Minimalizations of NFA Using the Universal Automaton

Libor Polák *
Department of Mathematics, Masaryk University
Janáčkovo nám. 2a, 662 95 Brno, Czech Republic
http://www.math.muni.cz/~polak

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Introduction

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Let $B = (\beta_{ij})$ be a matrix of type m/n with entries from $\{0, 1\}$ where $\beta_{ij} = 1$ if and only if $u_j v_i \in L$. This matrix is called the **basic matrix** of the language L. Adding to the columns of B new ones which are componentwise meets of sets of columns of B $(0 \wedge 0 = 0 \wedge 1 = 1 \wedge 0 = 0,\ 1 \wedge 1 = 1)$ we get the matrix C which is called the **universal matrix** of L.

The **universal automaton** of a language L is a (non-deterministic) automaton $\mathcal{U} = (\mathsf{U}\,, A, E, I, T)$ where
$(p, a, q) \in E$ if and only if $a^{-1}p \supseteq q$,

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Proposition. (i) $\mathcal{U}$ accepts L,
(ii) for each non-deterministic automaton $\mathcal{V} = (V, A, G, J, W)$ accepting a subset of L, the mapping

$$\phi : q \mapsto \bigcap \{ \; u^{-1}L \mid q \in J \cdot u \; \}$$

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Results

Let $\mathcal{U} = (\mathsf{U}, A, E, I, T)$ be the universal automaton of a regular language $L \subseteq A^*$. Each $P \subseteq \mathsf{U}$ induces a subautomaton $\mathcal{U}_P = (P, A, E_P, I \cap P, T \cap P)$ of $\mathcal{U}$ where $E_P = \{(p, a, q) \in E \mid p, q \in P\}$. Clearly, the language accepted by $\mathcal{U}_P$ is a subset of L. We formulate several conditions on a subset P of U:

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(A) the automaton $\mathcal{U}_P$ **accepts** the language L,

(C) $\mathcal{U}_P$ is **complete**, i.e.

$(\forall\, p \in P)(\forall\, a \in A)(\exists\, q \in P)\ q \subseteq a^{-1}p,$

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Theorem. The following implications between our conditions hold.

(i) (D) $\implies$ (K) & (C).

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Moreover, (iii) P_b satisfies (A).

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Moreover, (iii) P_b satisfies (A).

(iv) Both P_l and P_b satisfy the condition (LM).

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(v) For each $P \subseteq$ U satisfying (A) we have $P_0 \subseteq P$.

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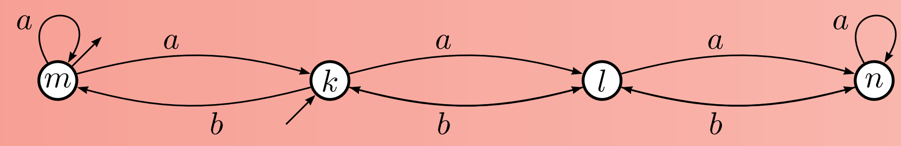
(iv) Both P_l and P_b satisfy the condition (LM).

(v) For each $P \subseteq$ U satisfying (A) we have $P_0 \subseteq P$.

(vi) None of the implications in (i) and (ii) can be reversed.

Kiel Example: Consider an automaton $\mathcal{A}$:



