Příklady z integrálního počtu v reálném oboru

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Obsah

1 Primitivní funkce

Ve všech výsledcích je vynechána integrační konstanta.

Příklad 1 (základní vlastnosti)**.** Vypočtěte integrály

(a) $\displaystyle\int \sqrt{x}\,\mathrm{d}x,$ $\quad[\frac{2}{3}\sqrt{x^3}]$

(b) $\displaystyle\int \frac{\mathrm{d}x}{x^2},$ $\quad[-\frac{1}{x}]$

(c) $\displaystyle\int a^x\,\mathrm{e}^x\,\mathrm{d}x,$ $\quad[\frac{(a\,\mathrm{e})^x}{1+\ln a}]$

(d) $\displaystyle\int \left(\frac{1-x}{x}\right)^2 \mathrm{d}x,$ $\quad[x-2\ln x-\frac{1}{x}]$

(e) $\displaystyle\int \frac{3\cdot 2^x-2\cdot 3^x}{2^x}\,\mathrm{d}x,$ $\quad[3x-\frac{2\left(\frac{3}{2}\right)^x}{\ln\frac{3}{2}}]$

(f) $\displaystyle\int \frac{\sqrt{x}-2\sqrt[3]{x^2}+1}{\sqrt[4]{x}}\,\mathrm{d}x,$ $\quad[\frac{4}{5}x\sqrt[4]{x}-\frac{24}{17}x\,\sqrt[12]{x^5}+\frac{4}{3}\sqrt[4]{x^3}]$

(g) $\displaystyle\int \frac{\sqrt{x^4+x^{-4}+2}}{x^3}\,\mathrm{d}x,$ $\quad[\ln|x|-\frac{1}{4x^4}]$

(h) $\displaystyle\int \frac{x^2}{1+x^2}\,\mathrm{d}x,$ $\quad[x-\operatorname{arctg}x]$

(i) $\displaystyle\int \frac{\mathrm{e}^{3x}+1}{\mathrm{e}^x+1}\,\mathrm{d}x,$ $\quad[\frac{1}{2}\,\mathrm{e}^{2x}-\mathrm{e}^x+x]$

(j) $\displaystyle\int \frac{\cos 2x}{\cos^2 x\sin^2 x}\,\mathrm{d}x,$ $\quad[-\operatorname{cotg}x-\operatorname{tg}x=-\frac{1}{\sin x\cos x}]$

(k) $\displaystyle\int \operatorname{tg}^2 x\,\mathrm{d}x,$ $\quad[\operatorname{tg}x-x]$

(l) $\displaystyle\int \operatorname{cotg}^2 x\,\mathrm{d}x,$ $\quad[-x-\operatorname{cotg}x]$

(m) $\displaystyle\int 2\sin^2\frac{x}{2}\,\mathrm{d}x,$ $\quad[x-\sin x]$

(n) $\displaystyle\int \frac{\mathrm{d}x}{\cos 2x+\sin^2 x},$ $\quad[\operatorname{tg}x]$

(o) $\displaystyle\int \operatorname{tgh}^2 x\,\mathrm{d}x,$ $\quad[x-\operatorname{tgh}x]$

(p) $\displaystyle\int \operatorname{cotgh}^2 x\,\mathrm{d}x,$ $\quad[x-\operatorname{cotgh}x]$

(q) $\displaystyle\int \frac{1+2x^2}{x^2(1+x^2)}\,\mathrm{d}x,$ $\quad[\operatorname{arctg}x-\frac{1}{x}]$

(r) $\displaystyle\int \frac{(1+x)^2}{x(1+x^2)}\,\mathrm{d}x,$ $[\ln|x| + 2\operatorname{arctg} x]$

(s) $\displaystyle\int \frac{1+\cos^2 x}{1+\cos 2x}\,\mathrm{d}x,$ $[\frac{1}{2}(\operatorname{tg} x + x)]$

(t) $\displaystyle\int \frac{x^4}{1+x^2}\,\mathrm{d}x.$ $[\frac{x^3}{3} - x + \operatorname{arctg} x]$

Příklad 2. Buď $\displaystyle\int f(x)\,\mathrm{d}x = F(x) + C$. Ukažte, že platí

$$\int f(ax+b)\,\mathrm{d}x = \frac{1}{a}F(ax+b) + C, \qquad a, b \in \mathbb{R}, \quad a \neq 0.$$

Příklad 3 (substituce)**.** Vypočtěte integrály

(a) $\displaystyle\int \frac{\mathrm{d}x}{2x+a},$ $[\frac{1}{2}\ln|2x+a|]$

(b) $\displaystyle\int (x+1)^{15}\,\mathrm{d}x,$ $[\frac{1}{16}(x+1)^{16}]$

(c) $\displaystyle\int (3-2x)^4\,\mathrm{d}x,$ $[-\frac{(3-2x)^5}{10}]$

(d) $\displaystyle\int (\mathrm{e}^{-x} + \mathrm{e}^{-2x})\,\mathrm{d}x,$ $[-\mathrm{e}^{-x} - \frac{1}{2}\mathrm{e}^{-2x}]$

(e) $\displaystyle\int (\sin 5x - \sin 5\alpha)\,\mathrm{d}x,$ $[-\frac{1}{5}\cos 5x - x\sin 5\alpha]$

(f) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{3-2x}},$ $[-\sqrt{3-2x}]$

(g) $\displaystyle\int \frac{\mathrm{d}x}{\cos^2 5x},$ $[\frac{1}{5}\operatorname{tg} 5x]$

(h) $\displaystyle\int \frac{\mathrm{d}x}{1-10x},$ $[-\frac{1}{10}\ln|1-10x|]$

(i) $\displaystyle\int \frac{\mathrm{d}x}{\sin^2\left(2x+\frac{\pi}{4}\right)},$ $[-\frac{1}{2}\operatorname{cotg}\left(2x+\frac{\pi}{4}\right)$ nebo $\frac{\sin 2x}{\sin 2x + \cos 2x}]$

(j) $\displaystyle\int \frac{\mathrm{d}x}{1+\cos x},$ $[\operatorname{tg}\frac{x}{2}]$

(k) $\displaystyle\int \frac{\mathrm{d}x}{1+\sin x},$ $[-\operatorname{tg}\left(\frac{\pi}{4}-\frac{x}{2}\right)$ nebo $\frac{1-\cos x+\sin x}{1+\sin x}]$

(l) $\displaystyle\int [\sinh(2x+1) + \cosh(2x-1)]\,\mathrm{d}x,$ $[\frac{1}{2\mathrm{e}}\cosh 2x + \frac{\mathrm{e}}{2}\sinh 2x]$

(m) $\displaystyle\int \frac{\mathrm{d}x}{\cosh^2\frac{x}{2}},$ $[2\operatorname{tgh}\frac{x}{2}]$

(n) $\displaystyle\int \frac{\mathrm{d}x}{\sinh^2 \frac{x}{2}}$, $[-2 \operatorname{cotgh} \frac{x}{2}]$

(o) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{4-9x^2}}$. $[\frac{1}{3} \arcsin \frac{3x}{2}]$

Příklad 4 (substituce)**.** Vypočtěte integrály

(a) $\displaystyle\int \frac{\mathrm{e}^{2x}}{1-3\,\mathrm{e}^{2x}}\,\mathrm{d}x$, $[-\frac{1}{6} \ln|1-3\,\mathrm{e}^{2x}|]$

(b) $\displaystyle\int \operatorname{cotg} x \,\mathrm{d}x$, $[\ln|\sin x|]$

(c) $\displaystyle\int \operatorname{tg} x \,\mathrm{d}x$, $[-\ln|\cos x|]$

(d) $\displaystyle\int \sin^2 x \cos x \,\mathrm{d}x$, $[\frac{1}{3}\sin^3 x]$

(e) $\displaystyle\int \frac{\mathrm{d}x}{x(1+\ln x)}$, $[\ln|1+\ln x|]$

(f) $\displaystyle\int \frac{\cos 2x}{\sin x \cos x}\,\mathrm{d}x$, $[\ln|\sin 2x|]$

(g) $\displaystyle\int \frac{\cos x}{1+2\sin x}\,\mathrm{d}x$, $[\frac{1}{2}\ln|1+2\sin x|]$

(h) $\displaystyle\int \frac{x}{\sqrt{x^2+1}}\,\mathrm{d}x$, $[\sqrt{x^2+1}]$

(i) $\displaystyle\int \frac{x^3}{\sqrt[3]{x^4+1}}\,\mathrm{d}x$, $[\frac{3}{8}\sqrt[3]{(x^4+1)^2}]$

(j) $\displaystyle\int \frac{x}{\sqrt{1-x^2}}\,\mathrm{d}x$, $[-\sqrt{1-x^2}]$

(k) $\displaystyle\int x\,\mathrm{e}^{-x^2}\,\mathrm{d}x$, $[-\frac{1}{2}\,\mathrm{e}^{-x^2}]$

