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MINISTRY OF EDUCATION, YOUTH AND SPORTS
OP Education for Competitivness
INVESTMENTS IN EDUCATION DEVELOPMENT

Population dynamics – a source of diversity

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Masaryk University, Faculty of science
Department of Mathematics and Statistics

7th Summer School on Computational Biology
September 15, 2011

Introduction

A bit of history

Leonhard Euler: *Introductio in analysin infinitorum* (1748)

Geometric growth of population

Thomas Robert Malthus: *An Essay on the Principle of Population* (1798)

- The increase of population is necessarily limited by the means of subsistence,
- population does invariably increase when the means of subsistence increase,
- the superior power of population is repressed, and the actual population kept equal to the means of subsistence, by misery and vice.

A bit of history

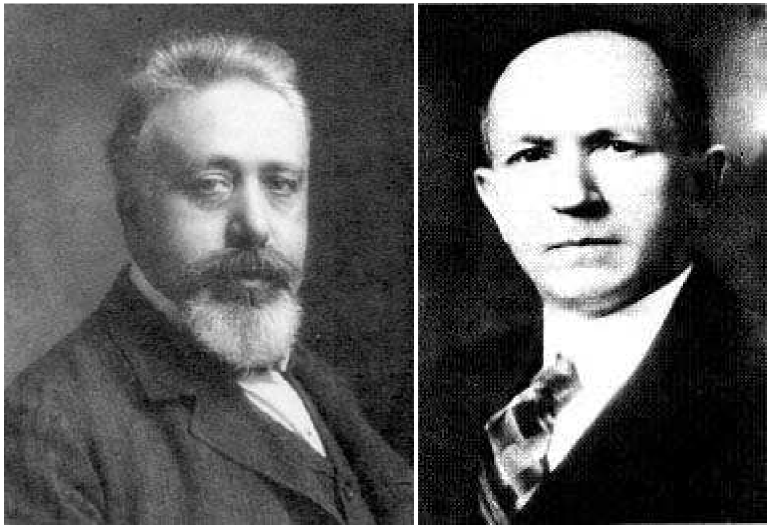
Pierre-François Verhulst: *Recherches mathématiques sur la loie d'accroissement de la population* (1845)

Self-limited (logistic) growth of population

A bit of history

Alfred J. Lotka: *Analytical note on certain rhythmic relations in organic systems* (1920)

A bit of history



Alfred J. Lotka: *Analytical note on certain rhythmic relations in organic systems* (1920)
Vito Volterra: *Variazioni e fluttuazioni del numero d'individui in specie animali conviventi.* (1926)

A bit of history

Alfred J. Lotka: *Elements of physical biology* (1925)
Vito Volterra: *Leçons sur la théorie mathématique de la lutte pur la vie* (1931)

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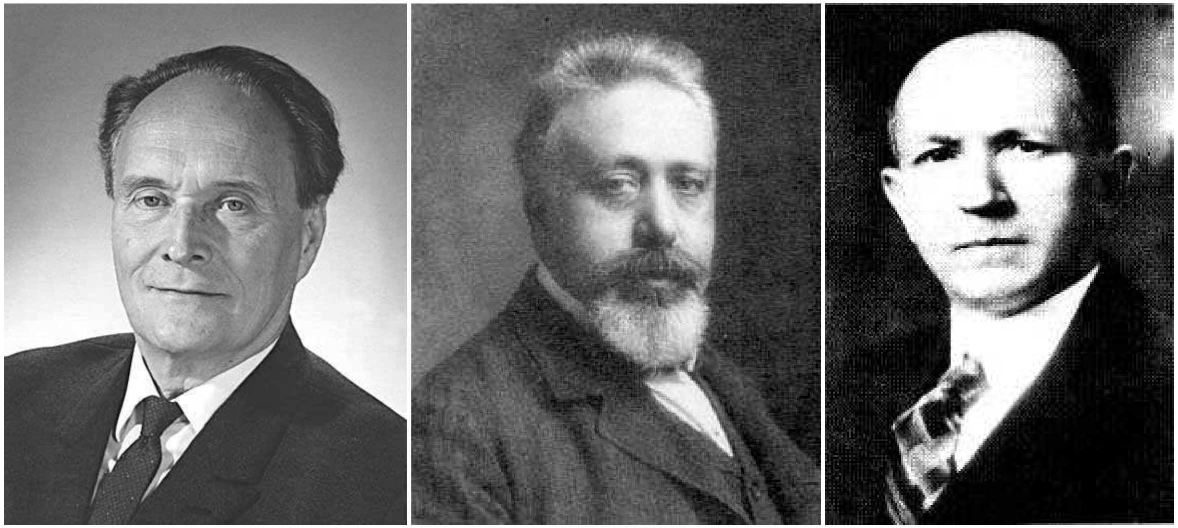