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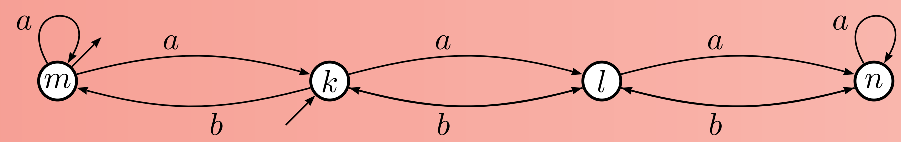
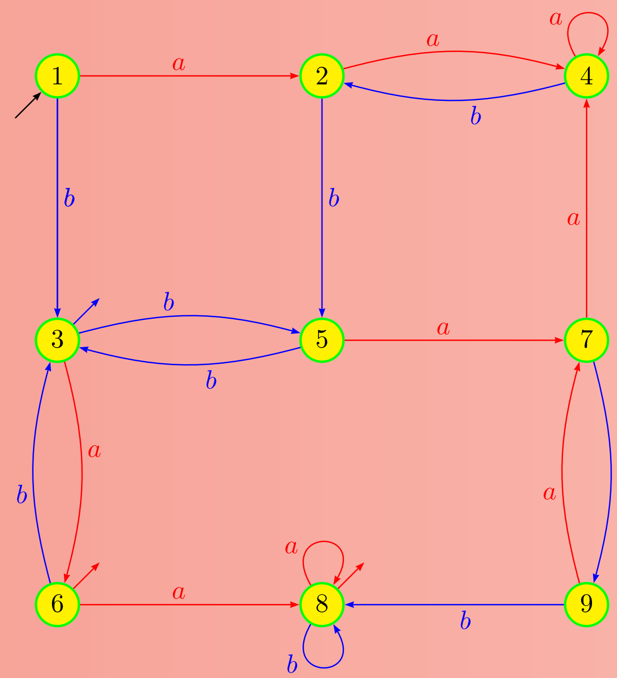


After determinization we get the minimal complete DFA $\mathcal{D}$:

$1 = \{k\},\ 2 = \{l\},\ 3 = \{l, m\},\ 4 = \{n\},\ 5 = \{k, n\}, \ldots$



We get the basic matrix by the determinization of $\overline{\mathcal{D}}$.

	1	2	3	4	5	6	7	8	9
1	0	0	1	0	0	1	0	1	0

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	1	2	3	4	5	6	7	8	9
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b	1	0	0	0	1	1	0	1	1

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b^2	0	1	1	0	0	0	1	1	1

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b^2	0	1	1	0	0	0	1	1	1
b^3	1	0	0	1	1	1	1	1	1

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b^3	1	0	0	1	1	1	1	1	1
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ab^3	0	1	1	1	1	1	1	1	1
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a^2b^3	1	1	1	1	1	1	1	1	1
	$\uparrow$	$\uparrow$	$\uparrow$	$\uparrow$		$\uparrow$			

The universal matrix.

	1	2	3	4	5	6	7	8	9	p	q	r	s
1	0	0	1	0	0	1	0	1	0	0	0	0	1
b	1	0	0	0	1	1	0	1	1	0	0	0	0
b^2	0	1	1	0	0	0	1	1	1	0	0	0	0
b^3	1	0	0	1	1	1	1	1	1	0	1	0	0
ab^3	0	1	1	1	1	1	1	1	1	0	0	1	1
a^2b^3	1	1	1	1	1	1	1	1	1	1	1	1	1

		1	2	3	4	5	6	7	8	9	p	q	r	s
$\longrightarrow$	1	0	0	1	0	0	1	0	1	0	0	0	0	1
$\longrightarrow$	b	1	0	0	0	1	1	0	1	1	0	0	0	0
$\longrightarrow$	b^2	0	1	1	0	0	0	1	1	1	0	0	0	0
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$\longrightarrow$	ab^3	0	1	1	1	1	1	1	1	1	0	0	1	1
	a^2b^3	1	1	1	1	1	1	1	1	1	1	1	1	1

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$\longrightarrow$	1	0	0	1	0	0	1	0	1	0	0	0	0	1
$\longrightarrow$	b	1	0	0	0	1	1	0	1	1	0	0	0	0
$\longrightarrow$	b^2	0	1	1	0	0	0	1	1	1	0	0	0	0
$\longrightarrow$	b^3	1	0	0	1	1	1	1	1	1	0	1	0	0
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		$\uparrow$	$\uparrow$									$\uparrow$	$\uparrow$	$\uparrow$