(l) $\displaystyle\int \frac{\mathrm{d}x}{\mathrm{e}^x+\mathrm{e}^{-x}}$, $[\operatorname{arctg} \mathrm{e}^x]$

(m) $\displaystyle\int \mathrm{e}^{\sin x}\cos x\,\mathrm{d}x$, $[\mathrm{e}^{\sin x}]$

(n) $\displaystyle\int \frac{\ln^2 x}{x}\,\mathrm{d}x$, $[\frac{1}{3}\ln^3 x]$

(o) $\displaystyle\int \frac{x}{x^4+1}\,\mathrm{d}x$, $[\frac{1}{2}\operatorname{arctg} x^2]$

(p) $\displaystyle\int \frac{2^x}{\sqrt{1-4^x}}\,\mathrm{d}x$, $[\frac{\arcsin 2^x}{\ln 2}]$

(q) $\displaystyle\int \frac{\sin x}{\sqrt{\cos^3 x}}\,\mathrm{d}x$, $[\frac{2}{\sqrt{\cos x}}]$

(r) $\displaystyle\int \frac{\sin x + \cos x}{\sqrt[3]{\sin x - \cos x}}\,\mathrm{d}x,$ $[\frac{3}{2}\sqrt[3]{1-\sin 2x}]$

(s) $\displaystyle\int \frac{\mathrm{d}x}{\sin^2 x\sqrt[4]{\operatorname{cotg} x}},$ $[-\frac{4}{3}\sqrt[4]{\operatorname{cotg}^3 x}]$

(t) $\displaystyle\int \frac{\operatorname{arctg} x}{1+x^2}\,\mathrm{d}x,$ $[\frac{1}{2}\operatorname{arctg}^2 x]$

(u) $\displaystyle\int \frac{x^4}{(x^5+1)^4}\,\mathrm{d}x,$ $[-\frac{1}{15(x^5+1)^3}]$

(v) $\displaystyle\int \frac{\mathrm{d}x}{(x-1)^2+4},$ $[\frac{1}{2}\operatorname{arctg}\frac{x-1}{2}]$

(w) $\displaystyle\int \frac{2x-\sqrt{\arcsin x}}{\sqrt{1-x^2}}\,\mathrm{d}x,$ $[-2\sqrt{1-x^2}-\frac{2}{3}\sqrt{\arcsin^3 x}]$

(x) $\displaystyle\int \frac{\mathrm{d}x}{4x^2+4x+5},$ $[\frac{1}{4}\operatorname{arctg} x+\frac{1}{2}]$

(y) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{2-6x-9x^2}}.$ $[\frac{1}{3}\arcsin\frac{3x+1}{\sqrt{3}}]$

Příklad 5 (per–partes)**.** Vypočtěte integrály

(a) $\displaystyle\int x\cos x\,\mathrm{d}x,$ $[x\sin x+\cos x]$

(b) $\displaystyle\int x\sin 2x\,\mathrm{d}x,$ $[\frac{1}{4}\sin 2x-\frac{1}{2}x\cos 2x]$

(c) $\displaystyle\int x\,\mathrm{e}^{-x}\,\mathrm{d}x,$ $[-\mathrm{e}^{-x}(x+1)]$

(d) $\displaystyle\int \ln x\,\mathrm{d}x,$ $[x(\ln x-1)]$

(e) $\displaystyle\int x^n\ln x\,\mathrm{d}x \quad (n\neq 1),$ $[\frac{x^{n+1}}{n+1}\left(\ln x-\frac{1}{n+1}\right)]$

(f) $\displaystyle\int 3^x x\,\mathrm{d}x,$ $[\frac{3^x}{\ln^2 3}(x\ln 3-1)]$

(g) $\displaystyle\int \operatorname{arctg} x\,\mathrm{d}x,$ $[x\operatorname{arctg} x-\frac{1}{2}\ln(1+x^2)]$

(h) $\displaystyle\int \arcsin x\,\mathrm{d}x,$ $[x\arcsin x+\sqrt{1-x^2}]$

(i) $\displaystyle\int x^2\,\mathrm{e}^{-2x}\,\mathrm{d}x,$ $[-\frac{1}{2}\mathrm{e}^{-2x}\left(x^2+x+\frac{1}{2}\right)]$

(j) $\displaystyle\int x^3\,\mathrm{e}^{-x^2}\,\mathrm{d}x,$ $[-\frac{x^2+1}{2}\mathrm{e}^{-x^2}]$

(k) $\displaystyle\int \ln(x+\sqrt{1+x^2})\,\mathrm{d}x,$ $[x\ln x(x+\sqrt{1+x^2})-\sqrt{1+x^2}]$

(l) $\displaystyle\int \ln^2 x \,\mathrm{d}x,$ $[x(\ln^2 x - 2\ln x + 2)]$

(m) $\displaystyle\int (\arcsin x)^2 \,\mathrm{d}x,$ $[x \arcsin^2 x + 2\sqrt{1-x^2}\arcsin x - 2x]$

(n) $\displaystyle\int \mathrm{e}^{3x}(\sin 2x - \cos 2x) \,\mathrm{d}x,$ $[\frac{\mathrm{e}^{3x}}{13}(\sin 2x - 5\cos 2x)]$

(o) $\displaystyle\int \frac{x}{\cos^2 x} \,\mathrm{d}x,$ $[x \operatorname{tg} x + \ln|\cos x|]$

(p) $\displaystyle\int \operatorname{arctg}\sqrt{x} \,\mathrm{d}x,$ $[-\sqrt{x} + (1+x)\operatorname{arctg}\sqrt{x}]$

(q) $\displaystyle\int \sqrt{a^2-x^2} \,\mathrm{d}x,$ $[\frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\arcsin\frac{x}{|a|} = \frac{x}{2}\sqrt{a^2-x^2} + \frac{a^2}{2}\operatorname{arctg}\frac{x}{\sqrt{a^2-x^2}}]$

(r) $\displaystyle\int x\sin^2 x \,\mathrm{d}x,$ $[\frac{x^2}{4} - \frac{x}{4}\sin 2x - \frac{\cos 2x}{8}]$

(s) $\displaystyle\int \frac{\mathrm{d}x}{(a^2+x^2)^2},$ $[\frac{x}{2a^2(a^2+x^2)} + \frac{1}{2a^3}\operatorname{arctg}\frac{x}{a}]$

(t) $\displaystyle\int \frac{x\,\mathrm{e}^{\operatorname{arctg} x}}{(1+x^2)^{\frac{3}{2}}} \,\mathrm{d}x.$ $[\frac{(x-1)\,\mathrm{e}^{\operatorname{arctg} x}}{2\sqrt{1+x^2}}]$

Příklad 6 (rozklad na parciální zlomky)**.** Vypočtěte integrály

(a) $\displaystyle\int \frac{x\,\mathrm{d}x}{(x+1)(2x+1)},$ $[\ln\frac{|x+1|}{\sqrt{|2x+1|}}]$

(b) $\displaystyle\int \frac{2x^2+41x-91}{(x-1)(x+3)(x-4)} \,\mathrm{d}x,$ $[\ln\left|\frac{(x-1)^4(x-4)^5}{(x+3)^7}\right|]$

(c) $\displaystyle\int \frac{x^5+x^4-8}{x^3-4x} \,\mathrm{d}x,$ $[\frac{x^3}{3} + \frac{x^2}{2} + 4x + \ln\left|\frac{x^2(x-2)^5}{(x+2)^3}\right|]$

(d) $\displaystyle\int \frac{32x\,\mathrm{d}x}{(2x-1)(4x^2-16x+15)},$ $[\ln\left|\frac{(2x-1)(2x-5)^5}{(2x-3)^6}\right|]$

(e) $\displaystyle\int \frac{2x^2-5}{x^4-5x^2+6} \,\mathrm{d}x,$ $[\frac{1}{2\sqrt{2}}\ln\left|\frac{x-\sqrt{2}}{x+\sqrt{2}}\right| + \frac{1}{2\sqrt{3}}\ln\left|\frac{x-\sqrt{3}}{x+\sqrt{3}}\right|]$

(f) $\displaystyle\int \frac{x^3-6x^2+11x-5}{(x-2)^4} \,\mathrm{d}x,$ $[\ln|x-2| + \frac{3x-8}{6(x-2)^3}]$

(g) $\displaystyle\int \frac{x^2\,\mathrm{d}x}{(x+2)^2(x+4)^2},$ $[2\ln\left|\frac{x+4}{x+2}\right| - \frac{5x+12}{x^2+6x+8}]$

(h) $\displaystyle\int \frac{1}{8}\left(\frac{x-1}{x+1}\right)^4 \,\mathrm{d}x,$ $[\frac{x}{8} - \ln|x+1| - \frac{9x^2+12x+5}{3(x+1)^3}]$

