QUANTALES

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In this paper we survey aspects of the theory of quantales, starting from its connection to locales and C*-algebras and finishing with recent developments involving the notions simplicity and spatiality.

1. Introduction

The term *quantale* was suggested by C.J. Mulvey at the Oberwolfach Category Meeting (see [14]) as "a quantization" of the term *locale*. Locales form an order-theoretic counterpart of topological spaces and are therefore able to describe commutative C*-algebras. More precisely, a commutative C*-algebra A is completely determined by its lattice $I(A)$ of closed ideals which is order isomorphic to the lattice of open sets of its spectrum. The aim of C.J. Mulvey was to find a substitute of locales which could play the same role for general C*-algebras. He was motivated by C.A. Akemann [1, 2] and R. Giles and H. Kummer [7] who had found a structure on the lattice $R(A)$ of all closed right ideals of a C*-algebra A which makes it possible to determine A. The suggestion of C.J. Mulvey was to consider $R(A)$ as a lattice equipped with a multiplication. In the commutative case, this multiplication is the meet operation, i.e., the quantale $R(A)$ is the locale $I(A)$.

The concept of a quantale is, of course, much older. It goes back to 1930's when M. Ward and R.P. Dilworth realized that the theory of ideals in rings can be conveniently formulated in terms of a lattice of ideals equipped with a multiplication (see [24, 25, 26, 5]). Even, as L. Skula has informed us, this idea is contained in W. Krull [12]. An important moment in the development of the theory of quantales was the realization that quantales give a semantics for propositional linear logic in the same way as Boolean algebras give a semantics for classical propositional logic (see [8]). In fact, the appearance of quantales in linear logic was anticipated by the use of multiplicative structures in relevance logic and Lambek calculus. The existing literature contains a monograph of K. Rosenthal [20] devoted to quantales. Our aim is to survey the theory of quantales, mainly in their connections with C*-algebras. We also add some recent developments.

2. Locales and quantales

In what follows a complete lattice will be called a *sup-lattice*. *Sup-lattice homomorphisms* are maps between sup-lattices preserving arbitrary joins. Evidently, any sup-lattice isomorphism is a complete lattice isomorphism (i.e., it preserves all meets). A sup-lattice homomorphism will be called *strong* if it preserves the top element 1. Any sup-lattice homomorphism $f : L_1 \to L_2$ between sup-lattices has a right adjoint $f_* : L_2 \to L_1$ defined by

$$f_*(a) = \bigvee\{s \in L_1 | f(s) \leq a\}$$

for all $a \in L_2$. Then f_* preserves arbitrary meets.

A *frame* is a sup-lattice L in which the infinite distributive law

$$a \wedge \bigvee S = \{a \wedge s | s \in S\}$$

holds for all $a \in L, S \subseteq L$. In fact, frames are exactly complete Heyting algebras and the Heyting implication is given by

$$a \Rightarrow b = \bigvee\{s \in L | s \wedge a \leq b\}.$$

Frame homomorphisms are defined as sup-lattice homomorphisms between frames preserving finite meets (including the empty one), in particular they preserve the distinguished elements 0 (the bottom) and 1 (the top). The study of frames was motivated by the following observation. Let X be a topological space and $\Omega(X)$ be the complete lattice of open subsets of X. Since finite meets and arbitrary joins correspond to the set-theoretic operations of intersection and union, $\Omega(X)$ is a frame. Moreover, if $f : X \to Y$ is a continuous map of spaces, the induced map $f^{-1} : \Omega(Y) \to \Omega(X)$ is a frame homomorphism. The functor $X \mapsto \Omega(X), f \mapsto f^{-1}$ is contravariant and thus the category of "generalized spaces" should be the opposite of the category of frames. Its objects are then called *locales* and morphisms *locale maps.*

Recall that a point of a space X may be identified with a continuous map from the one-point space $\mathbf{1}$ to X. Therefore, a *point* of a frame L is, alternatively, a frame homomorphism $h : L \to \mathbf{2}$ into the two-element frame $\mathbf{2} = \{0, 1\} \cong \Omega(\mathbf{1})$. It is easy to see that points $h : L \to \mathbf{2}$ correspond to *prime* elements $p \in L$, i.e., to elements $1 \neq p \in L$ such that

$$x \wedge y \leq p \implies \quad x \leq p \quad \text{or} \quad y \leq p$$

for any $x, y \in L$. This correspondence is given by

$$p = \bigvee\{x \in L | h(x) = 0\}.$$

The spectrum $P(L)$ of a frame L is a space whose elements are primes of L and whose open sets are the sets $\{p \in P(L) | x \nleq p\}$ where $x \in L$. L is said to be *spatial* if the surjective frame homomorphism $x \mapsto \{p \in P(L) | x \nleq p\}$ is an isomorphism. This happens if and only if any element $x \in L$ is a meet of primes, i.e. if L has "enough" points to separate distinct elements of L.

