On pseudovarieties of semiring homomorphisms

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Syntactic structures

An **idempotent semiring** is a structure $(S, \cdot, \vee)$ where
$(S, \cdot)$ is a monoid with the neutral element 1,
$(S, \vee)$ is a semilattice with the smallest element 0,
$(\ \forall\ a, b, c \in S\ )(\ a(b \vee c) = ab \vee ac$ and $(a \vee b)c = ac \vee bc\ )$,
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Let $L \subseteq A^*$ be a regular language. We define the following relations :

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The factor-structures $(\mathsf{O}\,(L), \cdot, 1{\approx_L}) = (A^*, \cdot, 1)/{\approx_L}$ and $(\mathsf{S}\,(L), \cdot, \vee) = (A^{\square}, \cdot, \cup)/{\sim_L}$ are called the **syntactic monoid** and the **syntactic semiring** of the language L.

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$$\kappa_L : u \mapsto u \approx_L \ \text{ and } \phi_L : \{u_1, \ldots, u_k\} \ \mapsto \{u_1, \ldots, u_k\} \sim_L$$

are called **syntactic monoid/semiring homomorphisms**.

The monoid $(\mathsf{O}\,(L), \cdot)$ is ordered by the relation

$$v \approx_L \leq \; u \approx_L \quad \text{iff} \;\; (\, \forall \, x, y \in A^* \,)\, (\, xuy \in L \; \Rightarrow \; xvy \in L \,)$$

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Ex. finite unions of $A^*a_1A^*a_2\dots a_kA^* \mapsto$ Mod $(x \leq 1)$

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$\mathbb{D}$-pseudovarieties of semiring homomorphisms

Let $\boldsymbol{S}$ be the class of all finite idempotent semirings. Let

$$\mathfrak{S} = \{\ \phi : A^{\square} \twoheadrightarrow S \mid A \text{ is a finite set and } S \in \boldsymbol{S}\ \}$$

be the class of all surjective semiring homomorphisms from a finitely generated free idempotent semiring onto a finite idempotent semiring.

generated free idempotent semiring onto a finite idempotent semiring. We consider a category $\mathbb{D}$ of homomorphisms between finitely generated free idempotent semirings, i.e. objects are all semirings $A^{\Box}$ where A is a finite set and the sets of morphisms $\mathbb{D}(B^{\Box}, A^{\Box})$ consist of certain semiring homomorphisms from $B^{\Box}$ to $A^{\Box}$.

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Basic examples are the category $\mathbb{D}_{all}$ of all semiring homomorphisms, and the categories $\mathbb{D}_{mne}$, $\mathbb{D}_{mi}$, $\mathbb{D}_{ml}$ and $\mathbb{D}_l$ of all **multi-non-erasing**, **monoid induced**, **multi-literal** and **literal** homomorphisms, respectively. For finite alphabets A, B :
$f \in \mathbb{D}_{mne}(B^\square, A^\square)$ iff for each $b \in B$ there are $u_1, \ldots, u_k \in A^+$ such that $f(\{b\}) = \{u_1, \ldots, u_k\}$,
$f \in \mathbb{D}_{mi}(B^\square, A^\square)$ iff for each $b \in B$ there is $u \in A^*$ such that

$f(\{b\}) = \{u\}$,
$f \in \mathbb{D}_{ml}(B^{\Box}, A^{\Box})$ iff for each $b \in B$ there are $a_1, \ldots, a_k \in A$ such that $f(\{b\}) = \{a_1, \ldots, a_k\}$, and
$f \in \mathbb{D}_{l}(B^{\Box}, A^{\Box})$ iff for each $b \in B$ there is $a \in A$ such that $f(\{b\}) = \{a\}$.

