INVESTMENTS IN EDUCATION DEVELOPMENT

Deterministic models of natural selection and their relation to ecology

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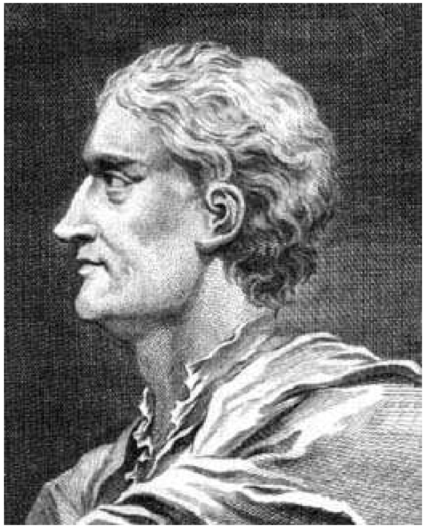
Introduction

Theoretical background

Newton law of motion

$$\begin{aligned} \boldsymbol{p} &= m\boldsymbol{x}', \\ \boldsymbol{F} &= m\boldsymbol{x}'' \end{aligned}$$



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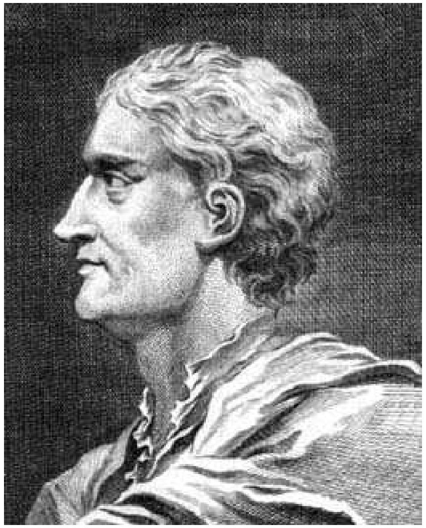

Simple rearrangement:

$$\boldsymbol{x}' = \frac{1}{m}\boldsymbol{p}$$

$$\boldsymbol{F} = m\boldsymbol{x}'' = m\left(\boldsymbol{x}'\right)' = m\left(\frac{\boldsymbol{p}}{m}\right)' = \boldsymbol{p}'$$

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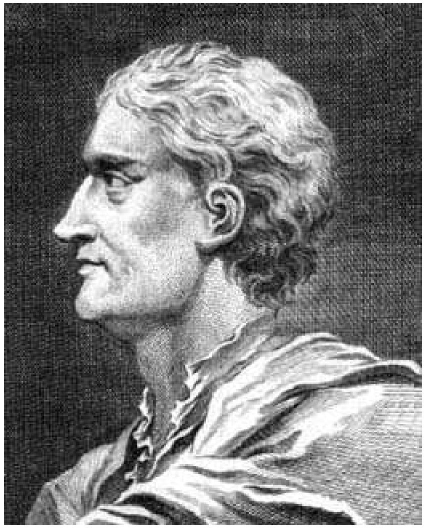

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Hence

$$\boldsymbol{x}' = \frac{1}{m}\boldsymbol{p}$$

$$\boldsymbol{p}' = \boldsymbol{F}(\boldsymbol{x})$$

Bipartite system

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In general:

$$\boldsymbol{x}' = \boldsymbol{f}(\boldsymbol{y})$$
$$\boldsymbol{y}' = \boldsymbol{g}(\boldsymbol{x})$$

Newton law – field of central force

$$\boldsymbol{x}' = \frac{1}{m}\boldsymbol{p}$$

$$\boldsymbol{p}' = \boldsymbol{F}(\boldsymbol{x})$$

Central force

$$\boldsymbol{F}(\boldsymbol{x}) = \frac{cm}{\|\boldsymbol{x}\|^3}\boldsymbol{x}$$

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Consequently, the system is of the form

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Kinetic energy: $\frac{1}{2}m\|\boldsymbol{x}'\|^2 = \frac{1}{2m}\|\boldsymbol{p}\|^2$

Potential energy: $\frac{cm}{\|\boldsymbol{x}\|}$

Total energy (Hamiltonian):

$$H(\boldsymbol{x}, \boldsymbol{p}) = \frac{1}{2m}\|\boldsymbol{p}\|^2 + \frac{cm}{\|\boldsymbol{x}\|}$$

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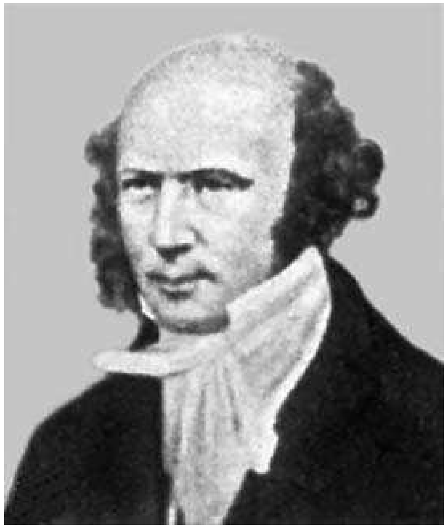

Hamiltonian: $H(\boldsymbol{x},\boldsymbol{p}) = \dfrac{1}{2m}\|\boldsymbol{p}\|^2 + \dfrac{cm}{\|\boldsymbol{x}\|}$

The following holds:

$$\nabla_{\boldsymbol{x}} H(\boldsymbol{x},\boldsymbol{p}) = \frac{\partial H}{\partial \boldsymbol{x}} = -\frac{cm}{\|\boldsymbol{x}\|^3}\boldsymbol{x} = -\boldsymbol{p}'$$

$$\nabla_{\boldsymbol{p}} H(\boldsymbol{x},\boldsymbol{p}) = \frac{\partial H}{\partial \boldsymbol{p}} = \frac{1}{m}\boldsymbol{p} = \boldsymbol{x}'$$

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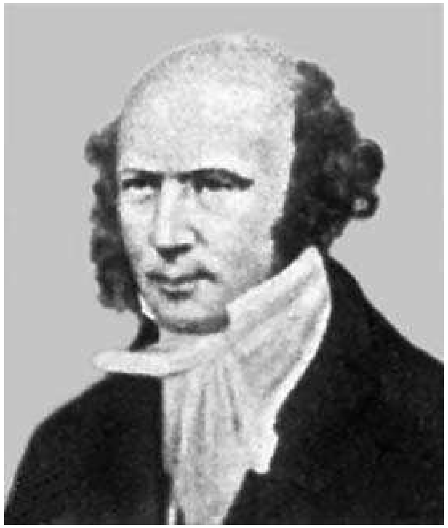