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Vito Volterra: *Leçons sur la théorie mathématique de la lutte pur la vie* (1931)
Georgij F. Gause: *The Struggle for existence* (1934)

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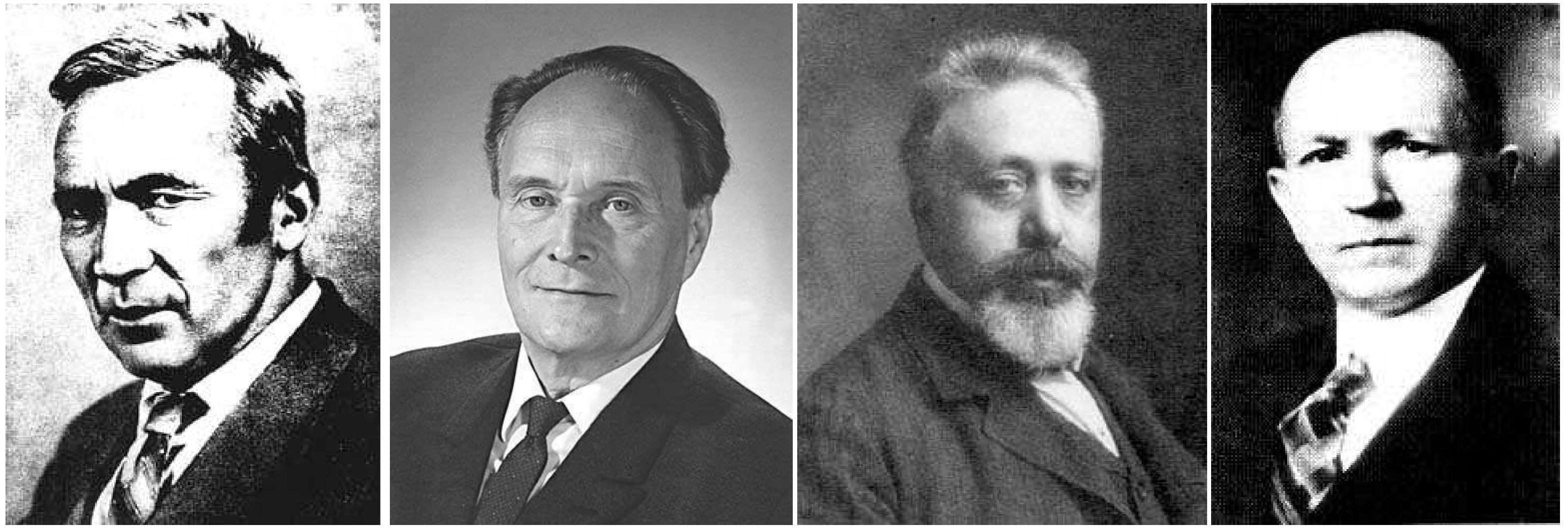

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Vito Volterra: *Leçons sur la théorie mathématique de la lutte pur la vie* (1931)
Georgij F. Gause: *The Struggle for existence* (1934)
Andrej N. Kolmogorov: *Sula teoria di Volterra della lota per l'esistenza* (1936)

A bit of history

Charles S. Elton: *Animal ecology* (1927)

food chain, ecological niche, pyramid of numbers

A bit of history

Charles S. Elton: *Animal ecology* (1927)
Alexander J. Nicholson: *The balance of animal population* (1933)

density (in)dependent growth

Charles S. Elton: *Animal ecology* (1927)
Alexander J. Nicholson: *The balance of animal population* (1933)

Ch. S. Elton, A. J. Nicholson: *The ten-year cycle in numbers of lynx in Canada* (1942)

A bit of history



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Alexander J. Nicholson: *The balance of animal population* (1933)

Ch. S. Elton, A. J. Nicholson: *The ten-year cycle in numbers of lynx in Canada* (1942)

Ernest T. Seton: *The arctic prairies* (1912)

A bit of history



Charles S. Elton: *Animal ecology* (1927)
Alexander J. Nicholson: *The balance of animal population* (1933)