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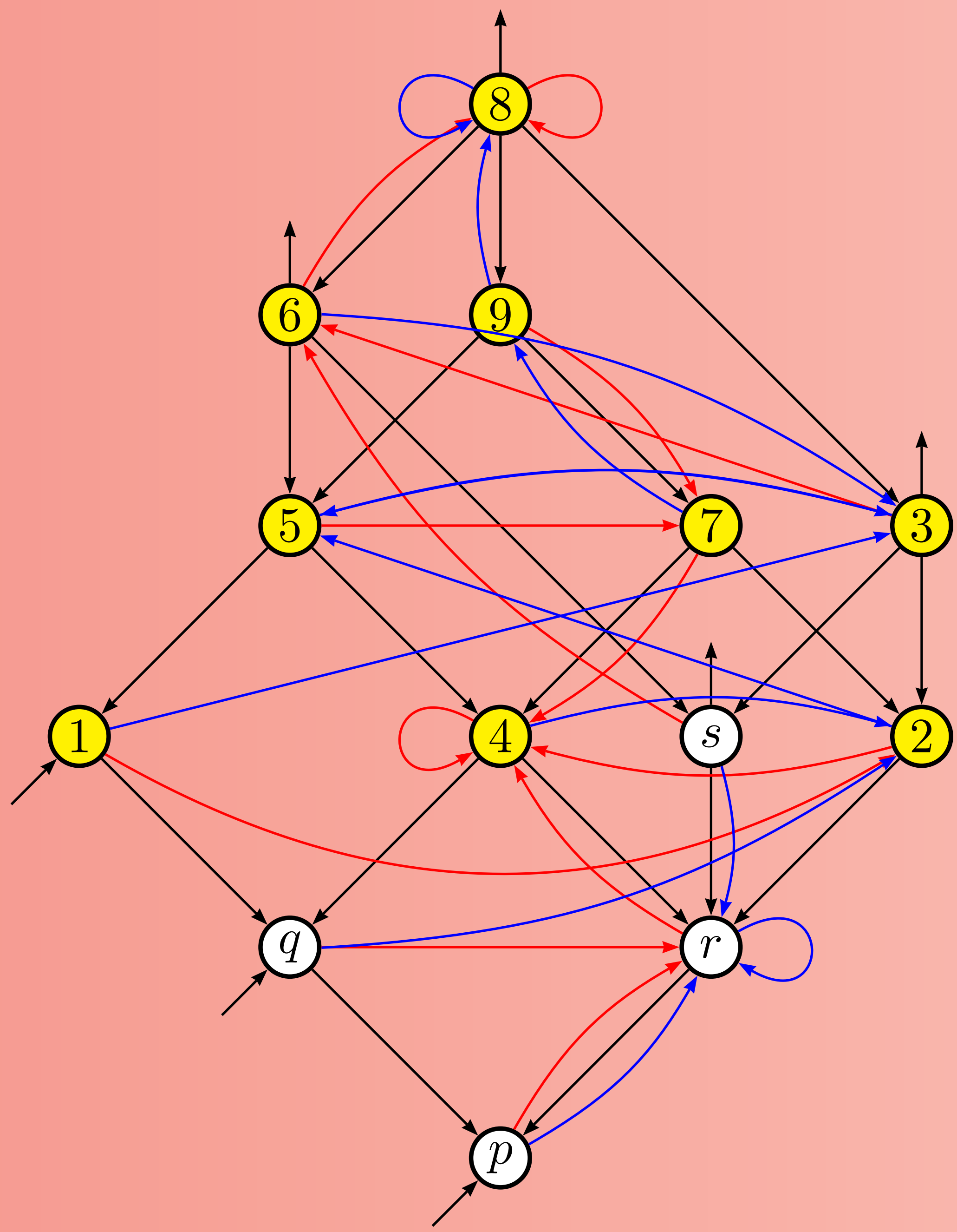
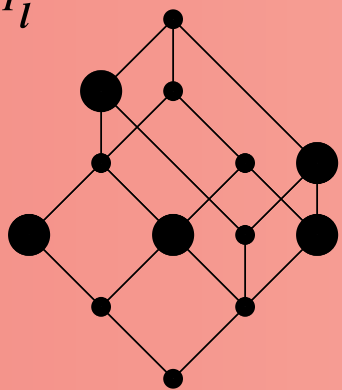
The universal automaton follows.

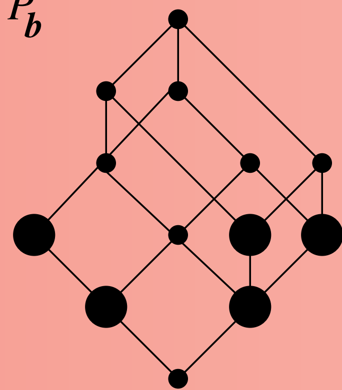

8
6
9
5
7
3
1
4
s
2
q
r
p

Consider the following choices for the set P.