(i) $\displaystyle\int \frac{x^2-2x+3}{(x-1)(x^3-4x^2+3x)} \,\mathrm{d}x,$ $[\frac{1}{x-1} + \ln\frac{\sqrt{|(x-1)(x-3)|}}{|x|}]$

(j) $\displaystyle\int \frac{x^3-2x^2+4}{x^3(x-2)^2} \,\mathrm{d}x,$ $[\frac{1}{4}\ln\left|\frac{x}{x-2}\right| - \frac{3x^2+3x+2}{2x^2(x-2)}]$

(k) $\displaystyle\int \frac{\mathrm{d}x}{x(x^2+1)}$, $[\ln \frac{|x|}{\sqrt{x^2+1}}]$

(l) $\displaystyle\int \frac{\mathrm{d}x}{1+x^3}$, $[\frac{1}{6}\ln\frac{(x+1)^2}{x^2-x+1} + \frac{1}{\sqrt{3}}\operatorname{arctg}\frac{2x-1}{\sqrt{3}}]$

(m) $\displaystyle\int \frac{x^4+1}{x^3-x^2+x-1}\,\mathrm{d}x$, $[\frac{(x+1)^2}{2} + \ln\frac{|x-1|}{\sqrt{x^2+1}} - \operatorname{arctg} x]$

(n) $\displaystyle\int \frac{\mathrm{d}x}{(x^2+1)(x^2+x)}$, $[\frac{1}{4}\ln\frac{x^4}{(x+1)^2(x^2+1)} - \frac{1}{2}\operatorname{arctg} x]$

(o) $\displaystyle\int \frac{3x^2+x+3}{(x-1)^3(x^2+1)}\,\mathrm{d}x$, $[\frac{1}{4}\left[\ln\frac{\sqrt{x^2+1}}{|x-1|} + \operatorname{arctg} x - \frac{7}{(x-1)^2}\right]]$

(p) $\displaystyle\int \frac{x^3+x-1}{(x^2+2)^2}\,\mathrm{d}x$, $[\frac{2-x}{4(x^2+2)} + \frac{1}{2}\ln(x^2+2) - \frac{1}{4\sqrt{2}}\operatorname{arctg}\frac{x}{\sqrt{2}}]$

(q) $\displaystyle\int \frac{5x^2-12}{(x^2-6x+13)^2}\,\mathrm{d}x$, $[\frac{13x-159}{8(x^2-6x+13)} + \frac{53}{16}\operatorname{arctg}\frac{x-3}{2}]$

(r) $\displaystyle\int \frac{\mathrm{d}x}{(x^2+9)^3}$, $[\frac{x(x^2+15)}{216(x^2+9)^2} + \frac{1}{648}\operatorname{arctg}\frac{x}{3}]$

(s) $\displaystyle\int \frac{2x}{(1+x)(1+x^2)^2}\,\mathrm{d}x$, $[\frac{x-1}{2(x^2+1)} + \frac{1}{4}\ln\frac{1+x^2}{(1+x)^2}]$

(t) $\displaystyle\int \frac{x^9}{(x^4-1)^2}\,\mathrm{d}x$, $[\frac{x^2}{4}\frac{2x^4-3}{x^4-1} + \frac{3}{8}\ln\left|\frac{x^2-1}{x^2+1}\right|]$

(u) $\displaystyle\int \frac{x^8}{x^8-1}\,\mathrm{d}x$. $[x - \frac{1}{4}\operatorname{arctg} x + \frac{1}{4\sqrt{2}}\left[\operatorname{arctg}(1-\sqrt{2}x) - \operatorname{arctg}(1+\sqrt{2}x)\right] +$
$+ \frac{1}{8}\left[\ln\left|\frac{x-1}{x+1}\right| + \frac{1}{\sqrt{2}}\ln\frac{x^2-\sqrt{2}x+1}{x^2+\sqrt{2}x+1}\right]]$

Příklad 7 (odmocniny)**.** Vypočtěte integrály

(a) $\displaystyle\int \frac{x+1}{\sqrt[3]{3x+1}}\,\mathrm{d}x$, $[\frac{x+2}{5}\sqrt[3]{(3x+1)^2}]$

(b) $\displaystyle\int \frac{1}{x}\sqrt{\frac{1-x}{1+x}}\,\mathrm{d}x$, $[\ln\left|\frac{\sqrt{1+x}-\sqrt{1-x}}{\sqrt{1+x}+\sqrt{1-x}}\right| + 2\operatorname{arctg}\sqrt{\frac{1-x}{1+x}}]$

(c) $\displaystyle\int \sqrt{\frac{1-\sqrt{x}}{1+\sqrt{x}}}\,\mathrm{d}x$, $[(\sqrt{x}-2)\sqrt{1-x} - \arcsin\sqrt{x}]$

(d) $\displaystyle\int \frac{\sqrt{x+1}-\sqrt{x-1}}{\sqrt{x+1}+\sqrt{x-1}}\,\mathrm{d}x$, $[\frac{1}{2}\left[x^2 - x\sqrt{x^2-1} + \ln(x+\sqrt{x^2-1})\right]]$

(e) $\displaystyle\int \frac{x\sqrt[3]{2+x}}{x+\sqrt[3]{2+x}}\,\mathrm{d}x$, $[\frac{3}{4}t^4 - \frac{3}{2}t^2 - \frac{3}{4}\ln|t-1| + \frac{15}{8}\ln(t^2+t+2) - \frac{27}{4\sqrt{7}}\operatorname{arctg}\frac{2t+1}{\sqrt{7}};$
$t = \sqrt[3]{2+x}]$

(f) $\displaystyle\int \frac{\mathrm{d}x}{x(\sqrt{x}+\sqrt[5]{x^2})}$, $[\ln\frac{x}{(1+\sqrt[10]{x})^{10}} + \frac{10}{\sqrt[10]{x}} - \frac{5}{\sqrt[5]{x}} + \frac{10}{3\sqrt[10]{x^3}} - \frac{5}{2\sqrt[5]{x^2}}]$

(g) $\displaystyle\int \frac{x}{\sqrt{x+1}+\sqrt[3]{x+1}}\,\mathrm{d}x,$ $\quad [-x-\frac{3}{2}(x+1)^{\frac{2}{3}}+\frac{6}{5}(x+1)^{\frac{5}{6}}+\frac{6}{7}(x+1)^{\frac{7}{6}}-$ $-\frac{3}{4}(x+1)^{\frac{4}{3}}+\frac{2}{3}(x+1)^{\frac{3}{2}}]$

(h) $\displaystyle\int \frac{\mathrm{d}x}{(1+\sqrt[4]{x})^3\sqrt{x}},$ $\quad [-2\frac{1+2\sqrt[4]{x}}{(1+\sqrt[4]{x})^2}]$

(i) $\displaystyle\int \frac{1-\sqrt{x+1}}{1+\sqrt[3]{x+1}}\,\mathrm{d}x,$ $\quad [6t-3t^2-2t^3+\frac{3}{2}t^4+\frac{6}{5}t^5-\frac{6}{7}t^7+3\ln(1+t^2)-6\operatorname{arctg} t;$ $t=\sqrt[6]{x+1}]$

(j) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt[3]{(x+1)^2(x-1)^4}},$ $\quad [-\frac{3}{2}\sqrt[3]{\frac{x+1}{x-1}}]$

(k) $\displaystyle\int \frac{\mathrm{d}x}{x+\sqrt{x^2+x+1}},$ $\quad [\frac{3}{2(2z+1)}+\frac{1}{2}\ln\frac{z^4}{|2z+1|^3};\ z=x+\sqrt{x^2+x+1}]$

(l) $\displaystyle\int \frac{\mathrm{d}x}{1+\sqrt{1-2x-x^2}},$ $\quad [\ln\left|\frac{z-1}{z}\right|-2\operatorname{arctg} z;\ z=\frac{1+\sqrt{1-2x-x^2}}{x}]$

(m) $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt{2+x-x^2}},$ $\quad [-\frac{1}{\sqrt{2}}\ln\left|\frac{\sqrt{2+x-x^2}+\sqrt{2}}{x}+\frac{1}{2\sqrt{2}}\right|]$

(n) $\displaystyle\int \frac{\sqrt{2x+x^2}}{x^2}\,\mathrm{d}x,$ $\quad [\ln|x+1+\sqrt{2x+x^2}|-\frac{4}{x+\sqrt{2x+x^2}}]$

(o) $\displaystyle\int \frac{\mathrm{d}x}{(x-1)\sqrt{x^2+x+1}},$ $\quad [-\frac{1}{\sqrt{3}}\ln\left|\frac{3+3x+2\sqrt{3(x^2+x+1)}}{x-1}\right|]$