Ch. S. Elton, A. J. Nicholson: *The ten-year cycle in numbers of lynx in Canada* (1942)

Crawford S. Holling: *The functional response of predator to prey density and its role in mimicry and population regulation* (1965)

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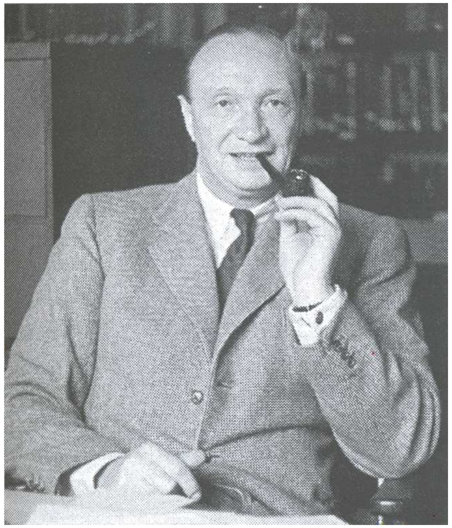
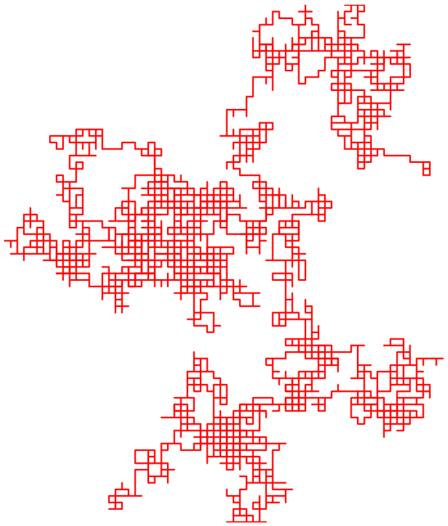
A bit of history



Patrick H. Leslie: *On the use of matrices in certain population mathematics* (1945)

structured population

A bit of history



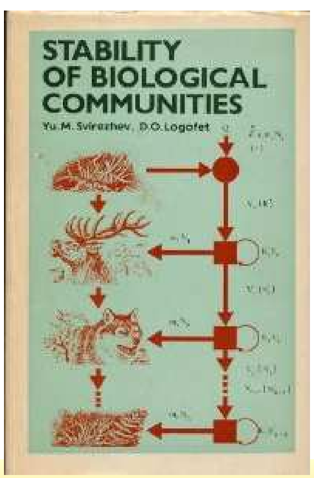
John G. Skellam: *Random Dispersal in Theoretical Populations* (1951)

dispersal of population in space, random walk, biological invasion

- Pielou E. C.: *An Introduction to Mathematical Ecology.* J.Willey&Sons, New York, NY, 1969

Sources, textbooks

- Pielou E. C.: *An Introduction to Mathematical Ecology.* J.Willey&Sons, New York, NY, 1969
- Свирежев, Ю. М. – Логофет, Д. О. *Устойчивость биологическич сообществ.* Москва: Наука, 1978.



Sources, textbooks

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Sources, textbooks
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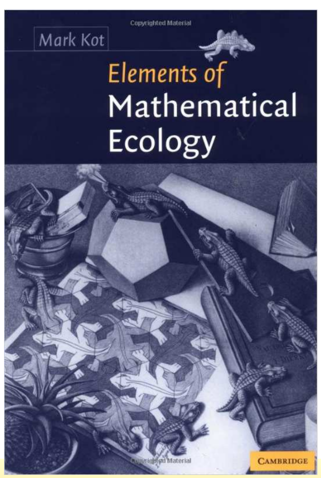
General theory of population dynamics
Models with discrete time
Models with continuous time



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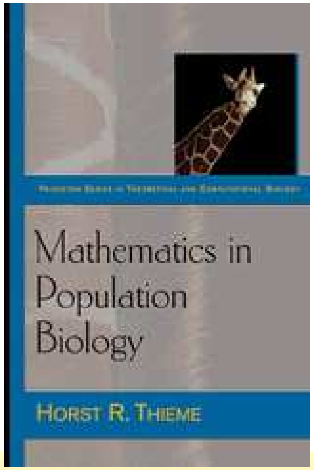

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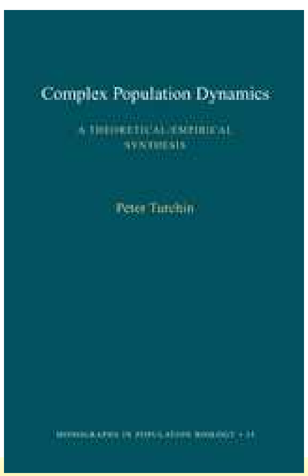