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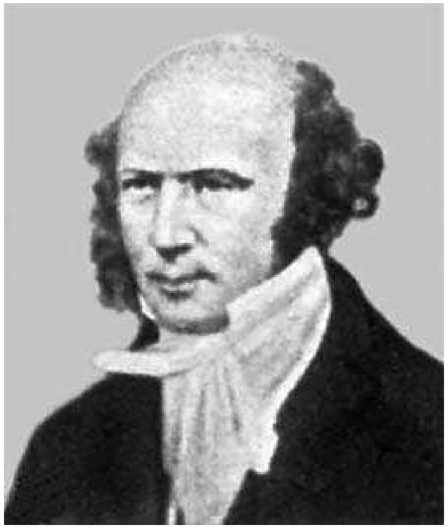

Hamiltonian: $H(\boldsymbol{x},\boldsymbol{p}) = \dfrac{1}{2m}\|\boldsymbol{p}\|^2 + \dfrac{cm}{\|\boldsymbol{x}\|}$

or, in a vector form

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{p}\end{pmatrix}' = \begin{pmatrix}\mathsf{O} & \mathsf{E}\\ -\mathsf{E} & \mathsf{O}\end{pmatrix}\begin{pmatrix}\nabla_{\boldsymbol{x}}H(\boldsymbol{x},\boldsymbol{p})\\ \nabla_{\boldsymbol{p}}H(\boldsymbol{x},\boldsymbol{p})\end{pmatrix}.$$

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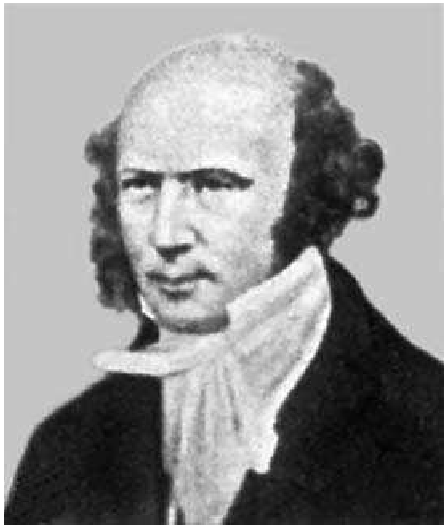

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Moreover: $\dfrac{\partial}{\partial t}H(\boldsymbol{x},\boldsymbol{p}) = \left(\dfrac{\partial H}{\partial \boldsymbol{x}}\right)^{\mathsf{T}}\boldsymbol{x}' + \left(\dfrac{\partial H}{\partial \boldsymbol{p}}\right)^{\mathsf{T}}\boldsymbol{p}' = 0$

$$\boldsymbol{x}' = \mathsf{J}(\boldsymbol{x})\nabla H(\boldsymbol{x}) \quad \text{where } \mathsf{J}(\boldsymbol{x}) = -\mathsf{J}(\boldsymbol{x})^{\mathsf{T}}$$

Hamiltonian system

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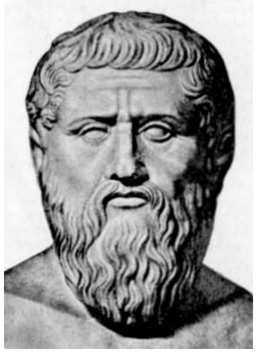
Plato, Timaios 28a: "First then, in my judgment, we must make a distinction and ask,
What is that which always is and has no becoming;
and what is that which is always becoming and never is?"



Nejprve jest podle mého mínění stanoviti tuto rozluku: co jest to, co stále jest, ale vzniku nemá, a co jest to, co stále vzniká, ale nikdy není jsoucí.

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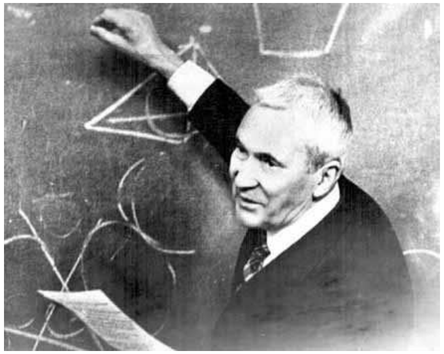

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  the simplest option – all of the functions f_i are linear.


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Example: classical predator-prey model

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x' &= x(r - by),\\
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$$H(x,y) = cx + by - \ln x^s y^r$$

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$$\begin{pmatrix} x \\ y \end{pmatrix}' = \begin{pmatrix} -xy\left(b - \frac{r}{y}\right) \\ xy\left(c - \frac{s}{x}\right) \end{pmatrix} = \begin{pmatrix} 0 & -xy \\ xy & 0 \end{pmatrix} \begin{pmatrix} \dfrac{\partial}{\partial x} H(x,y) \\ \dfrac{\partial}{\partial y} H(x,y) \end{pmatrix}$$

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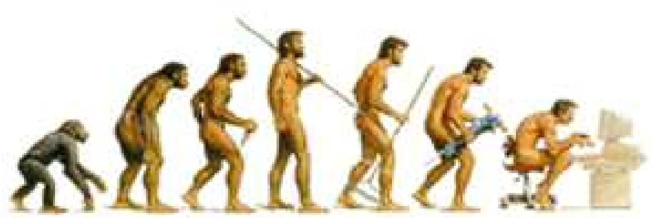
Hamiltonian system

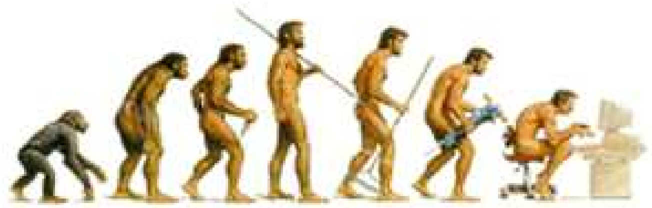
Replicator equation I

The fitest species survives and propagates.



Derivation of the equation I

A change of the relative abundance of species in a community is proportional to difference of its fitness and an overall fitness.



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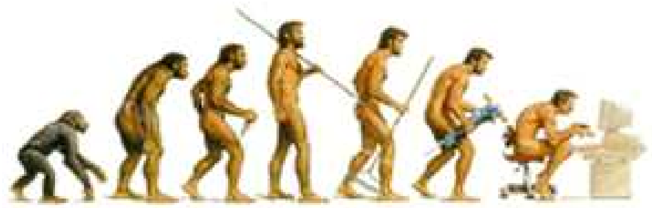

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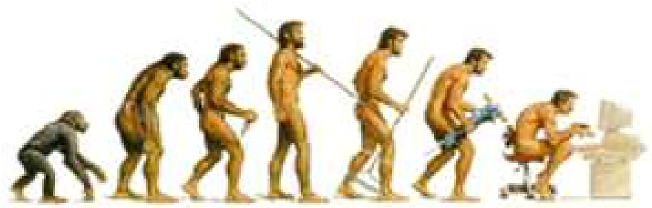
$$\sum_{i=1}^{n} x_i = 1$$