(p) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt{(4+x^2)^3}},$ $\quad [\frac{x}{4\sqrt{4+x^2}}]$

(q) $\displaystyle\int \frac{x^2}{\sqrt{(2-x^2)^3}}\,\mathrm{d}x,$ $\quad [\frac{x}{\sqrt{2-x^2}}-\arcsin\frac{x}{\sqrt{2}}]$

(r) $\displaystyle\int \frac{1+\sqrt{1-x^2}}{1-\sqrt{1-x^2}}\,\mathrm{d}x,$ $\quad [-\frac{2+x^2}{x}-\frac{2}{x}\sqrt{1-x^2}-2\arcsin x]$

(s) $\displaystyle\int \frac{\sqrt{1+x^2}}{2+x^2}\,\mathrm{d}x,$ $\quad [\frac{1}{2\sqrt{2}}\ln\frac{\sqrt{2x^2+2}-x}{\sqrt{2x^2+2}+x}+\ln(x+\sqrt{x^2+1})]$

(t) $\displaystyle\int x^{-1}(1+\sqrt[3]{x})^{-3}\,\mathrm{d}x,$ $\quad [3\left[\ln\left|\frac{\sqrt[3]{x}}{1+\sqrt[3]{x}}\right|+\frac{2\sqrt[3]{x}+3}{2(1+\sqrt[3]{x})^2}\right]]$

(u) $\displaystyle\int x^5\sqrt[3]{(1+x^3)^2}\,\mathrm{d}x,$ $\quad [\frac{5x^6+2x^3-3}{40}\sqrt[3]{(1+x^3)^2}]$

(v) $\displaystyle\int \frac{\mathrm{d}x}{x\sqrt[3]{1+x^5}},$ $\quad [\frac{1}{5}\ln\frac{|u-1|}{\sqrt{u^2+u+1}}+\frac{\sqrt{3}}{5}\operatorname{arctg}\frac{1+2u}{\sqrt{3}};\ u=\sqrt[3]{1+x^5}]$

(w) $\displaystyle\int \frac{\mathrm{d}x}{\sqrt[4]{1+x^4}},$ $\quad [\frac{1}{4}\ln\left|\frac{z+1}{z-1}\right|-\frac{1}{2}\operatorname{arctg} z;\ z=\frac{\sqrt[4]{1+x^4}}{x}]$

(x) $\displaystyle\int \sqrt[3]{3x-x^3}\,\mathrm{d}x.$ $\quad [\frac{3z}{2(z^3+1)}-\frac{1}{4}\ln\frac{(z+1)^2}{z^2-z+1}-\frac{\sqrt{3}}{2}\operatorname{arctg}\frac{2z-1}{\sqrt{3}};\ z=\frac{\sqrt[3]{3x-x^3}}{x}]$

Příklad 8 (goniometrické funkce)**.** Vypočtěte integrály

(a) $\displaystyle\int \sin^3 x \cos^2 x \,\mathrm{d}x,$ $\quad [\frac{1}{5}\cos^5 x - \frac{1}{3}\cos^3 x]$

(b) $\displaystyle\int \frac{\sin^3 x}{\cos^4 x}\,\mathrm{d}x,$ $\quad [\frac{1}{3\cos^3 x} - \frac{1}{\cos x}]$

(c) $\displaystyle\int \frac{\mathrm{d}x}{\cos x \sin^3 x},$ $\quad [\ln|\operatorname{tg} x| - \frac{1}{2\sin^2 x}]$

(d) $\displaystyle\int \frac{\sin^4 x}{\cos^2 x}\,\mathrm{d}x,$ $\quad [\operatorname{tg} x + \frac{1}{4}\sin 2x - \frac{3}{2}x]$

(e) $\displaystyle\int \frac{\mathrm{d}x}{\sin^4 x \cos^4 x},$ $\quad [\frac{(\operatorname{tg}^2 x - 1)(\operatorname{tg}^4 x + 10\operatorname{tg}^2 x + 1)}{3\operatorname{tg}^3 x}]$

(f) $\displaystyle\int \cos^5 x \,\mathrm{d}x,$ $\quad [\sin x - \frac{2}{3}\sin^3 x + \frac{1}{5}\sin^5 x]$

(g) $\displaystyle\int \sin^6 x \,\mathrm{d}x,$ $\quad [\frac{5}{16}x - \frac{1}{4}\sin 2x + \frac{3}{64}\sin 4x + \frac{1}{48}\sin^3 2x]$

(h) $\displaystyle\int \operatorname{cotg}^4 x \,\mathrm{d}x,$ $\quad [x - \frac{1}{3}\operatorname{cotg}^3 x + \operatorname{cotg} x]$

(i) $\displaystyle\int \operatorname{tg}^5 x \,\mathrm{d}x,$ $\quad [\frac{1}{4}\operatorname{tg}^4 x - \frac{1}{2}\operatorname{tg}^2 x - \ln|\cos x|]$

(j) $\displaystyle\int \frac{\cos^4 x + \sin^4 x}{\cos^2 x - \sin^2 x}\,\mathrm{d}x,$ $\quad [\frac{1}{4}\left(\ln\left|\frac{1+\operatorname{tg} x}{1-\operatorname{tg} x}\right| + \sin 2x\right)]$

(k) $\displaystyle\int \sin 5x \cos x \,\mathrm{d}x,$ $\quad [-\frac{1}{8}\cos 4x - \frac{1}{12}\cos 6x]$

(l) $\displaystyle\int \cos x \cos 2x \cos 3x \,\mathrm{d}x,$ $\quad [\frac{x}{4} + \frac{\sin 2x}{8} + \frac{\sin 4x}{16} + \frac{\sin 6x}{24}]$

(m) $\displaystyle\int \sin^3 2x \cos^2 3x \,\mathrm{d}x,$ $\quad [-\frac{3}{16}\cos 2x + \frac{3}{64}\cos 4x + \frac{1}{48}\cos 6x - \frac{3}{128}\cos 8x + \frac{1}{192}\cos 12x]$

(n) $\displaystyle\int \frac{\mathrm{d}x}{\sin x + \cos x},$ $\quad [\frac{\sqrt{2}}{2}\ln\left|\operatorname{tg}\left(\frac{\pi}{8} + \frac{x}{2}\right)\right|]$

(o) $\displaystyle\int \frac{\mathrm{d}x}{1 + \operatorname{tg} x},$ $\quad [\frac{1}{2}(x + \ln|\sin x + \cos x|)]$

(p) $\displaystyle\int \frac{\mathrm{d}x}{5 - 3\cos x},$ $\quad [\frac{1}{2}\operatorname{arctg}\left(2\operatorname{tg}\frac{x}{2}\right)]$

(q) $\displaystyle\int \frac{\mathrm{d}x}{5 + 4\sin x},$ $\quad [\frac{2}{3}\operatorname{arctg}\frac{5\operatorname{tg}\frac{x}{2} + 4}{3}]$

(r) $\displaystyle\int \frac{\sin^2 x}{1 + \sin^2 x}\,\mathrm{d}x,$ $\quad [x - \frac{1}{\sqrt{2}}\operatorname{arctg}(\sqrt{2}\operatorname{tg} x)]$

(s) $\displaystyle\int \frac{\mathrm{d}x}{a^2\sin^2 x + b^2\cos^2 x},\quad a, b \in \mathbb{R},$ $\quad [\frac{1}{ab}\operatorname{arctg}\left(\frac{a\operatorname{tg} x}{b}\right)]$

(t) $\displaystyle\int \frac{\mathrm{d}x}{1 + \varepsilon\cos x},\quad \varepsilon > 0,$ $\quad [\frac{2}{\sqrt{1-\varepsilon^2}}\operatorname{arctg}\left(\sqrt{\frac{1-\varepsilon}{1+\varepsilon}}\operatorname{tg}\frac{x}{2}\right);\ 0 < \varepsilon < 1$

$\frac{1}{\sqrt{\varepsilon^2-1}} \ln \frac{\varepsilon+\cos x+\sqrt{\varepsilon^2-1}\sin x}{1+\varepsilon\cos x};\ \varepsilon > 1]$

(u) $\displaystyle\int \frac{\sin x \cos x}{\sin x + \cos x}\,\mathrm{d}x,$ $[\frac{1}{2}(\sin x - \cos x) - \frac{1}{2\sqrt{2}} \ln\left|\operatorname{tg}\left(\frac{x}{2}+\frac{\pi}{8}\right)\right|]$

(v) $\displaystyle\int \frac{\mathrm{d}x}{\sin^4 x + \cos^4 x},$ $[\frac{1}{\sqrt{2}} \operatorname{arctg}\left(\frac{\operatorname{tg} 2x}{\sqrt{2}}\right)]$