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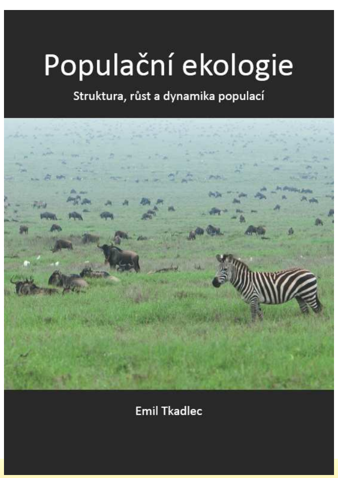

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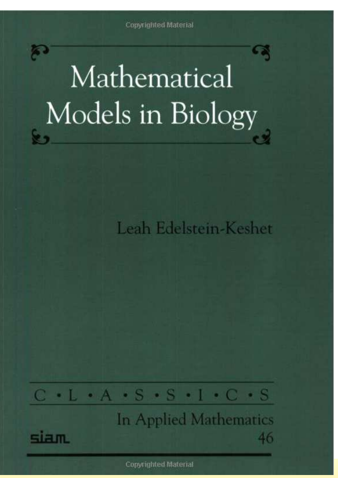

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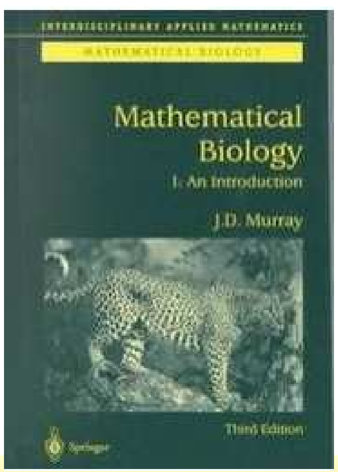

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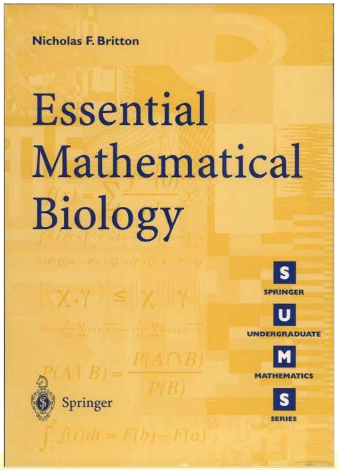

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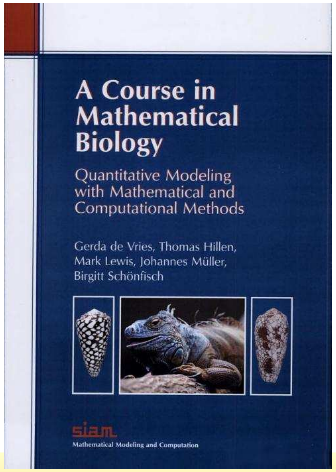