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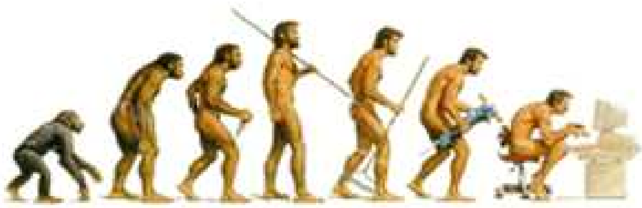

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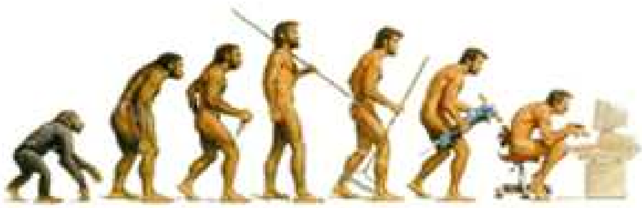

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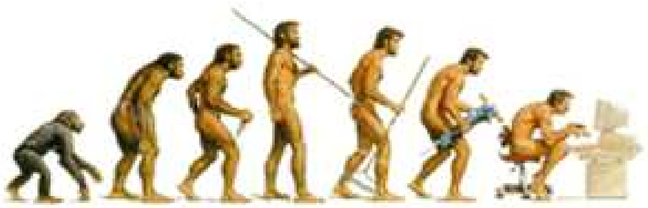

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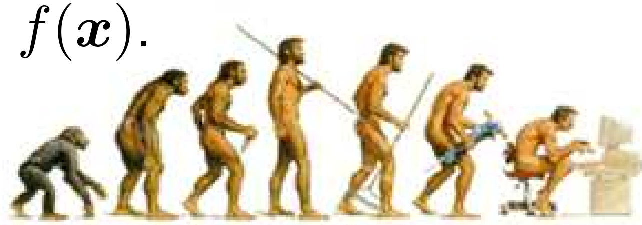

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$$\text{time change } x_i \sim f_i(\boldsymbol{x}) - \sum_{j=1}^{n} x_j f_j(\boldsymbol{x}), \quad i = 1, 2, \ldots, n$$

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$$\frac{x_i'}{x} = c\left(f_i(\boldsymbol{x}) - \sum_{j=1}^{n} x_j f_j(\boldsymbol{x})\right), \quad i = 1, 2, \ldots, n$$

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Community consisting of k populations

Derivation of the equation II

Community consisting of k populations

$n_i = n_i(t)$ . . . size of the i-th population

$r_i = r_i(\boldsymbol{n})$ . . . growth rate of the i-th population

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$N = \sum_{j=1}^{k} n_j$ ...size of community

$x_i = n_i/N$ ...relative abundance of the i-th population

$f_i(\boldsymbol{x}) = r_i(N\boldsymbol{x}) = r_i(\boldsymbol{n})$

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$N' = \sum_{j=1}^{k} n_j' = \sum_{j=1}^{k} r_j(\boldsymbol{n})n_j = N \sum_{j=1}^{k} x_j f_j(\boldsymbol{x})$

Derivation of the equation II

Community consisting of k populations

$n_i = n_i(t)$ . . . size of the i-th population

$r_i = r_i(\boldsymbol{n})$ . . . growth rate of the i-th population

$$n_i' = r_i(\boldsymbol{n})n_i, \quad i = 1, 2, \dots, k$$

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$$= x_i \left(f_i(\boldsymbol{x}) - \sum_{j=1}^{k} x_j f_j(\boldsymbol{x}) \right)$$

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Elementary properties of the replicator equation

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$$S_n = \left\{ \boldsymbol{x} \in \bar{\mathbb{R}}_+^n \ : \mathbf{1}^\mathsf{T} \boldsymbol{x} = 1 \right\} \qquad S_n^\circ = \left\{ \boldsymbol{x} \in \mathbb{R}_+^n \ : \mathbf{1}^\mathsf{T} \boldsymbol{x} = 1 \right\}$$

$$\partial S_n = S_n \setminus S_n^\circ$$

n-dimensional simplex, its interior and its boundary, respectively.

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- $\boldsymbol{x}(0) \in S_n^\circ \Rightarrow \boldsymbol{x}(t) \in S_n^\circ$ for all $t \geq 0$
- Let $\Psi : S_n \to \mathbb{R}$ be a continuous function. Put $g_i = f_i + \Psi$ for all $i \in \{1, 2, \ldots, n\}$. Then $\boldsymbol{x}$ solves the equation (1) if and only if it solves the equation

$$x_i' = x_i \left(g_i(\boldsymbol{x}) - \sum_{j=1}^{n} x_j g_j(\boldsymbol{x}) \right), \qquad i = 1, 2, \ldots, n.$$

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Theorem: Let there is a point $\hat{\boldsymbol{x}} \in S_n$ and its naighbourhood U such that

$$\sum_{i=1}^{n} \hat{x}_i f_i(\boldsymbol{x}) > \bar{f}(\boldsymbol{x}) \quad \text{for all } \boldsymbol{x} \in S_n \cap (U \setminus \{\hat{\boldsymbol{x}}\}).$$

Then $\hat{\boldsymbol{x}}$ is asymptoticaly stable equilibrium of the equation.

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$\hat{\boldsymbol{x}}$ . . . *evolutionary stable state.*

Equation with linear fitnesses

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$$\boldsymbol{x}' = \boldsymbol{x} \circ \big((\mathsf{E} - \boldsymbol{x} \mathbf{1}^{\mathsf{T}}) \mathsf{A} \boldsymbol{x} \big)$$

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{x})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{x}\big), \quad i = 1, 2, \ldots, n \tag{2}$$

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{x})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{x}\big), \quad i = 1, 2, \dots, n \tag{2}$$

- S_n, ∂S_n, S_n° are positive invariant sets of the equation (2).

Properties of the equation with linear fitnesses

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- S_n, ∂S_n, S_n° are positive invariant sets of the equation (2).
- Adding af a diagonal matrix to the matrix A or adding of a constant vector to a row (a column) of the matrix A does not change the solution of the equation (2).

Equivalence of the replicator equation and of the Lotka-Volterra ones

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{x})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{x}\big), \quad i = 1, 2, \ldots, n$$

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Put $b_{ij} = a_{nj} - a_{ij}$, $r_i = a_{in} - a_{nn}$ for $i, j = 1, 2, \ldots, n-1$. Transformation both of the independent variable (time) and of the functions x_i given by the equalities

$$\tau = \int_0^t x_n(s)\mathrm{d}s, \qquad y_j = \frac{x_j}{x_n}, \quad j = 1, 2, \ldots, n-1$$

maps the orbits of the replicator equation initialising in the interior of the simplex S_n° to the orbits of the Lotka-Volterra system

$$\frac{\mathrm{d}y_j}{\mathrm{d}\tau} = y_j\left(r_j - \sum_{k=1}^{n-1} b_{jk}y_k\right), \qquad j = 1, 2, \ldots, n-1$$

initialising in the interior of the positive orthant $\mathbb{R}_+^{n-1}$.