One can treat frames as generalized topological spaces. For instance, a frame L is called *compact* if, for any $S \subseteq L$ with $1 = \bigvee S$, there is a finite subset $S' \subseteq S$ such that $1 = \bigvee S'$. Let $a, b \in L$. We say that b is *rather below* a, and write $b \lhd a$, if there is an element $c \in L$ such that $c \wedge b = 0$, $c \vee a = 1$. A frame L in which each element is a join of elements rather below it is said to be *regular*. Then a topological space X is compact (regular) iff the frame $\Omega(X)$ is compact (regular). Consult P.T. Johnstone [9] and S. Vickers [23] for more information about frames.

Replacing in the definition of a frame the commutative operation $\wedge$ with a general associative binary operation, we obtain a "non-commutative generalized space". Namely, by a *quantale*, we shall mean a sup-lattice Q equipped with an associative multiplication $\cdot$ which distributes over arbitrary joins:

$$a \cdot \bigvee S = \bigvee \{a \cdot s | s \in S\} \quad \text{and} \quad \bigvee S \cdot a = \bigvee \{s \cdot a | s \in S\}$$

for all $a \in Q, S \subseteq Q$. A *quantale homomorphism* is defined as a sup-lattice homomorphism between quantales preserving the multiplication. If a quantale Q is a monoid then the unit will be denoted by e and such a Q is called *unital*. A quantale homomorphism is said to be *unital* if it preserves the unit e. Let us mention that, despite the fact that the term "quantale" is derived from that of a "locale", our quantale homomorphisms correspond to frame homomorphisms and not to locale maps.

Let x be an element of a quantale Q. Then x is called *idempotent* if $x \cdot x = x$ and *right-sided* (*left-sided*) if $x \cdot 1 \leq x$ ($1 \cdot x \leq x$). It is called *two-sided* if it is both right-sided and left-sided. The set of all right-sided (left-sided, two-sided, idempotent) elements of a quantale Q is denoted by $R(Q)$ ($L(Q), I(Q), E(Q)$). We shall say that a quantale Q is *right-sided* (*left-sided*, *two-sided*, *idempotent*, *right-idempotent*, *left-idempotent*) if $Q = R(Q)$ ($Q = L(Q), Q = I(Q), Q = E(Q), R(Q) \subseteq E(Q), L(Q) \subseteq E(Q)$). A subset $T \subseteq Q$ is called a *subquantale* of Q if it is closed under all joins and multiplication in Q. It may be remarked that $R(Q)$, $L(Q)$ and $I(Q)$ are subquantales of Q closed under arbitrary meets, $Q \cdot R(Q) \subseteq R(Q)$ and $L(Q) \cdot Q \subseteq L(Q)$. A *quantale congruence* on Q is an equivalence relation on Q which is a subquantale of $Q \times Q$. A subquantale of Q is said to be *strong* if it contains the top element of Q. The quantale $\mathbf{1} = \{0\}$ is called *trivial*.

Recall that frames are exactly two-sided idempotent quantales. Moreover, any frame homomorphism is a strong and unital quantale homomorphism. Other important examples are the quantales $Rid(R)$ ($Lid(R), Id(R)$) of all right-sided (left-sided, two-sided) ideals of a ring R, with set inclusion as partial order and the familiar multiplication of right-sided (left-sided, two-sided) ideals. Note that the ring R is not required to have a unit here; if it does have a unit then R is the right unit (left unit, unit) of $Rid(R)$ ($Lid(R), Id(R)$).

