Inaccessible cardinals and accessible categories

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Abstract model theory can be viewed as the study of subcategories of categories of *structures* $\mathbf{Str}\,\Sigma$. Here, Σ is a *signature*, i.e., a set of many-sorted infinitary function and relation symbols. Morphisms of Σ-structures are homomorphisms.

Any subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$ is *concrete*, i.e., it is equipped with a faithful functor $\mathcal{K} \to \mathbf{Set}$ where $\mathbf{Set}$ is the category of sets.

On the other hand, any concrete category $(\mathcal{K}, U)$ can be viewed as a subcategory of $\mathbf{Set}$ – the embedding $\mathcal{K} \to \mathbf{Set}$ sends an object K to the set $UK \times \{K\}$ and a morphism $f : K \to L$ to the mapping $UK \times \{K\} \to UL \times \{L\}$ sending (a, K) to $(U(f)(a), L)$.

Let (M) denote the non-existence of a proper class of measurable cardinals.

Theorem 1. (see Pultr, Trnková 1980) Assuming (M), any concrete category can be *fully embedded* to structures (i.e., it is isomorphic to a full subcategory of structures).

The homotopy category of topological spaces is not concrete (Freyd 1970).

The category $\mathcal{K}$ is called *bounded* if it has a small dense subcategory $\mathcal{A}$. This means that any $K \in \mathcal{K}$ is a colimit of its canonical diagram consisting of morphisms $A \to K$, $A \in \mathcal{A}$.

Equivalently, the canonical functor $E : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ is full and faithful where $EK = \mathcal{K}(-, K)$ (Isbell 1960).

Since $\mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ can be viewed as the category of many-sorted unary algebras, any bounded category can be fully embedded to structures.

The category **Top** of topological spaces and continuous maps is concrete but not bounded.

The category $\mathbf{Set}^{\mathrm{op}}$ is concrete because it is the category of complete atomic Boolean algebras. It is bounded iff (M) holds (Isbell 1960).

The category **Ord** of ordinals (as an ordered class), its dual and the large discrete category **Dis** are concrete but not bounded.

Vopěnka's principle (VP) states that **Dis** cannot be fully embedded to structures. It is equivalent to the fact that **Ord** cannot be fully embedded to structures.

Weak Vopěnka's principle (WVP) states that $\mathbf{Ord}^{\mathrm{op}}$ cannot be fully embedded to structures.

We have

$$(\mathrm{VP}) \Rightarrow (\mathrm{WVP}) \Rightarrow \neg(\mathrm{M})$$

(VP) is stronger than ¬(M) because it implies the existence of a proper class of extendible cardinals.

The position of (WVP) between (VP) and ¬(M) is open.

Theorem 2. (Adámek, JR, Trnková 1990) (VP) is equivalent to the fact that any full subcategory of $\mathbf{Str}\,\Sigma$ is bounded. Thus full subcategories of $\mathbf{Str}\,\Sigma$ are characterized as bounded categories.

A full subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$ is called *reflective* if any Σ-structure M allows a reflection $r : M \to M^*$ with M^* in $\mathcal{K}$ such that each homomorphism $f : M \to K$ with K in $\mathcal{K}$ uniquely factorizes through r.

Any reflective subcategory is closed under limits.

Theorem 3. (Adámek, JR, Trnková 1988) (WVP) is equivalent to the fact that any full subcategory of structures closed under limits is reflective.

Theorem 4. (Adámek, JR, Trnková 1990) (VP) is equivalent to the fact that any full subcategory of structures closed under colimits is coreflective.

Let λ be a regular cardinal. An object K of a category $\mathcal{K}$ is called λ-*presentable* if its hom-functor $\mathcal{K}(K, -) \to \mathbf{Set}$ preserves λ-*directed colimits*, i.e., colimits indexed by λ-directed posets.

A category $\mathcal{K}$ is called λ-*preaccessible* if it has a set $\mathcal{A}$ of λ-presentable objects such that every object in $\mathcal{K}$ is a λ-directed colimit of objects from $\mathcal{A}$.

$\mathcal{K}$ is called *preaccessible* if it is λ-preaccessible for some λ.

Any preaccessible category is bounded (with $\mathcal{A}$ as a small dense subcategory). Moreover, if $\mathcal{K}$ is λ-preaccessible then the full embedding $E : \mathcal{K} \to \mathbf{Set}^{\mathcal{A}^{\mathrm{op}}}$ preserves λ-directed colimits.

A λ-*accessible* category is a λ-preaccessible category having λ-directed colimits.

Let $U : \mathbf{Str}\,\Sigma \to \mathbf{Set}$ denote the underlying set functor.

We say that a subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$ has the *downward LST-property* if there is a regular cardinal λ such that for any object K of $\mathcal{K}$ and any subset $X \subseteq UK$ with $|X| < \lambda$ there is a substructure embedding $L \to K$ in $\mathcal{K}$ with $X \subseteq UL$ and $|UL| < \lambda$.

Theorem 5. (Adámek, JR 1995) Assuming (VP), any full subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$ is preaccessible and has the downward LST-property.

This related to the unpublished theorem of Stavi (see Magidor, Väänänen 2011).

Corollary 1. Assuming (VP), any full subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$ closed under λ-directed colimits for some λ is accessible.