Software

Software

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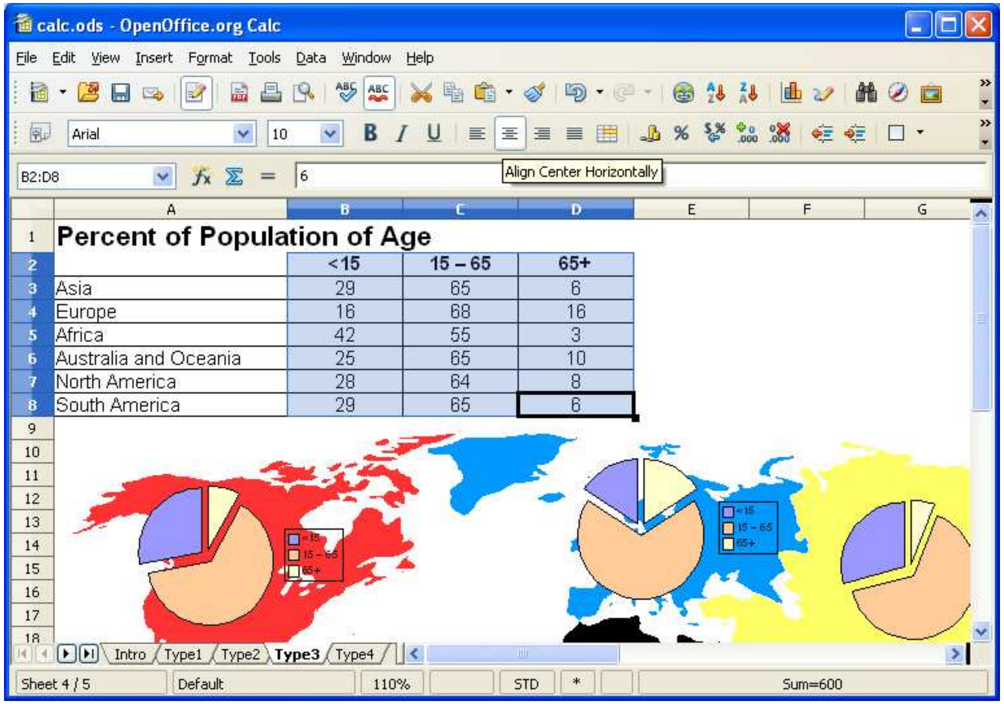
Software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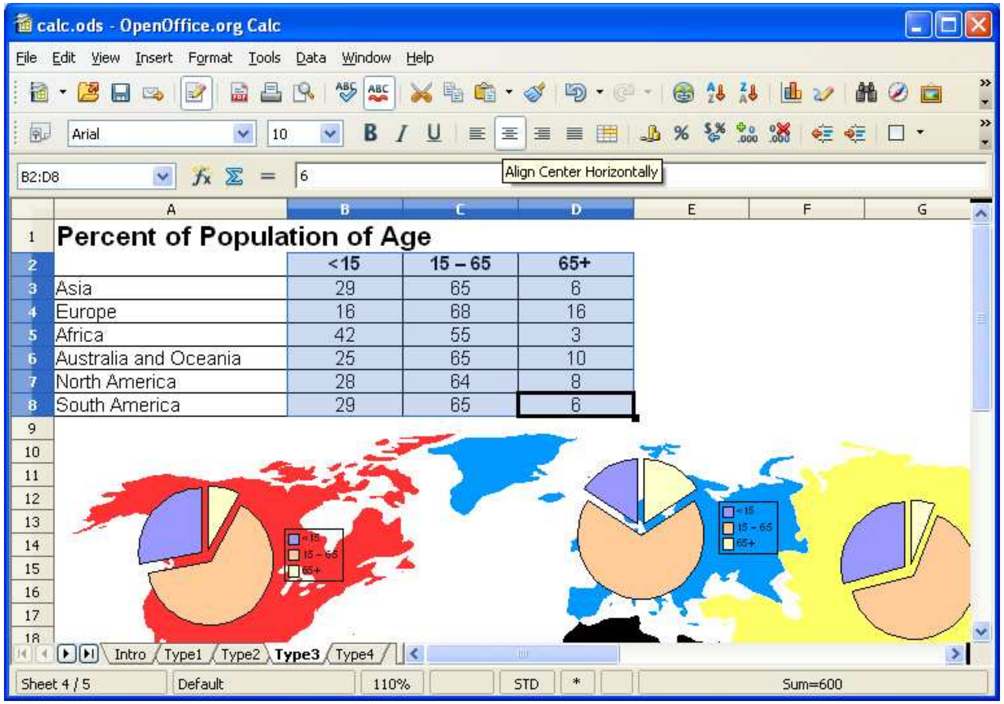
Software



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Software

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Henry M., Stevens H.: *A Primer of Ecology with R.* Springer: Dordrecht, Heidelberg, London, New York, 2009

General theory of population dynamics

Principles of population

1. A size of population sustains in the exponential increase or decrease until environmental conditions cause a change of the status.

Principles of population

1. A size of population sustains in the exponential increase or decrease until environmental conditions cause a change of the status.

2. The relative change of a population size equals to the impact of environmental conditions.

1st principle

A size of population sustains in the exponential increase or decrease until environmental conditions cause a change of the status.

$$x(t) = x_0 q^t$$

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$$x(t+1) = qx(t), \quad \frac{x(t+1) - x(t)}{x(t)} = q - 1$$