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Solution with the initial condition $y(0) = y_0 > 0$ is

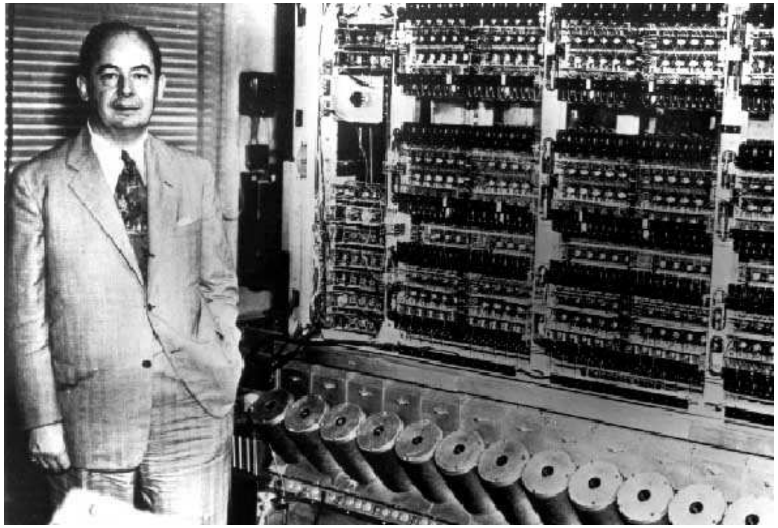
$$y(\tau) = \frac{(a_{12} - a_{22})y_0}{(a_{21} - a_{11})y_0 + \big(a_{12} - a_{22} - (a_{21} - a_{11})y_0\big)\mathrm{e}^{(a_{22} - a_{12})\tau}}$$

Matrix and bimatrix games

Normal form two player game:
quadruple $\mathcal{G} = (X, Y, u, v)$, where X, Y are finite sets and u, v are functions $X \times Y \to \mathbb{R}$.

Sets X and Y ... *sets of strategies of the first and of the second player, respectively*.

Functins u and v... *payoff function of the first and of the second player, respectively*.



Definition

Put

$$X = \{1, 2, \dots, n\} \quad Y = \{1, 2, \dots, m\}$$

$$a_{ij} = u(i, j) \quad b_{ji} = v(i, j)$$

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} & \dots & a_{1m} \\ a_{21} & a_{22} & \dots & a_{2m} \\ \vdots & \vdots & \ddots & \vdots \\ a_{n1} & a_{n2} & \dots & a_{nm} \end{pmatrix}, \quad \mathsf{B} = \begin{pmatrix} b_{11} & b_{12} & \dots & b_{1n} \\ b_{21} & b_{22} & \dots & b_{2n} \\ \vdots & \vdots & \ddots & \vdots \\ b_{m1} & b_{m2} & \dots & b_{mn} \end{pmatrix}.$$

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Using this notation, we have

$$u(i,j) = a_{ij} = \boldsymbol{e}_i^{\mathsf{T}} \mathsf{A} \boldsymbol{e}_j, \qquad v(i,j) = b_{ji} = \boldsymbol{e}_j^{\mathsf{T}} \mathsf{B} \boldsymbol{e}_i.$$

Matrix A, B … *payoff matrix.*

Definition

$$\mathcal{G} = (X, Y, u, v) = (\mathsf{A}, \mathsf{B})$$

A game can be represented by the table

		player 2			
		1	2	$\dots$	m
player 1	1	a_{11}, b_{11}	a_{12}, b_{21}	$\dots$	a_{1m}, b_{m1}
	2	a_{21}, b_{12}	a_{22}, b_{22}	$\dots$	a_{2m}, b_{m2}
	$\vdots$	$\vdots$	$\vdots$	$\ddots$	$\vdots$
	n	a_{n1}, b_{1n}	a_{n2}, b_{2n}	$\dots$	a_{nm}, b_{mn}

Definition

Probability extension of a bimatrix game $\mathcal{G} = (X, Y, u, v)$:

quadruple $\mathcal{G}^* = (X^*, Y^*, u^*, v^*)$;

$X^* = S_n$, $Y^* = S_m$,
u^*, v^* are functions $X^* \times Y^* \to \mathbb{R}$, defined by

$$u^*(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{x}^\mathsf{T} \mathsf{A} \boldsymbol{y}, \qquad v^*(\boldsymbol{x}, \boldsymbol{y}) = \boldsymbol{y}^\mathsf{T} \mathsf{B} \boldsymbol{x}.$$

X, Y . . . pure strategies

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Strategies

$\bar{\boldsymbol{x}} \in X^*$ ... *the best reply to the strategy* $\boldsymbol{y} \in Y^*$:

$$(\forall \boldsymbol{x} \in X^*)\ u^*(\bar{\boldsymbol{x}}, \boldsymbol{y}) = \bar{\boldsymbol{x}}^\mathsf{T} \mathsf{A} \boldsymbol{y} \geq \boldsymbol{x}^\mathsf{T} \mathsf{A} \boldsymbol{y} = u^*(\boldsymbol{x}, \boldsymbol{y})$$

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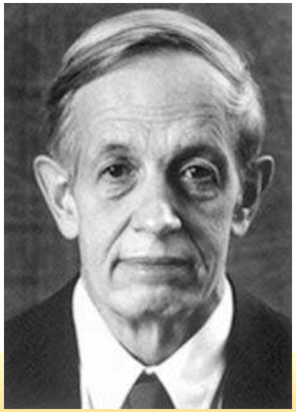
$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in X^* \times Y^*$ ... *(Nash) equilibrium*:

$$\forall(\boldsymbol{x} \in X^*)\forall(\boldsymbol{y} \in Y^*)\ \bar{\boldsymbol{x}}^\mathsf{T} \mathsf{A} \bar{\boldsymbol{y}} \geq \boldsymbol{x}^\mathsf{T} \mathsf{A} \bar{\boldsymbol{y}}, \quad \bar{\boldsymbol{y}}^\mathsf{T} \mathsf{B} \bar{\boldsymbol{x}} \geq \boldsymbol{y}^\mathsf{T} \mathsf{B} \bar{\boldsymbol{x}}$$

i.e. $\bar{\boldsymbol{x}}$ is the best reply to $\bar{\boldsymbol{y}}$ and at the same time $\bar{\boldsymbol{y}}$ is the best reply to $\bar{\boldsymbol{x}}$



Partnership game

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$\mathcal{G} = (\mathsf{A}, \mathsf{B})$, $c \in \mathbb{R}_+$

$\mathcal{G}$ is the c-*partnership game*:

$$(\exists \mathsf{D}, \boldsymbol{p}, \boldsymbol{q})\ \mathsf{A} = \mathsf{D} + \mathbf{1}\boldsymbol{q}^{\mathsf{T}}, \quad \mathsf{B} = c\mathsf{D}^{\mathsf{T}} + \mathbf{1}\boldsymbol{p}^{\mathsf{T}}$$

i.e. $a_{ij} = d_{ij} + q_j, \quad b_{ji} = cd_{ij} + p_i$

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$\mathcal{G}$ is the *identical interest game*:

$$c = 1, \quad \boldsymbol{p} = \boldsymbol{o} = \boldsymbol{q}$$

i.e. $\mathsf{A} = \mathsf{B}^\mathsf{T}$

Symmetric game

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$\bar{x} \in X^*$ is the *symmetric (Nash) equilibrium*:
$(\bar{\boldsymbol{x}}, \bar{\boldsymbol{x}})$, i.e.