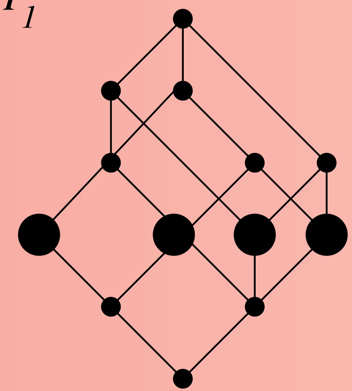
P_l



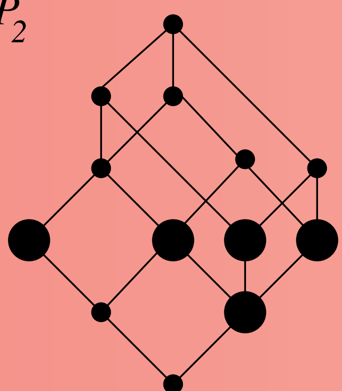
P_b



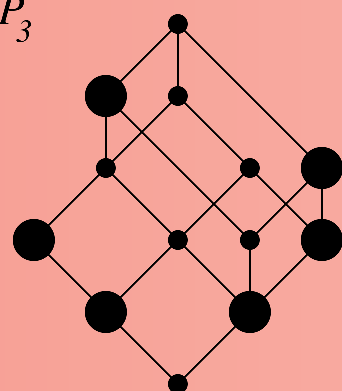
P_1



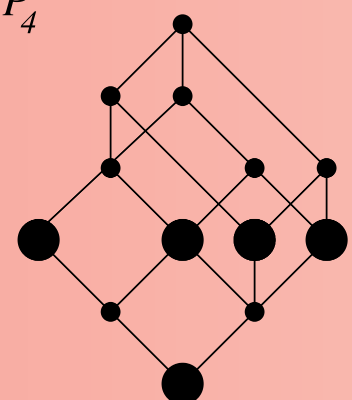
P_2



P_3



P_4



Here $P_0 = \{1, 2\}$ and the locally minimal sets of states with respect to (A) are exactly $P_l,\ P_b,\ P_1$ and P_3. Note also that P_1 is the ϕ-image of the automata we started with and P_2 is the image of its completion. Further, $P_b,\ P_2 \models (C),\ \neg(D),\ (K),\ P_1 \models \neg(C),\ \neg(D),\ \neg(K),\ P_4 \models (C),\ \neg(D),\ \neg(K)$.

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Waterloo Example:

The following table presents an (incomplete) DFA.

	$\downarrow$ 1	2	3	4	$\uparrow$ 5	$\uparrow$ 6	7	8
a	5	6	7	3	3	2	1	1
b	–	–	2	8	8	4	4	–