1^{st} principle

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$$x'(t) = x_0 q^t \ln q,$$

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$$x'(t) = (\ln q)x(t),$$

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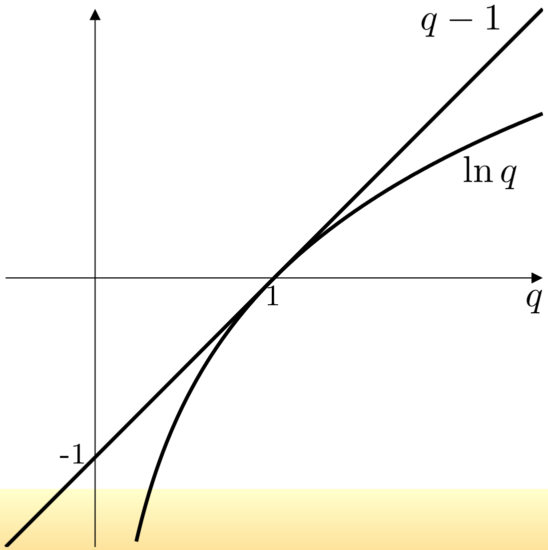


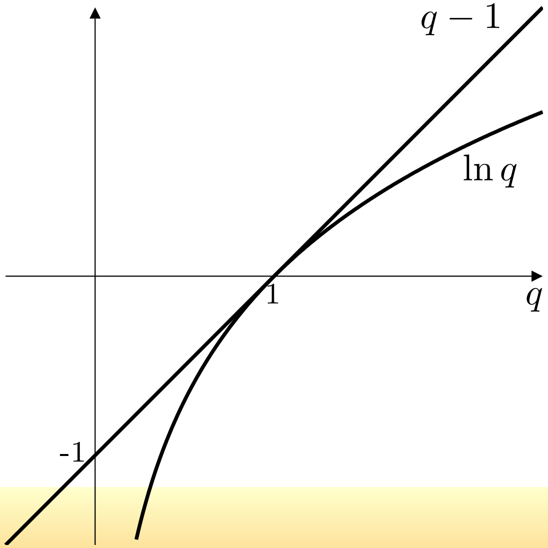
1^{st} principle

A size of population sustains in the exponential increase or decrease until environmental conditions cause a change of the status.

$$x(t) = x_0 q^t$$

$$x(t+1) = qx(t), \qquad \frac{x(t+1) - x(t)}{x(t)} = r$$

$$x'(t) = (\ln q)x(t), \qquad \frac{x'(t)}{x(t)} = r$$



2nd principle

The relative change of a population size equals to the impact of environmental conditions.

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$$\frac{x'(t)}{x(t)} = r$$

2nd principle

The relative change of a population size equals to the impact of environmental conditions.

$$\frac{x_i'(t)}{x_i(t)} = \varrho_i\big(x_1(t), x_2(t), \dots, x_n(t)\big), \quad i = 1, 2, \dots, n$$

2^{nd} principle

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$$\frac{x(t+1) - x(t)}{x(t)} = r$$

2^{nd} principle

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$$\frac{x_i(t+1) - x_i(t)}{x_i(t)} = \varrho_i\big(x_1(t), x_2(t), \dots, x_n(t)\big),$$

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$$i = 1, 2, \ldots, n$$

$$x(t+1) = qx(t)$$

2^{nd} principle

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$$\frac{x_i(t+1) - x_i(t)}{x_i(t)} = \varrho_i\big(x_1(t), x_2(t), \dots, x_n(t)\big),$$

$$i = 1, 2, \dots, n$$

$$x(t+1) = \mathrm{e}^r x(t)$$

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$$i = 1, 2, \dots, n$$

$$x_i(t+1) = \mathrm{e}^{\varrho_i\left(x_1(t), x_2(t), \dots, x_n(t)\right)} x(t), \quad i = 1, 2, \dots, n$$

“Fundamental equations”

Populations with overlapping generations

$$x_i' = x_i \varrho_i(x_1, x_2, \dots, x_n), \qquad i = 1, 2, \dots, n$$

Populations with non-overlapping generations

$$x_i(t+1) = \mathrm{e}^{\varrho_i\big(x_1(t), x_2(t), \dots, x_n(t)\big)} x_i(t), \qquad i = 1, 2, \dots, n$$

“Fundamental equations”

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ϱ_i linear functions – Lotka-Volterra systems
ϱ_i general functions – Kolmogorov systems

"Fundamental equations"

Populations with overlapping generations

$$x_i' = x_i\big(r_i - f_i(x_1, x_2, \ldots, x_n)\big), \qquad i = 1, 2, \ldots, n$$