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Replicator equation for a bimatrix game

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$$
\begin{aligned}
x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad && i = 1, 2, \dots, n, \\
y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad && j = 1, 2, \dots, m,
\end{aligned}
$$

$$
\begin{aligned}
x_i' &= x_i\Big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{y}\Big), && i = 1, 2, \ldots, n,\\
y_j' &= y_j\Big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^{\mathsf{T}}\mathsf{B}\boldsymbol{x}\Big), && j = 1, 2, \ldots, m,
\end{aligned}
$$

$$
\begin{aligned}
x_i' &= x_i(\boldsymbol{e}_i - \boldsymbol{x})^{\mathsf{T}}\mathsf{A}\boldsymbol{y}, && i = 1, 2, \ldots, n,\\
y_j' &= y_j(\boldsymbol{e}_j - \boldsymbol{y})^{\mathsf{T}}\mathsf{B}\boldsymbol{x}, && j = 1, 2, \ldots, m.
\end{aligned}
$$

Replicator equation for a bimatrix game

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), & i &= 1,2,\dots,n,\\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), & j &= 1,2,\dots,m, \end{aligned}$$

$$\begin{aligned} x_i' &= x_i(\boldsymbol{e}_i - \boldsymbol{x})^\mathsf{T}\mathsf{A}\boldsymbol{y}, & i &= 1,2,\dots,n,\\ y_j' &= y_j(\boldsymbol{e}_j - \boldsymbol{y})^\mathsf{T}\mathsf{B}\boldsymbol{x}, & j &= 1,2,\dots,m. \end{aligned}$$

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}' = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \circ \begin{pmatrix} \mathsf{O} & (\mathsf{E} - \boldsymbol{x}\mathbf{1}^\mathsf{T})\mathsf{A}\\ (\mathsf{E} - \boldsymbol{y}\mathbf{1}^\mathsf{T})\mathsf{B} & \mathsf{O}\end{pmatrix} \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}.$$

Replicator equation for a bimatrix game

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad & i &= 1,2,\dots,n, \\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad & j &= 1,2,\dots,m, \end{aligned}$$

$$\begin{aligned} x_i' &= x_i(\boldsymbol{e}_i - \boldsymbol{x})^\mathsf{T}\mathsf{A}\boldsymbol{y}, \qquad & i &= 1,2,\dots,n, \\ y_j' &= y_j(\boldsymbol{e}_j - \boldsymbol{y})^\mathsf{T}\mathsf{B}\boldsymbol{x}, \qquad & j &= 1,2,\dots,m. \end{aligned}$$

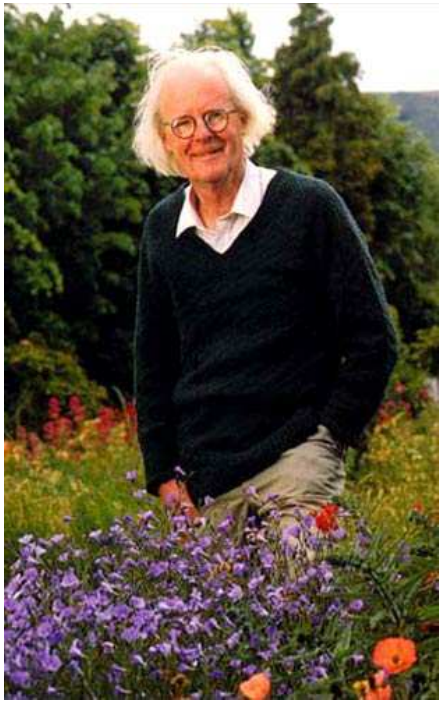
$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}' = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \circ \begin{pmatrix} \mathsf{O} & (\mathsf{E} - \boldsymbol{x}\mathbf{1}^\mathsf{T})\mathsf{A} \\ (\mathsf{E} - \boldsymbol{y}\mathbf{1}^\mathsf{T})\mathsf{B} & \mathsf{O} \end{pmatrix} \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}.$$

$$\begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}' = \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix} \circ \left[\mathsf{E} - \begin{pmatrix} \mathbf{1}\boldsymbol{x}^\mathsf{T} & \mathsf{O} \\ \mathsf{O} & \mathbf{1}\boldsymbol{y}^\mathsf{T} \end{pmatrix}\right] \begin{pmatrix} \mathsf{O} & \mathsf{A} \\ \mathsf{B} & \mathsf{O} \end{pmatrix} \begin{pmatrix}\boldsymbol{x}\\ \boldsymbol{y}\end{pmatrix}$$

Famous conflicts

	Hawk	Dove
Hawk	$\frac{1}{2}V - C$	V
Dove	0	$\frac{1}{2}V$

V – value of the resource
C – cost of the contest



The lizard *Uta stansburniana*

				
large teritory, several females		0	wins	loses
teritory with single female		loses	0	wins
no teritory		wins	loses	0

Mating strategies

The rock-scissors-paper game

			
	0	wins	loses
	loses	0	wins
	wins	loses	0

Mating strategies

	Rock	Scissors	Paper
Rock	0	1	-1
Scissors	-1	0	1
Paper	1	-1	0

Battle of sexes

Entrants	Strategies	
male	faithful	philanderer
female	coy	fast

V – value of the offspring
$2C$ – parental investment
c – cost of engagement period

Battle of sexes

Introduction
Theoretical background
Replicator equation I
Matrix and bimatrix games
Famous conflicts
Hawks and doves
Mating strategies
Battle of sexes
Replicator equation II
Alternative approaches

