. The above Corollary does not hold absolutely, i.e., without any restriction on set theory. In fact, Jan Reiterman [R] has proved that under the following assumption
non(M): there exist arbitrarily large measurable cardinals
a large lex functor $K : \mathbf{Set} \to \mathbf{Set}$ exists. That is, K lies in LEX(**Set**) – Lex(**Set**). We will see in the next section that this implies that the small equational hull of LFP is strictly smaller than PCONT.

VI. Equational Description

VI.1 Remark. We have described in the Introduction what operations and equations of LFP are. Let us add that every object

$$K \in \mathbb{D}^* \mathcal{K}$$

defines a $\mathcal{K}$-ary operation ω_K on LFP by the (standard) rule:

$$\omega_K^{\mathcal{A}} : \mathcal{A}^{\mathcal{K}} \to \mathcal{A}\,, \quad F \longmapsto \widehat{F}(K) \text{ for all } F : \mathcal{K} \to \mathcal{A}$$

(see Notation IV.2; recall that each locally finitely presentable category $\mathcal{A}$ is equivalent to LEX $\mathcal{B}$ following the Gabriel-Ulmer duality). This generalizes all examples from I.A:

$$\begin{aligned} ev_k &= \omega_{\eta_{\mathcal{K}}(k)}\,; \\ lim_{\mathcal{K}} &= \omega_K \text{ where } K \text{ is the limit of} \\ &\qquad\quad \eta_{\mathcal{K}} : \mathcal{K} \to \mathbb{D}^*\mathcal{K} \text{ in } \mathbb{D}^*\mathcal{K}\,; \\ colim_{\mathcal{K}} &= \omega_K \text{ where } K \text{ is the filtered colimit} \\ &\qquad\quad \text{of } \eta_{\mathcal{K}} \text{ in } \mathbb{D}^*\mathcal{K}\,. \end{aligned}$$

VI.2 Observation. (1) Every operation of LFP has the form ω_K for some $K \in \mathbb{D}^*\mathcal{K}$. In fact, let ω be a $\mathcal{K}$-ary operation, and let

$$K = \omega^{\mathbb{D}\mathcal{K}}(\eta_{\mathcal{K}}) \, .$$

For every functor $F : \mathcal{K} \to \mathcal{A}$, where $\mathcal{A} \in$ LFP, since $\widehat{F}$ preserves ω we have

$$\omega^{\mathcal{A}}(F) = \omega^{\mathcal{A}}(\widehat{F} \cdot \eta_{\mathcal{K}}) = \widehat{F}(\omega^{\mathbb{D}\mathcal{K}}(\eta_{\mathcal{K}})) = \omega_K^{\mathcal{A}}(F) \, .$$

(2) Every operation ω_K is a (large) filtered colimit of (large) limits of evaluations. In fact, $K \in \mathbb{D}^*\mathcal{K}$ is a lex functor form $\mathbf{Set}^{\mathcal{K}}$ to $\mathbf{Set}$, thus, K is a large filtered colimit of hom-functors, $K = \underset{\leftarrow}{colim}_{i \in I} \mathbf{Set}^{\mathcal{K}}(X_i, -)$. Express each $X_i \in \mathbf{Set}^{\mathcal{K}}$ as a colimit of hom-functors, $X_i = colim_{j \in J_i} \; \mathcal{K}(k_{ij}, -)$, to obtain $\mathbf{Set}^{\mathcal{K}}(X_i, -) = lim_{j \in J_i} \mathbf{Set}^{\mathcal{K}}(\mathcal{K}(k_{ij}, -), -) = \eta_{\mathcal{K}}(k_{ij})$. Thus,

$$K = \underset{\leftarrow}{colim}_{i \in I} lim_{j \in J_i} \eta_{\mathcal{K}}(k_{ij}) \, .$$