Populations with non-overlapping generations

$$x_i(t+1) = \mathrm{e}^{r_i - f_i\left(x_1(t), x_2(t), \ldots, x_n(t)\right)} x_i(t), \qquad i = 1, 2, \ldots, n$$

f_i linear functions – Lotka-Volterra systems
f_i general functions – Kolmogorov systems

$$f(0, 0, \ldots, 0) = 0$$

Populations with overlapping generations

$$x_i' = x_i\big(r_i - f_i(x_1, x_2, \ldots, x_n)\big), \qquad i = 1, 2, \ldots, n$$

Populations with non-overlapping generations

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Lotka-Volterra systems

$$f_i(x_1, x_2, \ldots, x_n) = \sum_{j=1}^{n} a_{ij} x_j = \mathsf{A}\boldsymbol{x}$$

Populations with overlapping generations

$$x_i' = x_i\big(r_i - f_i(x_1, x_2, \dots, x_n)\big), \qquad i = 1, 2, \dots, n$$

Populations with non-overlapping generations

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Lotka-Volterra systems

$$f_i(x_1, x_2, \dots, x_n) = \sum_{j=1}^{n} a_{ij} x_j = \mathsf{A}\boldsymbol{x}$$

$\mathsf{A} = (a_{ij})$ – community matrix

Models with discrete time

One population

$$x(t+1) = \mathrm{e}^r x(t),\ x(0) = x_0 \quad \Rightarrow \quad x(t) = x_0 \mathrm{e}^{rt}$$

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$$x(t+1) = \mathrm{e}^{p\left(x(t)\right)} x(t),\ x(0) = x_0$$

p decreasing – intra-specific competition

p increasing – Allee effect

One population

$$x(t+1) = \mathrm{e}^r x(t), \; x(0) = x_0 \quad \Rightarrow \quad x(t) = x_0 \mathrm{e}^{rt}$$

$$x(t+1) = \mathrm{e}^{p\left(x(t)\right)} x(t), \; x(0) = x_0$$

p decreasing – intra-specific competition
p increasing – Allee effect

$$p(x) = r\left(1 - \frac{x}{K}\right)$$ – logistic equation

$$x > K \; \Rightarrow \; \mathrm{e}^{p(x)} > 1, \qquad x < K \; \Rightarrow \; \mathrm{e}^{p(x)} < 1$$

K – carrying cappacity

One population

$$x(t+1) = \mathrm{e}^r x(t),\ x(0) = x_0 \quad \Rightarrow \quad x(t) = x_0 \mathrm{e}^{rt}$$

$$x(t+1) = \mathrm{e}^{p\left(x(t)\right)} x(t),\ x(0) = x_0$$

p decreasing – intra-specific competition
p increasing – Allee effect

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$$x > K \ \Rightarrow\ \mathrm{e}^{p(x)} > 1, \qquad x < K \ \Rightarrow\ \mathrm{e}^{p(x)} < 1$$

K – carrying cappacity

$\mathrm{e}^{p(x)} = \dfrac{q}{(1+(q-1)x/K)^b}$ – basic equation

$$
\begin{aligned}
x(t+1) &= x(t)\mathrm{e}^{r_1 - a_{11}x(t) - a_{12}y(t)} \\
y(t+1) &= y(t)\mathrm{e}^{r_2 - a_{21}x(t) - a_{22}y(t)}
\end{aligned}
$$

Two populations – Lotka-Volterra system

$$
\begin{aligned}
x(t+1) &= x(t)\mathrm{e}^{r_1 - a_{11}x(t) - a_{12}y(t)} \\
y(t+1) &= y(t)\mathrm{e}^{r_2 - a_{21}x(t) - a_{22}y(t)}
\end{aligned}
$$

$a_{ii} > 0$ – intra-specific competition
$a_{ii} < 0$ – Allee effect
$a_{ij} > 0$ – inter-specific competition
$a_{ij} < 0$ – mutualism
$a_{ij}a_{ji} < 0$ – predation

Models with continuous time

One population

$$x' = rx, \ x(0) = x_0 \quad \Rightarrow x(t) = x_0 \mathrm{e}^{rt}$$

One population

$$x' = rx,\ x(0) = x_0 \quad \Rightarrow x(t) = x_0 \mathrm{e}^{rt}$$

$$x' = xp(x),\ x(0) = x_0$$

p decreasing – intra-specific competition

p increasing – Allee effect

One population

$$x' = rx,\ x(0) = x_0 \quad \Rightarrow x(t) = x_0 \mathrm{e}^{rt}$$

$$x' = xp(x),\ x(0) = x_0$$

p decreasing – intra-specific competition

p increasing – Allee effect

$$p(x) = r\left(1 - \frac{x}{K}\right) \text{ – logistic equation}$$

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$$p(x) = r\left(1 - \frac{x}{K}\right) \text{ – logistic equation}$$

$$p(x) = rx^{a-1}\left(1 - \left(\frac{x}{K}\right)^b\right) \text{ – generalized logistic equation}$$

$$
\begin{aligned}
x' &= x(r_1 - a_{11}x - a_{12}y) \\
y' &= y(r_2 - a_{21}x - a_{22}y)
\end{aligned}
$$