Since $a \cdot - : Q \to Q$ and $- \cdot a : Q \to Q$ are sup-lattice homomorphisms for any $a \in Q$, they have right adjoints $a \to_r - : Q \to Q$ and $a \to_l - : Q \to Q$. Explicitly,

$$a \to_r x = \bigvee_{a \cdot y \leq x} y \quad \text{and} \quad a \to_l x = \bigvee_{y \cdot a \leq x} y.$$

For any $a \in Q$, we have $a \to_r 0 \in R(Q)$ and $a \to_l 0 \in L(Q)$. A quantale Q is *discrete* if

$$(a \to_l 0) \to_r 0 = a \quad \text{for any } a \in R(Q)$$

and

$$(b \to_r 0) \to_l 0 = b \quad \text{for any } b \in L(Q).$$

In any discrete quantale Q we have $L(Q) \simeq R(Q)^{op}$.

A quantale Q is said to be a *factor* if $I(Q) = \{0, 1\}$. Every sup-lattice L can be seen as a right-sided idempotent factor by providing it with the following multiplication:

$$x \cdot y = x \text{ if } y \neq 0 \text{ and } x \cdot 0 = 0,$$

for all $x, y \in L$. Conversely, any right-sided idempotent factor necessarily has such a multiplication. This follows from the fact that in right-sided idempotent quantales the multiplication is given by the formula

$$x \cdot y = x \wedge \widehat{y}$$

where $\widehat{y}$ is the smallest two-sided element greater than or equal to y. In what follows, whenever we consider a sup-lattice as a quantale we will have this multiplication in mind.

In particular, if H is a Hilbert space then the complete orthocomplemented (in fact orthomodular) lattice $L(H)$ of all closed subspaces of H is a right-sided idempotent factor. Recall that a lattice L is called *orthocomplemented* if there is a unary operation $^\perp : L \to L$ such that $a^{\perp\perp} = a$, $a \leq b$ implies $b^\perp \leq a^\perp$ and $a \vee a^\perp = 1$ for all $a, b \in L$.

Let L be a lattice. *Atoms* of L are elements $a \in L$ such that $x \leq a$ implies $x = 0$ or $x = a$, for all $x \in L$. Similarly, *dual atoms* of L are exactly atoms of the dual lattice L^{op}. L is *atomic* (*dually atomic*) if each element is a join (meet) of atoms (dual atoms). Let L_1, L_2 be dually atomic sup-lattices and $f : L_1 \to L_2$ be a sup-lattice homomorphism. We say that f is *surjective on dual atoms* if for any dual atom $x \in L_2$ there is a dual atom $y \in L_1$ such that $f(y) = x$.

Lemma 2.1 *Let $f : L_1 \to L_2$ be a strong sup-lattice homomorphism between dually atomic sup-lattices. Then f is surjective on dual atoms iff f_* is injective on dual atoms (i.e., it injectively maps dual atoms of L_2 into dual atoms of L_1).*

Proof. Let f be surjective on dual atoms and consider a dual atom $x \in L_2$. Since $x = f(y)$ for a dual atom $y \in L_1$ and $f(1) = 1$, we have $y = f_*(x)$. Since $f(y_1) = x_1 \neq x_2 = f(y_2)$ implies $y_1 \neq y_2$, f_* is injective on dual atoms.

Conversely, let f_* be injective on dual atoms. Let x be a dual atom in L_2. Then $f_*(x)$ is a dual atom in L_1. It suffices to show that $f(f_*(x)) = x$. We have that $f(f_*(x)) \leq x$. Assume that $f(f_*(x)) < x$. Then there is a dual atom $f(f_*(x)) < y \in L_2$, $x \neq y$. We have

$$y \geq f(f_*(x)) \vee f(f_*(y)) = f(f_*(x) \vee f_*(y)) = f(1) = 1,$$

which is a contradiction. □

3. Quantales and C*-algebras

Recall that a *C*-algebra* is a complex Banach algebra A with an involution * satisfying

$$||x^*x|| = ||x||^2$$

for all $x \in A$. C*-algebras are precisely closed subalgebras of the algebras $B(H)$ of all bounded linear operators on a Hilbert space H. Commutative C*-algebras can be characterized as algebras $C_0(X)$ of complex-valued continuous functions vanishing at infinity on locally compact regular topological spaces X. This is the well-known Gelfand duality. In particular, commutative unital C*-algebras correspond to the algebras $C(X)$ of complex-valued continuous functions on a compact regular topological space X. In both cases, we obtain a duality of relevant categories (cf. [6] or [11]).

Let A be a C*-algebra and denote by $R(A)$ its lattice of closed right ideals. Define the multiplication on $R(A)$ by taking for $J_1 \cdot J_2$ the closure $\overline{J_1J_2}$ in the norm topology of the algebraic product

$$J_1J_2 = \{xy | x \in J_1, y \in J_2\}$$

of $J_1, J_2 \in R(A)$. We then have the following basic result.