Entrants	Strategies	
male	faithful	philanderer
female	coy	fast

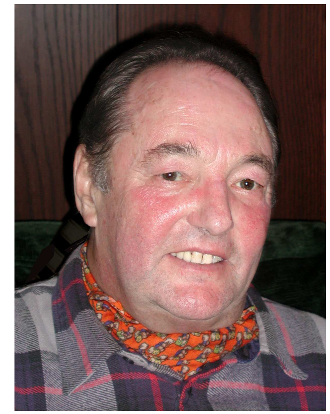
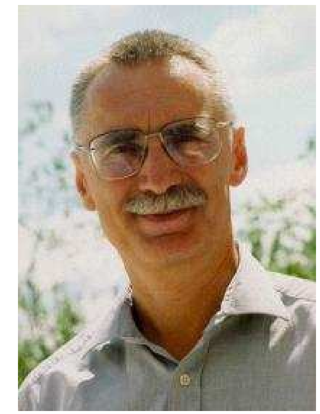
V – value of the offspring
$2C$ – parental investment
c – cost of engagement period

		female	
		coy	fast
male	faithful	$V - C - c$ $V - C - c$	$V - C$ $V - C$
	philanderer	0 0	$V - 2C$ V

Battle of sexes

Entrants	Strategies	
male	faithful	philanderer
female	coy	fast

V – value of the offspring
$2C$ – parental investment
c – cost of engagement period

		female	
		coy	fast
male	faithful	$V-C-c$ $V-C-c$	$V-C$ $V-C$
	philanderer	0 0	$V-2C$ V

Replicator equation II

$$
\begin{aligned}
x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{y}\big), \qquad & i = 1, 2, \dots, n \\
y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^{\mathsf{T}}\mathsf{B}\boldsymbol{x}\big), \qquad & j = 1, 2, \dots, m
\end{aligned}
$$

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{y}\big), & i &= 1,2,\ldots,n \\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^{\mathsf{T}}\mathsf{B}\boldsymbol{x}\big), & j &= 1,2,\ldots,m \end{aligned}$$

- $S_n \times S_m$, $\partial S_n \times \partial S_m$ $S_n^\circ \times S_m^\circ$ are positive invariant sets of the equations

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad i = 1, 2, \dots, n$$
$$y_j' = y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad j = 1, 2, \dots, m$$

- $S_n \times S_m$, $\partial S_n \times \partial S_m$ $S_n^\circ \times S_m^\circ$ are positive invariant sets of the equations,
- consequently, the $n + m$-dimensional system can be reduced to the $n + m - 2$-dimensional one:

$$x_i' = x_i(\boldsymbol{e}_i - \boldsymbol{x})^\mathsf{T}\left(\tilde{\mathsf{A}}\boldsymbol{y} - \hat{\boldsymbol{a}}\right), \quad i = 1, 2, \dots, n-1,$$
$$y_j' = y_j(\boldsymbol{e}_j - \boldsymbol{y})^\mathsf{T}\left(\tilde{\mathsf{B}}\boldsymbol{x} - \hat{\boldsymbol{b}}\right), \quad j = 1, 2, \dots, m-1.$$

where $\tilde{a}_{ij} = a_{ij} - a_{im} - a_{nj} + a_{nm}, \quad \hat{a}_i = a_{nm} - a_{im}$
$\tilde{b}_{ij} = b_{ij} - b_{in} - b_{mj} + b_{mn}, \quad \hat{b}_j = b_{mn} - b_{jn}.$

Example: $n = m = 2$

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

$$\mathsf{A} = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}, \qquad \mathsf{B} = \begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix}.$$

Reduced system:

$$\begin{aligned} x' &= x(1-x)(\alpha_1 y - \alpha_2) \\ y' &= y(1-y)(\beta_1 x - \beta_2) \end{aligned}$$

where $\alpha_1 = a_{11} - a_{12} - a_{21} + a_{22}$, $\alpha_2 = a_{22} - a_{12}$,
$\beta_1 = b_{11} - b_{12} - b_{21} + b_{22}$, $\beta_2 = b_{22} - b_{12}$

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where $\alpha_1 = a_{11} - a_{12} - a_{21} + a_{22}$, $\alpha_2 = a_{22} - a_{12}$,
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Phase space: $[0,1] \times [0,1]$

Reduced system:

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$\beta_1 = b_{11} - b_{12} - b_{21} + b_{22}$, $\beta_2 = b_{22} - b_{12}$

Phase space: $[0,1] \times [0,1]$

Stationary solutions: $(0,0)$, $(0,1)$, $(1,0)$, $(1,1)$ corresponds to the pure strategies.

If $\alpha_1 \neq 0,\ 0 < \dfrac{\alpha_2}{\alpha_1} < 1,\ \beta_1 \neq 0,\ 0 < \dfrac{\beta_2}{\beta_1} < 1,$

the interior stationary solution: $\left(\dfrac{\beta_2}{\beta_1}, \dfrac{\alpha_2}{\alpha_1}\right)$

corresponds to mixed strategies.

Reduced system:

$$\begin{aligned} x' &= x(1-x)(\alpha_1 y - \alpha_2) \\ y' &= y(1-y)(\beta_1 x - \beta_2) \end{aligned}$$

The variational matrix of the system:

$$J(0,0) = \begin{pmatrix} -\alpha_2 & 0 \\ 0 & -\beta_2 \end{pmatrix}, \quad J(0,1) = \begin{pmatrix} \alpha_1 - \alpha_2 & 0 \\ 0 & \beta_2 \end{pmatrix},$$

$$J(1,0) = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_1 - \beta_2 \end{pmatrix}, \quad J(1,1) = \begin{pmatrix} \alpha_2 - \alpha_1 & 0 \\ 0 & \beta_2 - \beta_1 \end{pmatrix},$$

Reduced system:

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$$J(1,0) = \begin{pmatrix} \alpha_2 & 0 \\ 0 & \beta_1 - \beta_2 \end{pmatrix}, \quad J(1,1) = \begin{pmatrix} \alpha_2 - \alpha_1 & 0 \\ 0 & \beta_2 - \beta_1 \end{pmatrix},$$

$$J\left(\frac{\beta_2}{\beta_1}, \frac{\alpha_2}{\alpha_1}\right) = \begin{pmatrix} 0 & \dfrac{\alpha_1 \beta_2 (\beta_1 - \beta_2)}{\beta_1^2} \\ \dfrac{\alpha_2 \beta_1 (\alpha_1 - \alpha_2)}{\alpha_1^2} & 0 \end{pmatrix}.$$

Reduced system:

$$\begin{aligned} x' &= x(1-x)(\alpha_1 y - \alpha_2) \\ y' &= y(1-y)(\beta_1 x - \beta_2) \end{aligned}$$

The stationary points corresponding to the pure strategies are saddle points or nodes, the stationary point corresponding to mixed strategies is saddle point or unstable focus.

$$
\begin{aligned}
x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad & i = 1, 2, \ldots, n\\
y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad & j = 1, 2, \ldots, m
\end{aligned}
$$

N . . . set of Nash equilibria of bimatrix game $\mathcal{G} = (\mathsf{A}, \mathsf{B})$

E . . . set of stationary solutions of the system

Stationary solutions of replicator equations

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad && i = 1, 2, \dots, n \\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad && j = 1, 2, \dots, m \end{aligned}$$