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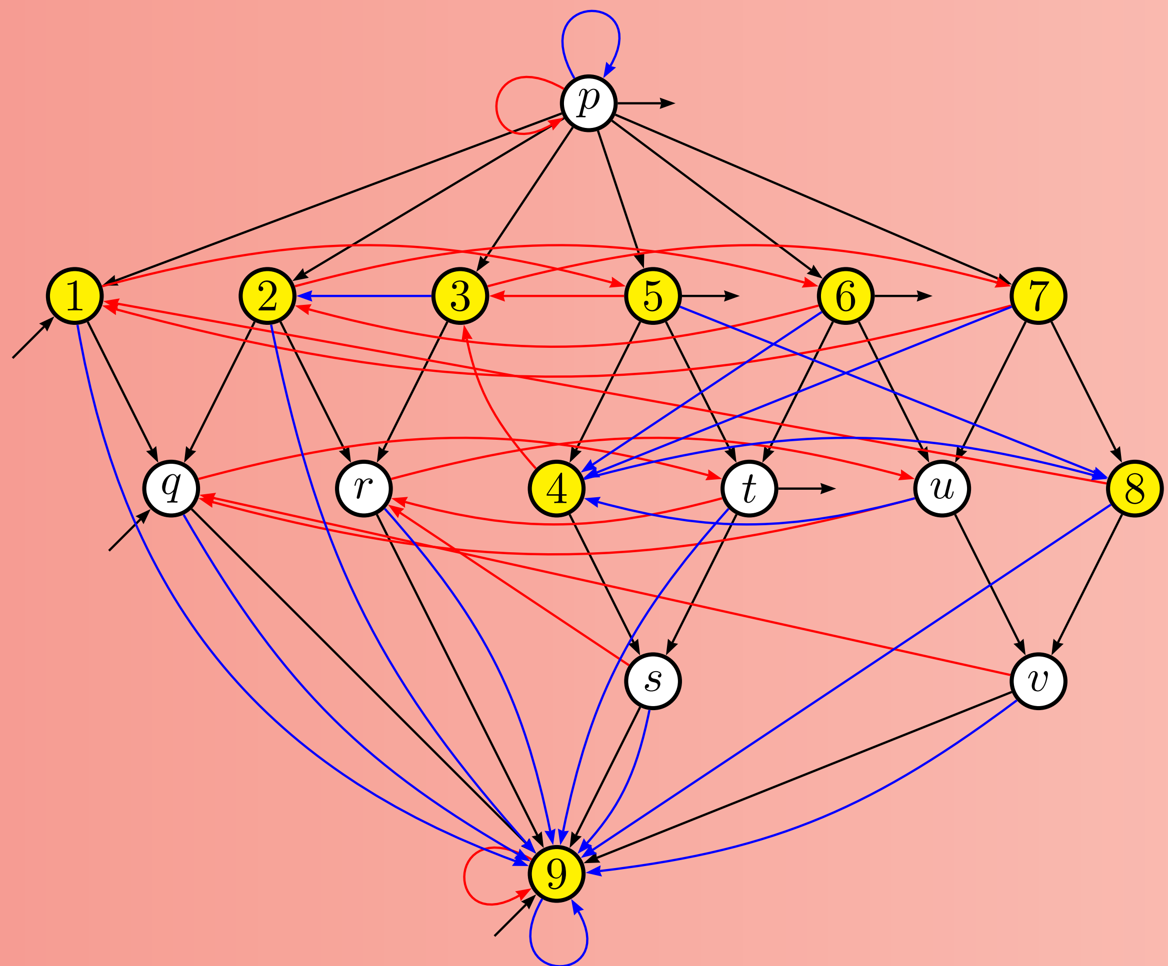
Its universal matrix follows.

	1	2	3	4	5	6	7	8	9	p	q	r	s	t	u	v
1	0	0	0	0	1	1	0	0	0	1	0	0	0	1	0	0
a	1	1	0	0	0	0	0	0	0	1	1	0	0	0	0	0
b	0	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
a^2	0	0	0	0	0	1	1	1	0	1	0	0	0	0	1	1
ba	0	0	1	0	0	0	0	0	0	1	0	0	0	0	0	0
a^3	0	1	1	0	0	0	0	0	0	1	0	1	0	0	0	0
aba	0	0	0	1	1	0	0	0	0	1	0	0	0	0	0	0
a^4	0	0	0	1	1	1	0	0	0	1	0	0	1	1	0	0
a^2ba	1	0	0	0	0	0	0	0	0	1	0	0	0	0	0	0
a^3ba	0	0	0	0	0	0	1	1	0	1	0	0	0	0	0	0
ba^4	0	0	0	0	0	1	1	0	0	1	0	0	0	0	1	0

The universal automaton is depicted below.



We have $P_0 = \{1, 3, 4, 8\}$ and the locally minimal sets of states with respect to (A) are exactly $P_l = P_0 \cup \{2, 5, 6, 7\},\ P_b = P_0 \cup \{q, r, t, u\},\ P_1 = P_0 \cup \{2, 5, 6, u\},\ P_2 = P_0 \cup \{2, 6, t, u\}$ and $P_3 = P_0 \cup \{2, 6, 7, t\}$. We have that $P_4 = P_0 \cup \{2, t, u\} \models (W), \neg(A)$.