Lemma 3.1 [14, 3] *$R(A)$ is an idempotent right-sided quantale for any C*-algebra A.*

Recall that $R(A)$ is dually atomic, dual atoms of $R(A)$ are maximal regular closed right ideals and that they correspond to pure states of A.

Any homomorphism $f : A \to B$ of C*-algebras induces a sup-lattice homomorphism $R(f) : R(A) \to R(B)$ given by taking $R(f)(J)$ as the closed right ideal generated by $f(J)$. But, in general, $R(f)$ does not preserve the multiplication. The reason is that $R(f)(J_1 \cdot J_2)$ is generated by $f(J_1J_2) = f(J_1)f(J_2)$ while $R(f)(I_1) \cdot R(f)(I_2)$ is generated by $f(J_1)Bf(J_2)$. To see the difference, it suffices to consider the embedding of diagonal matrices into the full matrix algebra $M_n = B(\mathbb{C}^n)$ for $n \geq 2$. $R(f)$ preserves multiplication if either f is surjective or both A and B are commutative.

Remarks 3.2 (see [4]): (i) A is commutative iff $R(A)$ is commutative, i.e., iff $R(A)$ is a frame.
(ii) A is *elementary*, i.e. isomorphic to the algebra $K(H)$ of all compact operators on a Hilbert space H, iff $R(A)$ is an atomic factor. In that case $R(A) \cong L(H)$.
(iii) A is *dual*, i.e. a C*-subalgebra of an elementary one, iff $R(A)$ is atomic. In that case $R(A) \cong \prod_{t\in T} L(H_t)$.

Let A be a C*-algebra and $p : A \to B(H)$ be a non-zero irreducible representation. This means that p is a C*-homomorphism such that $p \neq 0$ and the only closed linear subspaces of the Hilbert space H invariant under p are $\{0\}$ and H. Then $p(A)$ is dense in $B(H)$ in the weak topology and, consequently, for any $J \in R(A)$, the weak closure $cl(p(J))$ is a weakly closed right ideal in $B(H)$. Since the lattice of weakly closed right ideals of $B(H)$ is isomorphic to $L(H)$, we get a mapping [4]

$$\widetilde{p} : R(A) \to L(H)$$

Proposition 3.3 *Let $p : A \to B(H)$ be a non-zero irreducible representation of a C*-algebra A. Then $\widetilde{p} : R(A) \to L(H)$ is a strong homomorphism of quantales which is surjective on dual atoms.*

Proof. Following [4], $\widetilde{p}$ is a strong homomorphism of quantales. Due to Lemma 2.1, it suffices to prove that $\widetilde{p}_*$ is injective on dual atoms. Dual atoms of $L(H)$ are precisely orthocomplements $\langle x\rangle^\perp$ of one-dimensional subspaces of H. The weakly closed right ideal J_x of $B(H)$ corresponding to $\langle x\rangle^\perp$ is defined as

$$J_x = \{f \in B(H) | f(x) = 0\}.$$

Hence

$$\widetilde{p}_*(\langle x\rangle^\perp) = \{a \in A | cl(\{p(a)\}) \subseteq J_x\} = \{a \in A | p(a)(x) = 0\}.$$

Following [6] Theorem 2.9.5, $\widetilde{p}_*(\langle x\rangle^\perp)$ is a dual atom in $R(A)$.

Assume that $\langle x\rangle$ and $\langle y\rangle$ are distinct one-dimensional subspaces of H. Then x and y are linearly independent and, following the transitivity theorem ([11] 10.2.1), for any $0 \neq z \in H$ there is $a \in A$ such that $p(a)(x) = z$ and $p(a)(y) = 0$. Hence $\widetilde{p}_*(\langle x\rangle^\perp) \neq \widetilde{p}_*(\langle y\rangle^\perp)$ because $a \in \widetilde{p}_*(\langle y\rangle^\perp)$ but $a \notin \widetilde{p}_*(\langle x\rangle^\perp)$. □

Remark 3.4 As we will explain in Remark 5.4, surjectivity of $\widetilde{p}$ on dual atoms also follows from [16].

The main result of C.A. Akemann, R. Giles and H. Kummer is that a unital C*-algebra A is determined by the knowledge of homomorphisms $\widetilde{p} : R(A) \to L(H)$, $0 \neq p$ an irreducible representation. But, in general, we cannot recognize these homomorphisms $\widetilde{p}$ from the quantale $R(A)$ only.