(w) $\displaystyle\int \frac{\mathrm{d}x}{\sin^3 x + \cos^3 x},$ $[-\frac{2}{3}\operatorname{arctg}(\cos x - \sin x) + \frac{\sqrt{2}}{3} \ln \frac{\sqrt{2}-1+\operatorname{tg}\frac{x}{2}}{\sqrt{2}+1-\operatorname{tg}\frac{x}{2}}]$

(x) $\displaystyle\int \frac{\sin^2 x - \cos^2 x}{\sin^4 x + \cos^4 x}\,\mathrm{d}x,$ $[\frac{1}{2\sqrt{2}} \ln \frac{\sqrt{2}-\sin 2x}{\sqrt{2}+\sin 2x}]$

(y) $\displaystyle\int \frac{\mathrm{d}x}{a\sin x + b\cos x},\quad a, b \in \mathbb{R}.$ $[\frac{1}{\sqrt{a^2+b^2}} \ln\left|\frac{\operatorname{tg}\frac{x}{2}-t_1}{\operatorname{tg}\frac{x}{2}-t_2}\right|;\ t_1 = \frac{a-\sqrt{a^2+b^2}}{b};\ t_2 = \frac{a+\sqrt{a^2+b^2}}{b}]$

Příklad 9. Označme

$$I_n = \int \sin^n x\,\mathrm{d}x,\ K_n = \int \cos^n x\,\mathrm{d}x,\ L_n = \int \frac{\mathrm{d}x}{\sin^n x},\ M_n = \int \frac{\mathrm{d}x}{\cos^n x};\quad n \geq 2.$$

Ukažte, že platí

$$I_n = -\frac{\cos x \sin^{n-1} x}{n} + \frac{n-1}{n} I_{n-2}, \qquad K_n = \frac{\sin x \cos^{n-1} x}{n} + \frac{n-1}{n} K_{n-2},$$

$$L_n = -\frac{\cos x}{(n-1)\sin^{n-1} x} + \frac{n-2}{n-1} L_{n-2}, \qquad M_n = \frac{\sin x}{(n-1)\cos^{n-1} x} + \frac{n-2}{n-1} M_{n-2}.$$

Příklad 10 (hyperbolické funkce)**.** Vypočtěte integrály

(a) $\displaystyle\int \sinh^2 x\,\mathrm{d}x,$ $[\frac{\sinh 2x}{4} - \frac{x}{2}]$

(b) $\displaystyle\int \operatorname{tgh}^2 x\,\mathrm{d}x,$ $[x - \operatorname{tgh} x]$

(c) $\displaystyle\int \sinh^3 x\,\mathrm{d}x,$ $[\frac{1}{3}\cosh^3 x - \cosh x]$

(d) $\displaystyle\int \frac{\mathrm{d}x}{\sinh x},$ $[\ln\left|\operatorname{tg}\frac{x}{2}\right|]$

(e) $\displaystyle\int \frac{x}{\cosh^2 x}\,\mathrm{d}x.$ $[x \operatorname{tgh} x - \ln\cosh x]$

Příklad 11.

(a) $\displaystyle\int x^3\,\mathrm{e}^{3x}\,\mathrm{d}x,$ $[\mathrm{e}^{3x}\left(\frac{x^3}{3} - \frac{x^2}{3} + \frac{2x}{9} - \frac{2}{27}\right)]$

(b) $\displaystyle\int x^5 \sin 5x\,\mathrm{d}x,$ $[-\left(\frac{x^5}{5} - \frac{4x^3}{25} + \frac{24x}{625}\right)\cos 5x + \left(\frac{x^4}{5} - \frac{12x^2}{125} + \frac{24}{3125}\right)\sin 5x]$

(c) $\displaystyle\int x^7\,\mathrm{e}^{-x^2}\,\mathrm{d}x,$ $[-\frac{\mathrm{e}^{-x^2}}{2}(x^6 + 3x^4 + 6x^2 + 6)]$

(d) $\displaystyle\int e^{ax}\cos^2 bx\,dx,$ $\quad [e^{ax}\left[\frac{1}{2a}+\frac{a\cos 2bx+2b\sin 2bx}{2(a^2+4b^2)}\right]]$

(e) $\displaystyle\int x\,e^{x}\sin x\,dx,$ $\quad [\frac{e^x}{2}[x(\sin x-\cos x)+\cos x]]$

(f) $\displaystyle\int x^2\,e^{x}\cos x\,dx,$ $\quad [\frac{e^x}{2}[(x-1)^2\sin x+(x^2-1)\cos x]]$

Příklad 12. Vypočtěte integrály

(a) $\displaystyle\int \frac{e^{2x}}{1+e^x}\,dx,$ $\quad [e^x-\ln(1+e^x)]$

(b) $\displaystyle\int \frac{dx}{1+e^{\frac{x}{2}}+e^{\frac{x}{3}}+e^{\frac{x}{6}}},$ $\quad [x-3\ln\left[(1+e^{\frac{x}{6}})\sqrt{1+e^{\frac{x}{3}}}\right]-3\operatorname{arctg} e^{\frac{x}{6}}]$

(c) $\displaystyle\int \sqrt{\frac{e^x-1}{e^x+1}}\,dx,$ $\quad [\ln\left(e^x+\sqrt{e^{2x}-1}\right)+\arcsin e^{-x}]$

(d) $\displaystyle\int \frac{dx}{\sqrt{1+e^x+e^{2x}}},$ $\quad [x-\ln|2+e^x+2\sqrt{1+e^x+e^{2x}}|]$

(e) $\displaystyle\int \ln^2(x+\sqrt{1+x^2})\,dx,$ $\quad [x\ln^2(x+\sqrt{1+x^2})-2\sqrt{1+x^2}\ln(x+\sqrt{1+x^2})+2x]$

(f) $\displaystyle\int \ln(\sqrt{1-x}+\sqrt{1+x})\,dx,$ $\quad [-\frac{x}{2}+x\ln(\sqrt{1-x}+\sqrt{1+x})+\frac{1}{2}\arcsin x]$

(g) $\displaystyle\int \frac{\ln x}{(1+x^2)^{\frac{3}{2}}}\,dx,$ $\quad [\frac{x\ln x}{\sqrt{1+x^2}}-\ln(x+\sqrt{1+x^2})=\frac{x\ln x}{\sqrt{1+x^2}}-\operatorname{argsinh} x]$

(h) $\displaystyle\int \sqrt{x}\operatorname{arctg}\sqrt{x}\,dx,$ $\quad [-\frac{x}{3}+\frac{1}{3}\ln(1+x)+\frac{2x\sqrt{x}}{3}\operatorname{arctg}\sqrt{x}]$

(i) $\displaystyle\int x\ln\frac{1+x}{1-x}\,dx,$ $\quad [x+\frac{x^2-1}{2}\ln\frac{1+x}{1-x}]$

(j) $\displaystyle\int \frac{dx}{a^2-b^2\cos^2 x},\quad a,b\in\mathbb{R},$ $\quad [\frac{1}{a\sqrt{a^2-b^2}}\operatorname{arctg}\frac{a\operatorname{tg} x}{\sqrt{a^2-b^2}};\ a^2-b^2>0,$
$\frac{1}{2a\sqrt{b^2-a^2}}\ln\frac{a\operatorname{tg} x-\sqrt{b^2-a^2}}{a\operatorname{tg} x+\sqrt{b^2-a^2}};\ a^2-b^2<0]$

(k) $\displaystyle\int \frac{x\,e^x}{\sqrt{1+e^x}}\,dx,$ $\quad [(2x-4)\sqrt{1+e^x}+2\ln\frac{\sqrt{1+e^x}+1}{\sqrt{1+e^x}-1}]$

(l) $\displaystyle\int \frac{x\operatorname{arctg} x}{(1+x^2)^2}\,dx,$ $\quad [\frac{x+(x^2-1)\operatorname{arctg} x}{4(x^2+1)}]$

(m) $\displaystyle\int \frac{x\,e^x}{(1+x)^2}\,dx,$ $\quad [\frac{e^x}{1+x}]$

(n) $\displaystyle\int \frac{\operatorname{tg} x}{1+\operatorname{tg} x+\operatorname{tg}^2 x}\,dx,$ $\quad [x-\frac{2}{\sqrt{3}}\operatorname{arctg}\frac{1+2\operatorname{tg} x}{\sqrt{3}}]$

(o) $\displaystyle\int \sinh ax\cos bx\,dx,$ $\quad [\frac{a\cosh ax\cos bx+b\sinh ax\sin bx}{a^2+b^2}]$

(p) $\displaystyle\int |x|\,dx,$ $\quad [\frac{x|x|}{2}]$

(q) $\displaystyle\int \mathrm{e}^{-|x|}\,\mathrm{d}x,$ $[\mathrm{e}^{x} - 1 \text{ pro } x < 0;\ 1 - \mathrm{e}^{-x} \text{ pro } x \geq 0]$

(r) $\displaystyle\int \max\{1, x^2\}\,\mathrm{d}x.$ $[x \text{ pro } |x| \leq 1;\ \frac{x^3}{3} + \frac{2}{3}\operatorname{sgn} x \text{ pro } |x| > 1]$

2 Určitý integrál

Příklad 1.