N ... set of Nash equilibria of bimatrix game $\mathcal{G} = (\mathsf{A}, \mathsf{B})$
E ... set of stationary solutions of the system

$$N \subseteq E$$

Stationary solutions of replicator equations

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{y}\big), \qquad & i = 1, 2, \ldots, n \\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^{\mathsf{T}}\mathsf{B}\boldsymbol{x}\big), \qquad & j = 1, 2, \ldots, m \end{aligned}$$

N ... set of Nash equilibria of bimatrix game $\mathcal{G} = (\mathsf{A}, \mathsf{B})$
E ... set of stationary solutions of the system

$$N \subseteq E$$

$$(S_n^\circ \times S_m^\circ) \cap N = E \cap (S_n^\circ \times S_m^\circ)$$

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{x})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{x}\big), \quad i = 1, 2, \dots, n$$

N ... set of symmetric Nash equilibria of matrix game $\mathcal{G} = \mathsf{A}$

E ... set of stationary solutions of the equation

S ... set of stable stationary solutions of the equation

Stationary solutions of replicator equations

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{x})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{x}\big), \quad i = 1, 2, \ldots, n$$

N ... set of symmetric Nash equilibria of matrix game $\mathcal{G} = \mathsf{A}$
E ... set of stationary solutions of the equation
S ... set of stable stationary solutions of the equation

$$S \subseteq N \subseteq E$$

Bipartite system

$$
\begin{aligned}
x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad & i = 1, 2, \dots, n \\
y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad & j = 1, 2, \dots, m
\end{aligned}
$$

Bipartite system

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), \qquad & i = 1, 2, \dots, n \\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), \qquad & j = 1, 2, \dots, m \end{aligned}$$

Let us consider the system on $S_n^\circ \times S_m^\circ$

$$\begin{aligned} x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), & i &= 1,2,\ldots,n \\ y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), & j &= 1,2,\ldots,m \end{aligned}$$

Let us consider the system on $S_n^\circ \times S_m^\circ$

Substitution

$$\begin{aligned} \tilde{a}_{ij} &= a_{ij} - a_{nj},\ \hat{a}_i = a_{im} - a_{nm}, \\ \tilde{b}_{ji} &= b_{ji} - b_{mi},\ \hat{b}_j = b_{jn} - b_{mn}, \end{aligned} \qquad \xi_i = \frac{x_i}{x_n},\ \eta_j = \frac{y_j}{y_m},$$

$$\begin{aligned} \xi_i' &= \xi_i \frac{(\tilde{\mathsf{A}}\boldsymbol{\eta})_i + \hat{a}_i}{1 + \mathbf{1}^\mathsf{T}\boldsymbol{\eta}}, & i &= 1,2,\ldots,n-1, \\ \eta_j' &= \eta_j \frac{(\tilde{\mathsf{B}}\boldsymbol{\xi})_j + \hat{b}_j}{1 + \mathbf{1}^\mathsf{T}\boldsymbol{\xi}}, & j &= 1,2,\ldots,m-1. \end{aligned}$$

Bipartite system

$$
\begin{aligned}
x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{y}\big), \qquad & i &= 1,2,\dots,n \\
y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^{\mathsf{T}}\mathsf{B}\boldsymbol{x}\big), \qquad & j &= 1,2,\dots,m
\end{aligned}
$$

Let us consider the system on $S_n^\circ \times S_m^\circ$

Further substitution $u_i = \ln \xi_i,\ v_j = \ln \eta_j$

$$
u_i' = \frac{\sum\limits_{k=1}^{m-1} \tilde{a}_{ik} \mathrm{e}^{v_k} + \hat{a}_i}{1 + \sum\limits_{k=1}^{m-1} \mathrm{e}^{v_k}}, \qquad i = 1,2,\dots,n-1,
$$

$$
v_j' = \frac{\sum\limits_{k=1}^{n-1} \tilde{b}_{jk} \mathrm{e}^{u_k} + \hat{b}_i}{1 + \sum\limits_{k=1}^{n-1} \mathrm{e}^{u_k}}, \qquad j = 1,2,\dots,m-1.
$$

Hamiltonian system

$$
\begin{aligned}
x_i' &= x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}\big), & i &= 1, 2, \dots, n\\
y_j' &= y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}\big), & j &= 1, 2, \dots, m
\end{aligned}
$$

Let us consider the system on $S_n^\circ \times S_m^\circ$

Let (A, B) be c-partnership game and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in S_n^\circ \times S_m^\circ$ is the Nash equilibrium.
Then the function

$$
H(\boldsymbol{x}, \boldsymbol{y}) = c\sum_{i=1}^{n} \bar{x}_i \ln x_i - \sum_{j=1}^{m} \bar{y}_j \ln y_j
$$

is the invariant of the system.

Hamiltonian system

$$x_i' = x_i\big((\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^{\mathsf{T}}\mathsf{A}\boldsymbol{y}\big), \qquad i = 1, 2, \dots, n$$
$$y_j' = y_j\big((\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^{\mathsf{T}}\mathsf{B}\boldsymbol{x}\big), \qquad j = 1, 2, \dots, m$$

Let us consider the system on $S_n^\circ \times S_m^\circ$

Let (A, B) be c-partnership game and $(\bar{\boldsymbol{x}}, \bar{\boldsymbol{y}}) \in S_n^\circ \times S_m^\circ$ is the Nash equilibrium.

Substitution $r_{ij} = a_{ij} - a_{nj} - a_{im} + a_{nm}$,

$$u_i = \ln \frac{x_i}{x_n},\ v_j = \ln \frac{y_j}{y_m},$$

for $i = 1, 2, \dots, n-1$, $j = 1, 2, \dots, m-1$

transforms the replicator system to the Hamiltonian one:

$$\begin{pmatrix} \boldsymbol{u} \\ \boldsymbol{v} \end{pmatrix}' = \begin{pmatrix} \mathsf{O} & \mathsf{R} \\ -\mathsf{R}^{\mathsf{T}} & \mathsf{O} \end{pmatrix} \begin{pmatrix} \nabla_{\boldsymbol{u}} H \\ \nabla_{\boldsymbol{v}} H \end{pmatrix}$$

Alternative approaches

The types (behaviour patterns, strategies) do not replicate by inheritence but by imitation.

Imitation dynamics

The types (behaviour patterns, strategies) do not replicate by inheritence but by imitation.