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A class $\mathfrak{X} \subseteq \mathfrak{S}$ is a **$\mathbb{D}$-pseudovariety of semiring homomorphisms** if it satisfies :

(H) for each $(\phi : A^{\square} \twoheadrightarrow S) \in \mathfrak{X}$ and a surjective semiring homomorphism $\sigma : S \twoheadrightarrow T$ we have $\sigma\phi \in \mathfrak{X}$,

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(S$_{\mathbb{D}}$) for each $f \in \mathbb{D}(B^{\square}, A^{\square})$ and $(\phi : A^{\square} \twoheadrightarrow S) \in \mathfrak{X}$ we have $(\ \phi f : B^{\square} \twoheadrightarrow \mathrm{im}\,(\phi f)\ ) \in \ \mathfrak{X}$,

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(P) for each non-negative integer m and for each system

$\phi_1 : A^\square \twoheadrightarrow S_1, \ldots, \phi_m : A^\square \twoheadrightarrow S_m \in \mathfrak{X}$ we have
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(here
$(\phi_1, \ldots, \phi_m)(U) = (\phi_1(U), \ldots, \phi_m(U)) \in S_1 \times \cdots \times S_m,\ U \in A^\square$,
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For finite sets A and B, a semiring homomorphism
$f : (B^\Box, \cdot, \cup) \to (A^\Box, \cdot, \cup)$ and $K \subseteq A^*,\ L \subseteq B^*$ we define

$$f^{[-1]}(K) = \{\ v \in B^* \mid f(\{v\}) \subseteq K\ \}\ ,$$

$$f^{(-1)}(K) = \{\ v \in B^* \mid f(\{v\}) \cap K \neq \emptyset\ \}\ ,$$

$$f(L) = \bigcup \{\ f(\{v\}) \mid v \in L\ \}\ .$$

Further, for the complement K^{c} of a language $K \subseteq A^*$, it holds

$$f^{[-1]}(K^{\mathsf{c}}) = (f^{(-1)}(K))^{\mathsf{c}} \, .$$

For a multiliteral homomorphism $f : B^{\square} \to A^{\square}$ we define the **dual** multiliteral homomorphism $\widehat{f} : A^{\square} \to B^{\square}$ by $b \in \widehat{f}(\{a\})$ iff $a \in f(\{b\})$ for $a \in A, \ b \in B$. Notice that $f^{(-1)}(K) = \widehat{f}(K)$ for each $K \subseteq A^*$.

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A **class** of (regular) languages is an operator $\mathcal{L}$ assigning to every finite set A a set $\mathcal{L}(A)$ of regular languages over the alphabet A.

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A **class** of (regular) languages is an operator $\mathcal{L}$ assigning to every finite set A a set $\mathcal{L}(A)$ of regular languages over the alphabet A. Such a class is a **conjunctive variety** if
(i) each $\mathcal{L}(A)$ is closed with respect to finite intersections and quotients, and
(ii) for each finite sets A and B and $f : B^{\Box} \to A^{\Box},\ K \in \mathcal{L}(A)$

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It is a **$\mathbb{D}$-variety** if (i) is true and (ii) is satisfied for $f \in \mathbb{D}(B^{\square}, A^{\square})$.

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Finally, a class $\mathcal{L}$ is a **multiliteral variety** if
(i') each $\mathcal{L}(A)$ is closed with respect to finite unions and quotients, and
(ii') for each finite sets A and B and $f \in \mathbb{D}_{ml}(A^\square, B^\square),\ K \in \mathcal{L}(A)$ implies $f(K) \in \mathcal{L}(B)$.

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Define $\mathcal{L}^{\mathsf{c}}$ by $\mathcal{L}^{\mathsf{c}}(A) = \{\ L^{\mathsf{c}} \mid L \in \mathcal{L}(A)\ \}$.

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Fix a category $\mathbb{D}$. We can assign to any class of languages $\mathcal{L}$ the pseudovariety

$$\mathsf{S}\,(\mathcal{L}) = \langle\ \{\ (\mathsf{S}\,(L), \cdot, \vee) \mid A \text{ a finite set, } L \in \mathcal{L}(A)\ \}\ \rangle_{\mathcal{S}}$$

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Conversely, for a class $\mathcal{X}$ of idempotent semirings and a finite set A, we put

$$(\mathsf{L}\,(\mathcal{X}))(A) = \{\ L \subseteq A^* \mid (\mathsf{S}\,(L), \cdot, \vee) \in \mathcal{X}\ \}$$

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and similarly for a class $\mathfrak{X}$ of idempotent semiring homomorphisms and a finite set A, we put

$$(\mathbf{L}\,(\mathfrak{X}))(A) = \{ \, L \subseteq A^* \mid \phi_L \in \mathfrak{X} \, \}, \; (\mathbf{L}^{\mathsf{c}}(\mathfrak{X}))(A) = \{ \, L \subseteq A^* \mid \phi_{L^{\mathsf{c}}} \in \mathfrak{X} \, \}$$

Theorem (i) The assignments $\mathcal{L} \mapsto \mathsf{S}(\mathcal{L})$ and $\mathcal{X} \mapsto \mathsf{L}(\mathcal{X})$ are mutually inverse bijections between the conjunctive varieties of languages and pseudovarieties of finite idempotent semirings.