(a) $\displaystyle\int_{-1}^{8} \sqrt[3]{x}\,\mathrm{d}x,$ $[\frac{45}{4}]$

(b) $\displaystyle\int_{2}^{-13} \frac{\mathrm{d}x}{\sqrt[5]{(3-x)^4}},$ $[-5(\sqrt[5]{16} - 1)]$

(c) $\displaystyle\int_{0}^{16} \frac{\mathrm{d}x}{\sqrt{x+9} - \sqrt{x}},$ $[12]$

(d) $\displaystyle\int_{1}^{\mathrm{e}} \frac{\mathrm{d}x}{x\sqrt{1 - \ln^2 x}},$ $[\frac{\pi}{2}]$

(e) $\displaystyle\int_{0}^{\sqrt[n]{\frac{a}{2}}} \frac{x^{n-1}}{\sqrt{a^2 - x^{2n}}}\,\mathrm{d}x, \quad a \in \mathbb{R},\ n \in \mathbb{N},$ $[\frac{\pi}{6n}]$

(f) $\displaystyle\int_{\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{x^3}{\sqrt{\frac{5}{8} - x^4}}\right)^3 \mathrm{d}x,$ $[\frac{4}{3}]$

(g) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \frac{\mathrm{d}x}{1 + \cos x},$ $[2]$

(h) $\displaystyle\int_{0}^{\frac{\pi}{2}} \cos^5 x \sin 2x\,\mathrm{d}x,$ $[\frac{2}{7}]$

(i) $\displaystyle\int_{-\frac{\pi}{2}}^{-\frac{\pi}{4}} \frac{\cos^3 x}{\sqrt[3]{\sin x}}\,\mathrm{d}x,$ $[\frac{21}{16\sqrt[3]{2}} - \frac{9}{8}]$

(j) $\displaystyle\int_{1}^{2} x \log_2 x\,\mathrm{d}x,$ $[2 - \frac{3}{4\ln 2}]$

(k) $\displaystyle\int_{0}^{\mathrm{e}-1} \ln(x+1)\,\mathrm{d}x,$ $[1]$

(l) $\displaystyle\int_{0}^{a} \sqrt{a^2 - x^2}\,\mathrm{d}x, \quad a \in \mathbb{R},$ $[\frac{\pi a^2}{4}]$

(m) $\displaystyle\int_{0}^{\frac{\pi}{2}} \mathrm{e}^{2x} \cos x\,\mathrm{d}x,$ $[\frac{\mathrm{e}^{\pi} - 2}{5}]$

(n) $\displaystyle\int_{4}^{9} \frac{\sqrt{x}}{\sqrt{x} - 1}\,\mathrm{d}x,$ $[7 + 2\ln 2]$

(o) $\displaystyle\int_0^1 \frac{\sqrt{e^x}}{\sqrt{e^x+e^{-x}}}\,dx,$ $[\ln \frac{e+\sqrt{1+e^2}}{1+\sqrt{2}} = \operatorname{argsinh} e - \operatorname{argsinh} 1]$

(p) $\displaystyle\int_0^1 x^2\sqrt{1-x^2}\,dx,$ $[\frac{\pi}{16}]$

(q) $\displaystyle\int_1^{\sqrt{3}} \frac{\sqrt{1+x^2}}{x^2}\,dx,$ $[\sqrt{2}-\frac{2}{\sqrt{3}}+\ln\frac{2+\sqrt{3}}{1+\sqrt{2}}]$

(r) $\displaystyle\int_0^{-\ln 2} \sqrt{1-e^{2x}}\,dx,$ $[\frac{\sqrt{3}}{2}+\ln(2-\sqrt{3})]$

(s) $\displaystyle\int_{\sqrt{2}}^{2\sqrt{5}} \frac{dx}{x\sqrt{(x^2+5)^5}},$ $[\frac{1}{50\sqrt{5}}\ln\frac{3+\sqrt{5}}{2}-\frac{2^4\cdot 23}{3^4\cdot 5^4}]$

(t) $\displaystyle\int_0^1 \sqrt{2x+x^2}\,dx,$ $[\sqrt{3}-\frac{1}{2}\ln(2+\sqrt{3})]$

(u) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{2\cos x+3},$ $[\frac{2}{\sqrt{5}}\operatorname{arctg}\frac{1}{\sqrt{5}}]$

(v) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{1}{6}\sin^2 x},$ $[\frac{\pi}{2}\sqrt{\frac{6}{7}}]$

(w) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{\sin x\cos x}{a^2\cos^2 x+b^2\sin^2 x}\,dx, \quad a,b\in\mathbb{R},$ $[\frac{1}{a^2-b^2}\ln\left|\frac{a}{b}\right|]$

(x) $\displaystyle\int_0^{2\pi} \frac{dx}{1+\varepsilon\cos x}, \quad 0\le\varepsilon<1,$ $[\frac{2\pi}{\sqrt{1-\varepsilon^2}}]$

(y) $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{a^2\sin^2 x+b^2\cos^2 x}, \quad a,b\in\mathbb{R}\setminus\{0\}.$ $[\frac{\pi}{2|ab|}]$

Příklad 2. Buď

$$I_{m,n} = \int_0^{\frac{\pi}{2}} \sin^m x\cos^n x\,dx, \qquad m,n\ge 0.$$

Ukažte, že platí

$$I_{m,n} = \frac{n-1}{m+n}I_{m,n-2} = \frac{m-1}{m+n}I_{m-2,n}.$$

Užitím této formule vypočtěte

$$\int_0^1 x^p(1-x)^q\,dx, \qquad p,q\in\mathbb{N}.$$

$[\frac{p!q!}{(p+q+1)!}$; položte $x=\sin^2 t]$

Příklad 3. Vypočtěte následující integrály tak, že je vyjádříte jako limitu integrálních součtů

(a) $\displaystyle\int_{-1}^{2} x^2\,\mathrm{d}x,$ [3]

(b) $\displaystyle\int_{0}^{\frac{\pi}{2}} \sin x\,\mathrm{d}x.$ [1]

Příklad 4. Užitím určitého integrálu najděte

(a) $\displaystyle\lim_{n\to\infty}\sum_{i=1}^{n}\frac{1}{n+i},$ [$\ln 2$; užijte $\displaystyle\int_1^2 \frac{\mathrm{d}x}{x}$]

(b) $\displaystyle\lim_{n\to\infty} n\sum_{k=1}^{n}\frac{1}{n^2+k^2},$ [$\frac{\pi}{4}$; užijte $\displaystyle\int_0^1 \frac{\mathrm{d}x}{1+x^2}$]

(c) $\displaystyle\lim_{n\to\infty}\frac{1}{n^{p+1}}\sum_{k=1}^{n}k^p, \quad p>0.$ [$\frac{1}{p+1}$; užijte $\displaystyle\int_0^1 x^p\,\mathrm{d}x$]

Příklad 5 (derivace integrálu)**.** Najděte

(a) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_a^b \sin x^2\,\mathrm{d}x,$ [0]

(b) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}a}\int_a^b \sin x^2\,\mathrm{d}x,$ [$-\sin a^2$]

(c) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}b}\int_a^b \sin x^2\,\mathrm{d}x.$ [$\sin b^2$]

Příklad 6 (derivace podle meze)**.** Vypočtěte derivace

(a) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_0^x \frac{1-t+t^2}{1+t+t^2}\,\mathrm{d}t, \quad \text{pro } x=1,$ [$\frac{1}{3}$]

(b) $\displaystyle\frac{\mathrm{d}}{\mathrm{d}x}\int_x^5 \sqrt{1+t^2}\,\mathrm{d}t, \quad \text{pro } x=0;\ \text{pro } x=\frac{3}{4}.$ [-1; $-\frac{5}{4}$]

Příklad 7. Vypočtěte

(a) $\displaystyle\lim_{x\to 0}\frac{1}{x}\int_0^x \cos t^2\,\mathrm{d}t,$ [1]

(b) $\displaystyle\lim_{x\to\infty}\frac{1}{\sqrt{x^2+1}}\int_0^x \operatorname{arctg}^2 t\,\mathrm{d}t,$ [$\frac{\pi^2}{4}$]

(c) $\displaystyle\lim_{x\to\infty}\frac{\left(\int_0^x \mathrm{e}^{t^2}\,\mathrm{d}t\right)^2}{\int_0^x \mathrm{e}^{2t^2}\,\mathrm{d}t}.$ [0; užijte l'Hospitalova pravidla]