Events: S_{ij} ... an individual of the type j meets an individual of the type i

C_{ij} ... an individual of the type j addopts the type i

Imitation dynamics

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Events: S_{ij} ... an individual of the type j meets an individual of the type i

C_{ij} ... an individual of the type j addopts the type i

Probabilities of the events during a time interval of length Δt:

$$P(S_{ij}) \sim x_i, \quad P(C_{ij}|S_{ij}) \sim \Delta t, \quad P(C_{ij}|\neg S_{ij}) = 0.$$

Imitation dynamics

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$$P(S_{ij}) \sim x_i, \quad P(C_{ij}|S_{ij}) \sim \Delta t, \quad P(C_{ij}|\neg S_{ij}) = 0.$$

Probability that an individual of the type i addopts the type j:

$$P(C_{ij}) = P(S_{ij})P(C_{ij}|S_{ij}) \sim x_i \Delta t.$$

Imitation dynamics

The types (behaviour patterns, strategies) do not replicate by inheritence but by imitation.

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Probability that an individual of the type i addopts the type j:

$$P(C_{ij}) = P(S_{ij})P(C_{ij}|S_{ij}) \sim x_i \Delta t.$$

g_{ij} . . . rate of proportionality

N . . . size of population

Expected number of individuals of the type i after the time interval Δt:

$$Nx_i(t+\Delta t) = Nx_i(t) + \sum_{j=1}^{n} \big(Nx_i(t)\big) g_{ij} x_j(t) \Delta t - \\ - \sum_{k=1}^{n} \big(Nx_k(t)\big) g_{ki} x_i(t) \Delta t$$

$$x_i' = x_i \sum_{k=1}^{n} (g_{ik} - g_{ki}) x_k, \quad i = 1, 2, \ldots, n.$$

Imitation dynamics

$$x_i' = x_i \sum_{k=1}^{n} (g_{ik} - g_{ki}) x_k, \quad i = 1, 2, \ldots, n.$$

Probability of transition from the j-th type to the i-th depends on payoffs $(\mathsf{A}\boldsymbol{x})_i$, $(\mathsf{A}\boldsymbol{x})_j$:

$$g_{ij} = \varphi\big((\mathsf{A}\boldsymbol{x})_i, (\mathsf{A}\boldsymbol{x})_j\big)$$

$$x_i' = x_i \sum_{k=1}^{n} (g_{ik} - g_{ki}) x_k, \quad i = 1, 2, \ldots, n.$$

Probability of transition from the j-th type to the i-th depends on payoffs $(\mathsf{A}\boldsymbol{x})_i$, $(\mathsf{A}\boldsymbol{x})_j$:

$$g_{ij} = \varphi\big((\mathsf{A}\boldsymbol{x})_i, (\mathsf{A}\boldsymbol{x})_j\big)$$

A rule "imitate the better":

$$\varphi(u, v) = \begin{cases} 1, & u > v, \\ 0, & u \le v \end{cases}$$

$$x_i' = x_i \sum_{k=1}^{n} (g_{ik} - g_{ki}) x_k, \quad i = 1, 2, \dots, n.$$

Probability of transition from the j-th type to the i-th depends on payoffs $(\mathsf{A}\boldsymbol{x})_i$, $(\mathsf{A}\boldsymbol{x})_j$:

$$g_{ij} = \varphi\big((\mathsf{A}\boldsymbol{x})_i, (\mathsf{A}\boldsymbol{x})_j\big)$$

A rule "imitate the better with effort which increase with expected gain":

$$\varphi(u,v) = \begin{cases} (u-v)^\alpha, & u > v, \\ 0, & u \le v \end{cases} \qquad \alpha > 0$$

Imitation dynamics

$$x_i' = x_i \sum_{k=1}^{n} (g_{ik} - g_{ki}) x_k, \quad i = 1, 2, \ldots, n.$$

Probability of transition from the j-th type to the i-th depends on payoffs $(\mathsf{A}\boldsymbol{x})_i$, $(\mathsf{A}\boldsymbol{x})_j$:

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A rule "imitate the better with effort which increase with expected gain":

$$\varphi(u, v) = \begin{cases} (u - v)^\alpha, & u > v, \\ 0, & u \le v \end{cases} \qquad \alpha > 0$$

Replicator equation can be viewed as a particular case of the imitation dynamics equation for $\alpha = 1$.

Growth rate of each subpopulation (behaviour pattern, strategy) is proportional to “the gain”:

$$x_i(t+h) = c(t)\big(\mathsf{A}\boldsymbol{y}(t)\big)_i\, x_i(t),$$
$$y_j(t+h) = d(t)\big(\mathsf{B}\boldsymbol{x}(t)\big)_j\, y_j(t)$$

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For $\big(\boldsymbol{x}(t), \boldsymbol{y}(t)\big) \in S_n \times S_m \Rightarrow \big(\boldsymbol{x}(t), \boldsymbol{y}(t)\big) \in S_n \times S_m$, the following has to hold

$$c(t) = \frac{1}{\boldsymbol{x}(t)^{\mathsf{T}}\mathsf{A}\boldsymbol{y}(t)}, \quad d(t) = \frac{1}{\boldsymbol{y}(t)^{\mathsf{T}}\mathsf{B}\boldsymbol{x}(t)}, \quad a_{ij} > 0,\ b_{ij} > 0.$$

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Hence

$$x_i(t+h) = x_i(t)\frac{\big(\mathsf{A}\boldsymbol{y}(t)\big)_i}{\boldsymbol{x}(t)^{\mathsf{T}}\mathsf{A}\boldsymbol{y}(t)}, \quad y_j(t+h) = y_j(t)\frac{\big(\mathsf{B}\boldsymbol{x}(t)\big)_j}{\boldsymbol{y}(t)^{\mathsf{T}}\mathsf{B}\boldsymbol{x}(t)}$$

Discrete dynamics for bimatrix game

$$\Delta x_i(t) = x_i(t)\frac{(\mathsf{A}\boldsymbol{y}(t))_i - \boldsymbol{x}(t)^{\mathsf{T}}\mathsf{A}\boldsymbol{y}(t)}{\boldsymbol{x}(t)^{\mathsf{T}}\mathsf{A}\boldsymbol{y}(t)},$$

$$\Delta y_j(t) = y_j(t)\frac{(\mathsf{B}\boldsymbol{x}(t))_j - \boldsymbol{y}(t)^{\mathsf{T}}\mathsf{B}\boldsymbol{x}(t)}{\boldsymbol{y}(t)^{\mathsf{T}}\mathsf{B}\boldsymbol{x}(t)}$$

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Continuous analogy

$$x_i' = x_i\frac{(\mathsf{A}\boldsymbol{y})_i - \boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}}{\boldsymbol{x}^\mathsf{T}\mathsf{A}\boldsymbol{y}}, \qquad i = 1, 2, \ldots, n,$$

$$y_j' = y_j\frac{(\mathsf{B}\boldsymbol{x})_j - \boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}}{\boldsymbol{y}^\mathsf{T}\mathsf{B}\boldsymbol{x}}, \qquad j = 1, 2, \ldots, m.$$

Interpretation of matrices A and B in this case differs from the one for the introduced continuous replicator equation.