Theorem 6. (Adámek, JR, Trnková 1990) (VP) is equivalent to the fact that, for every full subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$, there is a regular cardinal λ such that the inclusion of $\mathcal{K}$ to $\mathbf{Str}\,\Sigma$ preserves λ-directed colimits.

Corollary 2. Assuming (VP), any full subcategory $\mathcal{K}$ of $\mathbf{Str}\,\Sigma$ closed under limits is accessible.

Following Theorem 3, $\mathcal{K}$ is reflective. Thus $\mathcal{K}$ is cocomplete and, following Theorem 6, $\mathcal{K}$ is closed in $\mathbf{Str}\,\Sigma$ under λ-directed colimits for some λ. By Corollary 1, $\mathcal{K}$ is accessible.

In many results above, the necessary large-cardinal hypotheses can be weakened if the category $\mathcal{K}$ has a lower complexity in the Lévy hierarchy (Bagaria, Casacuberta, Mathias, JR 2015). For instance, a full subcategory $\mathcal{K}$ of structures closed under limits is reflective provided that

1. $\mathcal{K}$ is $\mathbf{\Sigma_1}$
2. $\mathcal{K}$ is $\mathbf{\Sigma_2}$ and there is a proper class of supercompact cardinals
3. $\mathcal{K}$ is $\mathbf{\Sigma_3}$ and there is a proper class of extendible cardinals.

The same weakening is valid in Corollary 1.

A functor $F : \mathcal{K} \to \mathcal{L}$ is called *accessible* if $\mathcal{K}$, $\mathcal{L}$ are accessible categories and F preserves λ-directed colimits for some λ. The full subcategory of $\mathcal{L}$ consisting of subobjects of FK, K in $\mathcal{K}$ is called the *powerful image* of F.

Free abelian groups form the powerful image of the free abelian group functor $F : \mathbf{Set} \to \mathbf{Ab}$.

Makkai and Paré (1989) proved that, assuming the existence of a proper class of strongly compact cardinals, any powerful image of an accessible functor is accessible. Brooke-Taylor and JR (2015) weakened the assumption to the existence of a proper class of almost strongly compact cardinals (and to the λ-pure powerful image, i.e., to the closure of the image under λ-pure subobjects). Independently, Boney and Unger (2015) proved

Theorem 7. Every powerful image of an accessible functor is accessible iff there is a proper class of almost strongly compact cardinals.

They also showed that this is equivalent to the fact that every abstract elementary class is tame.

A cocomplete accessible category is called *locally presentable*.

If $\mathcal{K}$ is a locally presentable stable model category the its homotopy category $\mathrm{Ho}(\mathcal{K})$ is *triangulated*. Except trivial cases $\mathrm{Ho}(\mathcal{K})$ is not accessible and, even, not concrete. It has products and coproducts but it does not have equalizers and coequalizers. They are replaced by fibres and cofibres.

Theorem 8. (Casacuberta, Gutiérrez, JR 2015) Let $\mathcal{K}$ be a locally presentable stable model category. Assuming (VP), any full subcategory $\mathcal{L}$ of $\mathrm{Ho}(\mathcal{K})$ closed under products and fibres is reflective. Moreover, ${}^{\perp}\mathcal{L}$ is coreflective.

Problem 1. Is (VP) needed in Theorem 8?

Accessible categories are closed under constructions of limit type, which is more problematic for constructions of colimit type. For instance, for powerful images.

Theorem 9. (Paré, JR) Accessible categories are closed in **CAT** under directed colimits of accessible full embeddings. Assuming the existence of a proper class of strongly compact cardinals, they are closed under directed colimits of accessible embeddings.

Consider the following countable chain of locally finitely presentable categories and finitely accessible functors

$$\mathbf{Set} \xrightarrow{F_{01}} \mathbf{Set}^2 \xrightarrow{F_{12}} \dots \mathbf{Set}^n \xrightarrow{F_{nn+1}} \dots$$

where $F_{nn+1}(X_1, \dots, X_n) = (X_1, \dots, X_n, X_n)$. The colimit $\mathbf{Set}^{<\omega}$ in **CAT** consists of sequences $(X_n)_{n\in\omega}$ which are eventually constant. Similarly, morphisms are eventually constant sequences $(f_n)_{n\in\omega}$ of mappings. Following Theorem 9, $\mathbf{Set}^{<\omega}$ is accessible assuming the existence of an ω_1-strongly compact cardinal.

Problem 2. Does the accessibility of $\mathbf{Set}^{<\omega}$ depend on set theory?

Theorem 10. (Makkai, Pitts) Let $\mathcal{K}$ be an iso-full subcategory of $\mathbf{Str}\,\Sigma$ closed under limits and directed colimits. Then $\mathcal{K}$ is reflective.

The proof is model-theoretic and consists in showing that $\mathcal{K}$ is closed under elementary submodels. Having an elementary submodel L of K in $\mathcal{K}$, one saturates K by taking an ultrapower K^U by a good ultrafilter and shows that L is an equalizer of automorphisms of K^U. Since K^U is a directed colimit of powers of K, K^U belongs to $\mathcal{K}$ and thus L belongs to $\mathcal{K}$. To avoid GCH, one can take an iterated ultrapower.

For full subcategories, elementary embeddings can be replaced by pure embeddings and this proof generalizes to full subcategories closed under limits and λ-directed colimits (Adámek, JR 1989).

Problem 3. Is any iso-full subcategory of $\mathbf{Str}\,\Sigma$ closed under limits and λ-directed colimits accessible?