Příklad 8. Najděte $\int_a^b f(x)\,\mathrm{d}x$, je-li

(a) $a = 0, \quad b = 2, \quad f(x) = \begin{cases} x^2 & \text{pro} \quad 0 \le x \le 1, \\ 2 - x & \text{pro} \quad 1 < x \le 2, \end{cases}$ $[\frac{5}{6}]$

(b) $a = 0, \quad b = 1, \quad f(x) = \begin{cases} x & \text{pro} \quad 0 \le x \le t, \\ t\dfrac{1-x}{1-t} & \text{pro} \quad t < x \le 1, \end{cases}$ $[\frac{t}{2}]$

(c) $a = -2, \quad b = 2, \quad f(x) = \begin{cases} -x & \text{pro} \quad x \in \langle -2, -1 \rangle, \\ 0 & \text{pro} \quad x \in (-1, 0), \\ \sqrt{1-x^2} & \text{pro} \quad x \in \langle 0, 1 \rangle, \\ 2 - x & \text{pro} \quad x \in (1, 2 \rangle. \end{cases}$ $[2 + \frac{\pi}{4}]$

Příklad 9. Buďte $f(x)$ sudá a $g(x)$ lichá funkce, které jsou intergrovatelné na intervalu $\langle -a, a \rangle$. Ukažte, že

$$\int_{-a}^{a} f(x)\,\mathrm{d}x = 2\int_0^a f(x)\,\mathrm{d}x; \qquad \int_{-a}^{a} g(x)\,\mathrm{d}x = 0.$$

Příklad 10. Pomocí vět o střední hodnotě odhadněte integrál I

(a) $\displaystyle\int_0^{2\pi} \frac{\mathrm{d}x}{1 + \frac{1}{2}\cos x},$ $[\frac{4\pi}{3} \le I \le 4\pi]$

(b) $\displaystyle\int_0^1 \frac{x^9}{\sqrt{1+x}}\,\mathrm{d}x,$ $[\frac{1}{10\sqrt{2}} \le I \le \frac{1}{10}]$

(c) $\displaystyle\int_0^{100} \frac{\mathrm{e}^{-x}}{x + 100}\,\mathrm{d}x,$ $[\frac{1}{200} \le I \le \frac{1}{100}]$

(d) $\displaystyle\int_{100\pi}^{200\pi} \frac{\sin x}{x}\,\mathrm{d}x,$ $[|I| \le \frac{1}{50\pi}]$

(e) $\displaystyle\int_a^b \frac{\mathrm{e}^{-\alpha x}}{x}\,\mathrm{d}x.$ $[|I| \le \frac{2\,\mathrm{e}^{-\alpha a}}{a}]$

Příklad 11 (nevlastní integrál)**.** Vypočtěte integrály

(a) $\displaystyle\int_1^{\infty} \frac{\mathrm{d}x}{x^4},$ $[\frac{1}{3}]$

(b) $\displaystyle\int_0^{\infty} \mathrm{e}^{-ax}\,\mathrm{d}x, \quad a > 0,$ $[\frac{1}{a}]$

(c) $\displaystyle\int_{-\infty}^{+\infty} \frac{2x}{x^2+1}\,\mathrm{d}x,$ [diverguje]

(d) $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{x^2+2x+2},$ $[\pi]$

(e) $\displaystyle\int_2^{\infty} \frac{\ln x}{x}\,\mathrm{d}x,$ [diverguje]

(f) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x^2(x+1)}$, $[1-\ln 2]$

(g) $\displaystyle\int_0^\infty x\,\mathrm{e}^{-x^2}\,\mathrm{d}x$, $[\frac{1}{2}]$

(h) $\displaystyle\int_0^\infty x\sin x\,\mathrm{d}x$, [diverguje]

(i) $\displaystyle\int_0^\infty \mathrm{e}^{-ax}\cos bx\,\mathrm{d}x$, $[\frac{a}{a^2+b^2}$ pro $a>0$; diverguje pro $a\le 0]$

(j) $\displaystyle\int_1^\infty \frac{\operatorname{arctg} x}{x^2}\,\mathrm{d}x$, $[\frac{\pi}{4}+\frac{1}{2}\ln 2]$

(k) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{1+x^3}$, $[\frac{2\pi}{3\sqrt{3}}]$

(l) $\displaystyle\int_{-\infty}^{+\infty} \frac{\mathrm{d}x}{(x^2+x+1)^2}$, $[\frac{4\pi}{3\sqrt{3}}]$

(m) $\displaystyle\int_0^\infty \frac{\operatorname{arctg} x}{(1+x^2)^{\frac{3}{2}}}\,\mathrm{d}x$, $[\frac{\pi}{2}-1]$

(n) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\sqrt{1-x^2}}$, $[\frac{\pi}{2}]$

(o) $\displaystyle\int_1^2 \frac{x}{\sqrt{x-1}}\,\mathrm{d}x$, $[\frac{8}{3}]$

(p) $\displaystyle\int_1^{\mathrm{e}} \frac{\mathrm{d}x}{x\sqrt{\ln x}}$, $[2]$

(q) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{1-x^2+2\sqrt{1-x^2}}$, $[\frac{\pi}{3\sqrt{3}}]$

(r) $\displaystyle\int_{-1}^1 \frac{x-1}{\sqrt[3]{x^5}}\,\mathrm{d}x$, [diverguje]

(s) $\displaystyle\int_a^b \frac{\mathrm{d}x}{\sqrt{(x-a)(b-x)}}$, $\quad a<b$, $[\pi]$

(t) $\displaystyle\int_a^b \frac{x}{\sqrt{(x-a)(b-x)}}\,\mathrm{d}x$, $\quad a<b$, $[\frac{\pi(a+b)}{2}]$

(u) $\displaystyle\int_1^\infty \frac{\mathrm{d}x}{x\sqrt{x^2-1}}$, $[\frac{\pi}{2}]$

(v) $\displaystyle\int_0^\infty x^n\,\mathrm{e}^{-x}\,\mathrm{d}x$, $\quad n\in\mathbb{N}$, $[n!]$

(w) $\displaystyle\int_0^\infty x^{2n+1}\,\mathrm{e}^{-x^2}\,\mathrm{d}x$, $\quad n\in\mathbb{N}$, $[\frac{n!}{2}]$

(x) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{(a^2+x^2)^n}$, $\quad n\in\mathbb{N}$, $[\frac{(2n-3)!!}{(2n-2)!!}\frac{\pi}{2a^{2n-1}}]$

(y) $\displaystyle\int_0^1 \frac{x^n}{\sqrt{1-x^2}}\,\mathrm{d}x, \quad n \in \mathbb{N}.$ [$\frac{(2k-1)!!}{(2k)!!}\frac{\pi}{2}$ pro $n = 2k$; $\frac{(2k)!!}{(2k+1)!!}$ pro $n = 2k+1$]

Příklad 12. Vyšetřete konvergenci integrálů

(a) $\displaystyle\int_1^\infty \frac{\mathrm{e}^{-x}}{x}\,\mathrm{d}x,$ [konverguje]

(b) $\displaystyle\int_0^\infty \frac{\mathrm{d}x}{\sqrt{1+x^3}},$ [konverguje]

(c) $\displaystyle\int_2^\infty \frac{\mathrm{d}x}{\sqrt[3]{x^3-1}},$ [diverguje]

(d) $\displaystyle\int_1^\infty \frac{\sin x}{x^4}\,\mathrm{d}x,$ [konverguje]

(e) $\displaystyle\int_0^\infty \frac{x^2}{x^4-x^2+1}\,\mathrm{d}x,$ [konverguje]

(f) $\displaystyle\int_0^2 \frac{\mathrm{d}x}{\ln x},$ [diverguje]

(g) $\displaystyle\int_0^\infty x^{p-1}\,\mathrm{e}^{-x}\,\mathrm{d}x,$ [konverguje pro $p > 0$]

(h) $\displaystyle\int_0^\infty \frac{x^m}{1+x^n}\,\mathrm{d}x, \quad m, n \in \mathbb{N},$ [konverguje pro $n - m > 1$]

(i) $\displaystyle\int_0^1 \frac{\sqrt{x}}{\sqrt{1-x^4}}\,\mathrm{d}x,$ [konverguje]

(j) $\displaystyle\int_0^1 \frac{x^2}{\sqrt[3]{(1-x^2)^5}}\,\mathrm{d}x,$ [diverguje]

(k) $\displaystyle\int_0^1 \frac{\sqrt{x}}{\mathrm{e}^{\sin x}-1}\,\mathrm{d}x,$ [konverguje]

(l) $\displaystyle\int_0^1 \frac{\mathrm{d}x}{\mathrm{e}^x-\cos x}.$ [diverguje]