Proposition 3.5 [21] *Let A be a C*-algebra. Then the quantale $R(A)$ is determined by the Jordan algebra A_h of hermitian elements of A.*

If A is commutative then any non-zero irreducible representation $p : A \to B(H)$ is one-dimensional, i.e., $H \cong \mathbb{C}$. Hence $L(H) \cong \mathbf{2}$ and the $\widetilde{p}$ are precisely the points of a frame $R(A)$. But a much larger class of C*-algebras A can be reconstructed from $R(A)$.

Theorem 3.6 [4] *Any postliminary unital C*-algebra A is determined by its quantale $R(A)$.*

A consequence of [1], [2] and [7] is the following result

Proposition 3.7 [4] *Let A be a C*-algebra and J_1, J_2 be two distinct elements of $R(A)$. Then there exists a non-zero irreducible representation $p : A \to B(H)$ such that $\widetilde{p}(J_1) \neq \widetilde{p}(J_2)$.*

Remark 3.8 [21] One can put more structure on $R(A)$. Let $J_1 \circ J_2$ be the closure in the norm topology of $J_1^* J_2$. The new multiplication $\circ$ on $R(A)$ is commutative but not associative and $J_1 \circ J_2$ is always two-sided. The disjointness relation

$$J_1 \perp J_2 \text{ iff } J_1 \circ J_2 = 0$$

makes it possible to introduce topological properties, like regularity, for $R(A)$. One can then prove that $R(A)$ is compact regular for any unital C*-algebra A. But, still, $\circ$ is determined by the Jordan part A_h only.

4. Involutive quantales

An *involution* (see [15]) on a quantale Q is a unary operation * such that

$$\begin{aligned} a^{**} &= a, \\ (a \cdot b)^* &= b^* \cdot a^*, \\ (\textstyle\bigvee_{i \in I} a_i)^* &= \textstyle\bigvee_{i \in I} a_i^* \end{aligned}$$

for all $a, a_i, b \in Q$, $i \in I$. A quantale with involution is said to be *involutive.* An *involutive homomorphism* is a quantale homomorphism of involutive quantales which also preserves the involution. An involutive quantale Q is called *Gelfand* if $a \cdot a^* \cdot a = a$ for all $a \in R(Q)$. An element $a \in Q$ is said to be *hermitian* if $a = a^*$. We put $H(Q) = \{a \in Q | a = a^*\}$. Since the involution is a complete lattice isomorphism, $H(Q)$ is closed under arbitrary joins and meets. Moreover, the involution establishes a sup-lattice isomorphism between $R(Q)$ and $L(Q)$.

For an involutive quantale Q and $a \in R(Q)$ we put

$$a^{\perp} = \bigvee \{b \in Q | b^* \cdot a = 0\}.$$

Lemma 4.1 *An involutive quantale is discrete iff $a^{\perp\perp} = a$ for any $a \in R(Q)$.*

Proof. Let Q be an involutive quantale and $a \in R(Q)$. Then

$$(a \to_l 0)^* = (\bigvee\{b \in L(Q) | b \cdot a = 0\})^* = \bigvee\{b^* \in L(Q) | b \cdot a = 0\} = a^\perp$$

and

$$a^* \to_r 0 = (a \to_l 0)^* = a^\perp.$$

If Q is discrete then

$$a^{\perp\perp} = (a \to_l 0)^{*\perp} = (a \to_l 0) \to_r 0 = a$$

for any $a \in R(Q)$. Conversely, let $a^{\perp\perp} = a$ for any $a \in R(Q)$. Then

$$(a \to_l 0) \to_r 0 = a^{\perp *} \to_r 0 = a^{\perp\perp} = a$$

$$(a^* \to_r 0) \to_l 0 = a^\perp \to_l 0 = (a^{\perp\perp})^* = a^*$$

for any $a \in R(Q)$. □

An involutive quantale Q is said to be an *I-factor* if $I(Q) \cap H(Q) = \{0, 1\}$ i.e. if 0 and 1 are the only hermitian two-sided elements of Q.