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$a_{ii} > 0$ – intra-specific competition

$a_{ii} < 0$ – Allee effect

$a_{ij} > 0$ – inter-specific competition

$a_{ij} < 0$ – mutualism

$a_{ij}a_{ji} < 0$ – predation

Gause-type predator-prey system

Gause-type predator-prey system

$$
\begin{aligned}
x' &= xp(x), \\
y' &= -dy
\end{aligned}
$$

x – size of prey population
y – size of predator population
$p(x)$ – prey growth rate
d – predator death rate

Gause-type predator-prey system

$$\begin{aligned} x' &= xp(x) - S\varphi(x)y, \\ y' &= -dy \end{aligned}$$

x – size of prey population

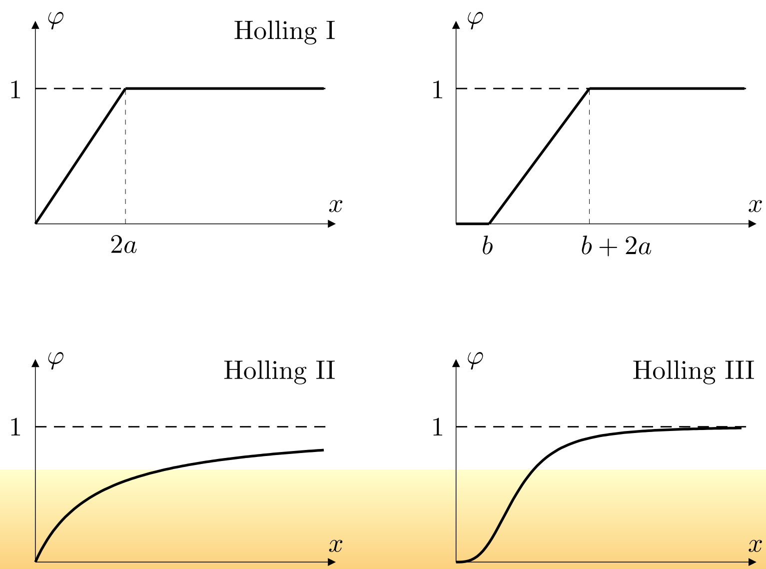
y – size of predator population

$p(x)$ – prey growth rate

d – predator death rate

$\varphi(x)$ – trophic function; increasing, $\varphi(0) = 0$, $\lim\limits_{x\to\infty} \varphi(x) = 1$

S – predator level of satiety



Gause-type predator-prey system

$$\begin{aligned} x' &= xp(x) - S\varphi(x)y, \\ y' &= -dy + \kappa S\varphi(x)y \end{aligned}$$

x – size of prey population

y – size of predator population

$p(x)$ – prey growth rate

d – predator death rate

$\varphi(x)$ – trophic function; increasing, $\varphi(0) = 0$, $\lim_{x\to\infty} \varphi(x) = 1$

S – predator level of satiety

κ – efficiency of conversion: prey into predator growth rate

Gause-type predator-prey system

$$
\begin{aligned}
x' &= x\left(p(x) - S\frac{\varphi(x)}{x}y\right), \\
y' &= y\big(-d + \kappa S\varphi(x)\big)
\end{aligned}
$$

$$
\begin{aligned}
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$$
\varphi(x) = \begin{cases} x/(2a), & x < 2a \\ 1, & x \geq 2a \end{cases}
$$

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$$
\varphi(x) = \frac{x^k}{x^k + a^k}
$$

$$
\varphi(x) = 1 - 2^{-(x/a)^k}
$$

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$$\varphi(x) = \begin{cases} x/(2a), & x < 2a \\ 1, & x \geq 2a \end{cases}$$ – Holling I

$$\varphi(x) = \frac{x^k}{x^k + a^k}$$ – Michaelis-Menten

$$\varphi(x) = 1 - 2^{-(x/a)^k}$$ – Ivlev

a – level of “half saturation”