3 Užití určitého integrálu

Příklad 1. Najděte obsah rovinného oboru, omezeného křivkami

(a) $ax = y^2;\ ay = x^2,$ [$\frac{a^2}{3}$]

(b) $y = 2x = x^2;\ x + y = 0,$ [$\frac{9}{2}$]

(c) $y = x;\ y = x + \sin^2 x, \quad 0 \le x \le \pi,$ [$\frac{\pi}{2}$]

(d) $y = -x^2 + 4x - 3$ a tečnami k této křivce v bodech $[0, -3]$ a $[3, 0]$, [$\frac{9}{4}$]

(e) $x^2 + y^2 = 8;\ y = \dfrac{x^2}{2}$ (dva obory), [$2\pi + \frac{4}{3};\ 6\pi - \frac{4}{3}$]

(f) $x^2 + y^2 = a^2$; $x^2 - 2y^2 = \dfrac{a^2}{4}$ (tři obory), $[a^2\left[\frac{\pi}{6} - \frac{\sqrt{2}}{8}\ln(\sqrt{3}+\sqrt{2})\right]$ dva obory; $a^2\left[\frac{2\pi}{3} + \frac{\sqrt{2}}{4}\ln(\sqrt{3}+\sqrt{2})\right]]$

(g) $y^2 = x^3$; $y = 8$; $x = 0$, [19,2]

(h) $y^2 = x^2(a^2 - x^2)$, $[\frac{4}{3}a^3]$

(i) $4(y^2 - x^2) + x^3 = 0$, $[\frac{128}{15}]$

(j) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ (astroida). $[\frac{3\pi a^2}{8}]$

Příklad 2. Najděte obsah rovinného oboru, omezeného parametricky zadanou křivkou $x = \psi(t)$, $y = \varphi(t)$, $t \in \langle\alpha, \beta\rangle$

(a) $x = a(t - \sin t)$, $y = a(1 - \cos t)$ (cykloida) $t \in \langle 0, 2\pi\rangle$; $y = 0$, $[3\pi a^2]$

(b) $x = 3t^2$, $y = 3t - t^3$, $[\frac{72}{5}\sqrt{3}]$

(c) $x = t^2 - 1$, $y = t^3 - t$, $[\frac{8}{15}]$

(d) $x = a(2\cos t - \cos 2t)$, $y = a(2\sin t - \sin 2t)$ (kardioida), $[6\pi a^2]$

(e) $x^3 + y^3 - 3axy = 0$ (Descartův list). $[\frac{3}{2}a^2$; předpokládejte, že $y = tx]$

Návod. Užijte vzorců

$$S = \left|\int_\alpha^\beta \psi(t)\varphi'(t)\,\mathrm{d}t\right|$$

nebo

$$(*) \qquad S = \frac{1}{2}\left|\int_\alpha^\beta (\varphi'(t)\psi(t) - \varphi(t)\psi'(t))\,\mathrm{d}t\right|.$$

Příklad 3. Užitím vzorce (∗) ukažte, že pro křivku zadanou v polárních souřadnicích platí

$$(\dagger) \qquad S = \frac{1}{2}\int_{\varphi_0}^{\varphi_1} r^2(\varphi)\,\mathrm{d}\varphi.$$

Příklad 4. Užitím vzorce (†) najděte obsah rovinného oboru, omezeného křivkou v polárním tvaru $r = r(\varphi)$

(a) $r^2 = a^2\cos 2\varphi$ (lemniskata), $[a^2]$

(b) $r = a\cos 5\varphi$ (pětilistá růže), $[\frac{\pi a^2}{4}]$

(c) $r = a\sin\varphi$ (čtyřlistá růže), $[\frac{\pi a^2}{4}]$

(d) $r = a|\sin 2\varphi|$ (čtyřlístek), $[\frac{\pi a^2}{2}]$

(e) $r = 2 + \cos 2\varphi$ vně křivky $r = 2 + \sin\varphi$. $[\frac{51\sqrt{3}}{16}]$

Příklad 5. Najděte délku oblouku křivky

(a) $y^2 = x^3$ pro přímku $x = \dfrac{4}{3}$, $[\frac{112}{27}]$

(b) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$, $[6a]$

(c) $y = a\cosh\dfrac{x}{a},\ 0 \le x \le b$, $[a\sinh\frac{b}{a}]$

(d) $y^2 = 2px,\ 0 \le x \le x_0$, $[2\sqrt{x_0\left(x_0+\frac{p}{2}\right)} + p\ln\frac{\sqrt{x_0}+\sqrt{x_0+\frac{p}{2}}}{\sqrt{\frac{p}{2}}}]$

(e) $y = \sqrt{x-x^2} + \arcsin\sqrt{x}$, $[2]$

(f) $x = a(t-\sin t),\ y = a(1-\cos t),\ 0 \le t \le 2\pi$, $[8a]$

(g) $x = a(\cos t + t\sin t),\ y = a(\sin t - t\cos t),\ t \in \langle 0, 2\pi\rangle$, $[2\pi^2 a]$

(h) $\left(\dfrac{x}{a}\right)^{\frac{4}{3}} + \left(\dfrac{y}{b}\right)^{\frac{4}{3}} = 1,\ a, b > 0$, $[4\frac{a^2+ab+b^2}{a+b}]$

(i) $r = a(1+\cos\varphi)$, $[8a]$

(j) $r = a\varphi$ (Archimedova spirála) $0 \le \varphi \le 2\pi$, $[\pi a\sqrt{1+4\pi^2} + \frac{a}{2}\ln\left(2\pi + \sqrt{1+4\pi^2}\right)]$

Příklad 6. Najděte objem rotačního tělesa, které vznikne rotací oboru

(a) $y = \sin x,\ y = 0,\ 0 \le x \le \pi$, kolem osy x; kolem osy y, $[\frac{\pi^2}{2};\ 2\pi^2]$

(b) $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1;\ y = \pm b$, kolem osy x; kolem osy y, $[\frac{4\pi ab^2}{3}(2\sqrt{2}-1);\ \frac{8\pi a^2 b}{3}]$

(c) $x^{\frac{2}{3}} + y^{\frac{2}{3}} = a^{\frac{2}{3}}$ kolem osy symetrie, $[\frac{32\pi a^3}{105}]$

(d) $x^2 + (y-b)^2 = a^2,\ 0 < a \le b$, kolem osy x, $[2\pi^2 a^2 b]$

(e) $x = a(t-\sin t),\ y = a(1-\cos t),\ 0 \le t \le 2\pi;\ y = 0$, kolem osy x; kolem osy y; kolem přímky $y = 2a$. $[5\pi^2 a^3;\ 6\pi^3 a^3;\ 7\pi^2 a^3]$

Příklad 7. Najděte obsah pláště rotačního tělesa, které vznikne rotací křivky

(a) $y = a\cosh\dfrac{x}{a},\ -a \le x \le a$, kolem osy x, $[\pi a^2(\sinh 2 + 2)]$

(b) $4x^2 + y^2 = 4$ kolem osy x; kolem osy y, $[4\pi\left(2 + \frac{\ln(2+\sqrt{3})}{\sqrt{3}}\right);\ 2\pi\left(1 + \frac{4\pi}{3\sqrt{3}}\right)]$

(c) $y = x\sqrt{\dfrac{x}{a}},\ 0 \le x \le a$, kolem osy x, $[\frac{4\pi a^2}{243}\left(21\sqrt{13} + 2\ln\frac{3+\sqrt{13}}{2}\right)]$

(d) $y = \operatorname{tg} x,\ 0 \le x \le \dfrac{\pi}{4}$, kolem osy x, $[\pi\left[\sqrt{5} - \sqrt{2} + \ln\frac{(\sqrt{2}+1)(\sqrt{5}-1)}{2}\right]]$

(e) $x^2 + (y-b)^2 = a^2,\ b \ge a$, kolem osy x, $[4\pi^2 ab]$

(f) $x = a(t-\sin t),\ y = a(1-\cos t),\ 0 \le t \le 2\pi$, kolem osy x; kolem osy y; kolem přímky $y = 2a$, $[\frac{64}{3}\pi a^2;\ 16\pi^2 a^2;\ \frac{32}{3}\pi a^2]$

(g) $x = a\cos^3 t,\ y = a\sin^3 t$ kolem osy x; kolem přímky $y = x$, $[\frac{12}{5}\pi a^2;\ \frac{3\pi}{5}a^2(4\sqrt{2}-1)]$

(h) $r = a(1+\cos\varphi)$ kolem polární osy, $[\frac{32}{5}\pi a^2]$

(i) $r^2 = a^2\cos 2\varphi$ kolem polární osy; kolem osy $\varphi = \dfrac{\pi}{2}$; kolem osy $\varphi = \dfrac{\pi}{4}$. $[2\pi a^2(2-\sqrt{2});\ 2\pi a^2\sqrt{2};\ 4\pi a^2]$