Examples 4.2 (see [15]): (i) Any frame L is a unital, involutive, commutative Gelfand quantale with multiplication given by the meet $\wedge$, with unit given by the top element 1 and $^* = \mathrm{id}_L$.
(ii) The set $\mathcal{R}el(X)$ of relations on a set X, ordered by inclusion, is a unital involutive quantale with multiplication given by the composition of relations

$$T_1 \cdot T_2 = \{(x, z) \in X \times X | \exists y \in X (x, y) \in T_1 \text{ and } (y, z) \in T_2\},$$

with the equality relation providing the unit and the involution defined as the inverse relation

$$T^* = \{(x, y) \in X \times X | (y, x) \in T\}.$$

The right-sided elements of $\mathcal{R}el(X)$ are those relations R_A, for $A \subseteq X$, given by

$$R_A = (X - A) \times X.$$

Similarly, the left-sided elements of $\mathcal{R}el(X)$ are those relations L_B, for $B \subseteq X$, given by

$$L_B = X \times B.$$

Note that $R_A^* = L_{X-A}$. Moreover, there is a well-known bijective correspondence between the relations on X and sup-preserving maps from the power set $\mathbf{2}^X$ to itself. Namely, for all $T \in \mathcal{R}el(X)$ and all $\varphi : \mathbf{2}^X \to \mathbf{2}^X$, $T \mapsto \varphi_T, \varphi \mapsto R_\varphi$ such that

$$\varphi_T(A) = \{x \in X | \exists y \in A \ (x, y) \in T\}, \quad R_\varphi = \{(x, y) \in X \times X | x \in \varphi(\{y\})\}.$$

For any sup-lattice S, we will denote by $\mathcal{Q}(S)$ the sup-lattice of all sup-lattice endomorphisms $f : S \to S$ (with the pointwise ordering $f \le g$ iff $f(x) \le g(x)$ for all $x \in S$). Then $\mathcal{Q}(S)$ is a unital quantale; multiplication is given by composition and the unit is given by id_S. Moreover, $\mathcal{Q}(S)$ is an involutive quantale provided that S is *a sup-lattice with a duality*, i.e., with a unary operation $'$ such that

$$\begin{array}{rcl} s'' & = & s \\ (\bigvee_{i\in I} s_i)' & = & \bigwedge_{i\in I} s_i' \end{array}$$

for all $s, s_i \in S$, $i \in I$. In fact, the involution * on $\mathcal{Q}(S)$ is then given by

$$\varphi^*(s) = \Big(\bigvee_{\varphi(t)\le s'} t \Big)'$$

(see [19]). In the special case when S is an orthocomplemented sup-lattice and $s' = s^\perp$ for $s \in S$ (i.e., duality is given by orthocomplements), the calculations were done in C.M. Mulvey and J.W. Pelletier [15] (the general case is completely analogous). Following Example 4.2 (ii), $\mathcal{R}el(X) \cong \mathcal{Q}(\mathbf{2}^X)$ for any set X.

Right-sided elements of $\mathcal{Q}(S)$ are precisely mappings

$$\rho_s(t) = \begin{cases} s & t \neq 0 \\ 0 & \text{otherwise} \end{cases}$$

where $s \in S$. Similarly, left-sided elements are precisely mappings

$$\lambda_s(t) = \begin{cases} 0 & t \le s \\ 1 & \text{otherwise.} \end{cases}$$

Thus $R(\mathcal{Q}(S)) \cong L(\mathcal{Q}(S))^{op}$. If S is equipped with a duality then $R(\mathcal{Q}(S)) \cong L(\mathcal{Q}(S))$. Moreover, $\rho_s^* = \lambda_{s'}$ and $\lambda_s^* = \rho_{s'}$. Since

$$(\lambda_u \circ \rho_v)(t) = \begin{cases} 0 & \text{if } t = 0 \text{ or } v \le u \\ 1 & \text{otherwise,} \end{cases}$$

we have that $\rho_s^\perp = \rho_{s'}$.

Proposition 4.3 [15, 19] *Let S be a sup-lattice with a duality. Then $\mathcal{Q}(S)$ is a right-idempotent, discrete involutive quantale. Moreover, S is orthocomplemented iff $\mathcal{Q}(S)$ is a Gelfand quantale.*

Proof. Everything, except that S is orthocomplemented whenever $\mathcal{Q}(S)$ is Gelfand, is in [15] and [19].