Now, as observed in IV.2, the functors $\widehat{F}$ preserve large filtered colimits and large limits. Consequently,

$$\omega_K = \underset{\leftarrow}{colim}_{i \in I} lim_{j \in J_i} ev_{k_{ij}} \, .$$

(3) We do not know how to work with all of these ("large" or "illegitimate") operations: as we have made clear in IV.3, we do not know how to describe the resulting category of algebras using all of those operations. Even when the arity $\mathcal{K}$ is small (and thus the limits in the above formula for ω_K are small), there may exist "large" operations, i.e., those which are not small filtered colimits of those limits of evaluations. But, as we make clear below, what we did was to describe the equational hull w.r.t. all small operations, i.e., all ω_K which are small filtered colimits of small limits of evaluations. And assuming the axiom (R), these are in fact all operations of small arities.

VI.3 Composed operations. Besides operations as described above we want to work with composed operations, e.g., a limit of filtered colimits. Recall that in the classical case mentioned above the discovery of [L$_1$] was that one can work with natural transformations

$$U^k \to U^n$$

(which, since $U^n = U \times U \times \cdots \times U$, are just n-tuples of k-ary operations). This makes the task of composing operations very easy: they were just composites of natural transformations between powers of U.

Analogously in our case of U : LFP $\to$ CAT: we work with *n-tuples of k-ary operations* (where n and k are categories) which we define as pseudonatural transformations

$$\omega : U^k \to U^n.$$

VI.4 Equations in LFP. Given operations

$$\omega_1, \omega_2 : U^k \longrightarrow U$$

of the same arity on LFP, we form an equation

$$\omega_1 \approx \omega_2$$

whose interpretation is not strict, of course. That is, we do not request that $\omega_1^{\mathcal{A}} = \omega_2^{\mathcal{A}}$ for all locally finitely presentable categories $\mathcal{A}$. Our interpretation is the following non-strict one:

Definition. For k-ary operations ω_1 and ω_2 on LFP we write

$$\omega_1 \approx \omega_2$$

provided that there is a collection

$$i^{\mathcal{A}} : \omega_1^{\mathcal{A}} \to \omega_2^{\mathcal{A}} \qquad (\mathcal{A} \in \mathrm{LFP})$$

of natural isomorphisms satisfying the following coherence condition: for every morphism $F : \mathcal{A} \to \mathcal{B}$ of LFP the following square

$$\begin{tikzcd}
F\omega_1^{\mathcal{A}} \arrow[r, "\widehat{\omega}_1^F"] \arrow[d, "F_i^{\mathcal{A}}"'] & \omega_1^{\mathcal{B}} F^k \arrow[d, "i^{\mathcal{B}} F^k"'] \\
F\omega_2^{\mathcal{A}} \arrow[r, "\widehat{\omega}_2^F"'] & \omega_2^{\mathcal{B}} F^k
\end{tikzcd}$$

commutes.

VI.5 Examples of equations. (1) Finite limits commute with filtered colimits. This is a property that all locally finitely presentable categories have. The commutation of k-lim (k finite) with f-$\underset{\rightarrow}{\mathrm{colim}}$ (f small, filtered) can be expressed by equation as follows: we can form an operation of arity $f \times k$ by composing the k-tuple

$$(f\text{-}\underset{\rightarrow}{\mathrm{colim}})^k : (\mathcal{A}^f)^k \to \mathcal{A}^k$$

with k-lim. And another operation of the same arity $f \times k$ by composing the f-tuple

$$(k\text{-}\lim)^f : (\mathcal{A}^k)^f \to \mathcal{A}^f$$

with f-lim. The fact that k-limits commute with f-colimits is expressed by the following equation

(E1) $\qquad (k\text{-}\lim) \cdot (f\text{-}\underset{\rightarrow}{\mathrm{colim}})^k \approx (f\text{-}\underset{\rightarrow}{\mathrm{colim}}) \cdot (k\text{-}\lim)^f \qquad$ (f filtered, k finite).