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Examples of multiliteral varieties of languages

We define the classes of languages $\mathcal{S}(n)$, $\mathcal{H}_{1/2}(k)$, $\mathcal{H}_{3/2}(k)$, $\mathcal{H}_{1/2}$, $\mathcal{H}_{3/2}$, $\mathcal{L}(k,m,l)$, $\mathcal{L}(m)$ below. It is obvious that all of them are multiliteral varieties of languages.

For a finite set A and $n, k, l \geq 0,\ m \geq 1$ put
$\mathcal{S}(n)(A) =$ class of all languages over A of star height $\leq n$,

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$0 \leq p \leq k,\ a_1, \ldots, a_p \in A$ and $B_0, \ldots, B_p \subseteq A$,
$\mathcal{L}(k, m, l) =$ all finite unions of $uB_1^* \ldots B_m^*v$, where
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$u, v \in A^*, |u| \leq k, |v| \leq\ l,\ B_1, \ldots, B_m \subseteq A$.
Further, put $\mathcal{H}_{1/2}(A) = \bigcup_{k\geq 0} \mathcal{H}_{1/2}(k)(A)$,

$$\mathcal{H}_{3/2}(A) = \bigcup_{k\geq 0} \mathcal{H}_{3/2}(k)(A),\ \mathcal{L}(m)(A) = \bigcup_{k,l\geq 0} \mathcal{L}(k, m, l)(A)\ .$$

The varieties $\mathcal{H}_{1/2}$ and $\mathcal{H}_{3/2}$ are members of the so-called

Straubing-Thérien hierarchy.

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Pseudoidentities

This section modifies the approach by Kunc concerning pseudovarieties of monoid homomorphisms to the case of semirings.

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Recall that $(\pi_S)_{S \in \boldsymbol{S}}$ is an n-**ary implicit operation** $(n \geq 0)$ for the class $\boldsymbol{S}$ of all finite idempotent semirings if $\pi_S : S^n \to S$ $(S \in \boldsymbol{S})$ is a mapping and for each semiring homomorphism $\sigma : S \to T$ and $s_1, \ldots, s_n \in S$ we have

$$\sigma(\pi_S(s_1, \ldots, s_n)) = \pi_T(\sigma(s_1), \ldots, \sigma(s_n)) \ .$$

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An n-**ary pseudoidentity** is an ordered pair $\pi = \rho$ of n-ary implicit operations. Let $v_1, v_2, \ldots$ be a fixed sequence of pairwise different variables and let $V_n = \{v_1, \ldots, v_n\}$ for each $n \geq 0$.

variables and let $V_n = \{v_1, \ldots, v_n\}$ for each $n \geq 0$.
Let $\mathbb{D}$ be a category of of homomorphisms of free finitely generated idempotent semirings. The pseudoidentity $\pi = \rho$ is **$\mathbb{D}$-satisfied** in a semiring homomorphism $(\phi : A^\square \twoheadrightarrow S) \in \mathfrak{S}$ if for each $f \in \mathbb{D}(V_n^\square, A^\square)$ we have

$$\pi_S((\phi f)(v_1), \ldots, (\phi f)(v_n)) = \rho_S((\phi f)(v_1), \ldots, (\phi f)(v_n)) \ .$$

Reiterman-type theorem

Theorem Let $\mathbb{D}$ be a category of homomorphisms of finitely generated free idempotent semirings containing all bijections. Then a class $\mathfrak{X} \subseteq \mathfrak{S}$ is a $\mathbb{D}$-pseudovariety if and only if there exists a set Σ of pseudoidentities such that $\mathfrak{X} = \mathsf{Mod}_{\mathbb{D}}(\Sigma)$.