In fact, the canonical isomorphisms

$$i_D^{\mathcal{A}} : k\text{-}\lim(f\text{-}\mathrm{colim})^k D \to f\text{-}\mathrm{colim}(k\text{-}\lim)^f D$$

where $\mathcal{A}$ ranges through locally finitely presentable categories and D through diagrams of the scheme $f \times k$ in $\mathcal{A}$, form natural isomorphisms

$$j^{\mathcal{A}} : k\text{-}\lim(f\text{-}\mathrm{colim})^k \to f\text{-}\mathrm{colim}(k\text{-}\lim)^f$$

which are obviously coherent.

(2) (Infinite) products distribute over filtered colimits. That is, given a set $D_i \cdot f_i \to \mathcal{K}$ ($i \in k$) of filtered diagrams in $\mathcal{K}$, denote by $D : \prod\limits_{i \in k} f_i \to \mathcal{K}$ the diagram obtained by forming products in $\mathcal{K}$:

$$(d_i)_{i \in k} \longmapsto D d_i\,.$$

then the canonical morphism $\operatorname{colim} D \to \prod_{i\in k} \operatorname{colim} D_i$ is invertible.

To express this as an equation, denote by f the product of the categories f_i (for $i \in k$) and by g their coproduct; observe that f is filtered (due to the Axiom of Choice).

The passage from $(D_i)_{i\in k}$ to $\Pi \operatorname{colim} D_i$ is the composite of the product operation $\prod_k : \mathcal{A}^k \to \mathcal{A}$ (k considered as a discrete category) with the product of the operations f_i-colim:

$$\mathcal{A}^g \cong \prod_{i\in k} \mathcal{A}^{f_i} \xrightarrow{\ \Pi(f_i\text{-}\underset{\rightarrow}{\mathrm{colim}})\ } \mathcal{A}^k \xrightarrow{\ \prod_k\ } \mathcal{A}$$

And the passage to $\operatorname{colim} D$ is expressed by using the trivial operations $o_i : \mathcal{A}^{f_i} \to \mathcal{A}^f$ corresponding to the i-th projection of f, and composing them with f-colim : $\mathcal{A}^f \to \mathcal{A}$:

$$\mathcal{A}^g \cong \prod_{i\in k} \mathcal{A}^{f_i} \xrightarrow{\ (o_i)_{i\in k}\ } \mathcal{A}^f \xrightarrow{\ f\text{-}\underset{\rightarrow}{\mathrm{colim}}\ } \mathcal{A}$$

This gives an equation expressing distributivity of filtered colimits over products as follows

$$\text{(E2)} \qquad \prod_k \cdot \prod_{i\in k} (f_i\text{-}\underset{\rightarrow}{\mathrm{colim}}) \approx \big(\prod_{i\in k} f_i\text{-}\underset{\rightarrow}{\mathrm{colim}}\big) \cdot (o_i)_{i\in k} \qquad (f_i \text{ filtered, } i \in k)\,.$$

Again, the naturality conditions for the above canonical morphisms are easy to verify.

VI.6 Conclusion. Following Theorem V.13 and Examples VI.5, precontinuous categories are equational over CAT and, under (R), they form a small equational hull of LFP (cf. Corollary V.17). And the equational laws (E1) and (E2) generate all equational laws between operations of small arity over LFP.

On the other hand, assuming non(M), PCONT is not a small equational hull of LFP: for the functor K of V.18, the unary operation ω_K (see VI.1) cannot be extended to the category Lex(**Set**), thus, Lex(**Set**) is *not* included in the equational hull of LFP, although it is precontinuous. In fact, express K in $\mathbf{Set}^{\mathbf{Set}}$ as a large chain colimit of small lex functors K_i, $i \in Ord$, then each of the operations ω_{K_i} (being small) defines a unary operation on Lex(**Set**), and if ω_K were extendible to Lex(**Set**), the chain $\omega_{K_i}^{\mathrm{Lex}(\mathbf{Set})}$ would have a colimit, which is not the case.

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