Assume that $\mathcal{Q}(S)$ is Gelfand and $a \in S$. Following [15], we have that

$$\rho_a \wedge \rho_a^\perp = 0.$$

Since $\rho_a^\perp = \rho_{a'}$, we have $\rho_a \wedge \rho_{a'} = 0$, i.e., $a \wedge a' = 0$. Moreover,

$$a \vee a' = (a \wedge a')' = 0' = 1$$

and we get that a' is the orthocomplement of a. □

Moreover, the joins of a right-sided and a left-sided element are of the form

$$(\rho_u \vee \lambda_v)(t) = \begin{cases} 0 & t = 0 \\ u & 0 \neq t \leq v \\ 1 & \text{otherwise.} \end{cases}$$

Lemma 4.4 [13] *Let S be a sup-lattice. Then any element of $\mathcal{Q}(S)$ is a meet of elements of the form $\rho_s \vee \lambda_t$, $s, t \in S$.*

Remark 4.5 The sup-lattice $\mathcal{Q}(S)$ can be described by means of a tensor product of sup-lattices

$$\mathcal{Q}(S) \cong (S \otimes S^{op})^{op}$$

(see [10]). This isomorphism is given by

$$f \mapsto \bigvee_{f(t) \leq s} t \otimes s.$$

Elements $\rho_s \vee \lambda_t$ correspond to $t \otimes s$, which explains Lemma 4.4. If S is a sup-lattice with a duality $'$ then the involution in $\mathcal{Q}(S)$ is induced by

$$(t \otimes s)^* = s' \otimes t'.$$

Remark 4.6 For any quantale Q, we get a map

$$\mu : Q \to \mathcal{Q}(R(Q))$$

given by $\mu(a)(c) = a \cdot c$, which clearly preserves multiplication and arbitrary joins. Moreover, if Q is unital then μ is unital as well. If Q is involutive and $R(Q)$ is a discrete quantale then μ is an involutive quantale homomorphism (see [16], [19]). Moreover, μ is strong iff Q is a factor.

5. Mulvey-Pelletier spectrum

Let A be a C*-algebra and denote by $M(A)$ its lattice of closed linear subspaces. Define a multiplication on $M(A)$ by taking for $X \cdot Y$ the closed subspace generated by XY. Then $M(A)$ is an involutive quantale. $R(A)$ is the quantale of right-sided elements of $M(A)$. Moreover, the multiplication $\circ$ from 3.8 is calculated as

$$J_1 \circ J_2 = J_1^* \cdot J_2$$

in $M(A)$. For a homomorphism $f : A \to B$ of C*-algebras, put

$$M(f(X)) = \overline{f(X)}$$

for any $X \in M(A)$. It is easy to see that $M : \mathcal{C}^* \to \mathcal{IQ}$ is a functor. Here $\mathcal{C}^*$ denotes the category of C*-algebras and $\mathcal{IQ}$ the category of involutive quantales.

Let $p : A \to B(H)$ be a non-zero irreducible representation of a C*-algebra A. For any $X \in M(A)$, put

$$\widehat{p}(X)(V) = \overline{\{\sum_{i=1}^{n} p(x_i)(v_i) | x_i \in X, v_i \in V, n \in \mathbb{N}\}}.$$

We then get a mapping $\widehat{p} : M(A) \to \mathcal{Q}(L(H))$.

Lemma 5.1 [16] *Let A be a C*-algebra. Then $\widehat{p} : M(A) \to \mathcal{Q}(L(H))$ is a strong involutive homomorphism of involutive quantales.*

Proof. It is easy to see that $\widehat{p}$ is a homomorphism of quantales. The fact that $\widehat{p}$ is strong follows from the transitivity theorem. Indeed, having $0 \neq X \in L(H)$ and $u \in H$, we can find $0 \neq x \in X$ and $a \in A$ such that $p(a)(x) = u$. Hence $\widehat{p}(A) = 1_{\mathcal{Q}(L(H))}$. It remains to prove that $\widehat{p}$ preserves involution, which we will not recall here. □

Remark 5.2 The restriction of $\widehat{p} : M(A) \to \mathcal{Q}(L(H))$ to right-sided elements is equal to $\widetilde{p} : R(A) \to L(H)$ from section 3. In fact, since $\widehat{p}$ is strong, $\widehat{p}(J) \in R(\mathcal{Q}(L(H))) \cong L(H)$ for any $J \in R(A)$. Hence $\widehat{p}(J) = \rho_X$ for some $X \in L(H)$. This is equivalent to the fact that $\widehat{p}(J)(H) = X$, which exactly means that $cl(p(J))$ is generated by the projection on X. Hence $\widetilde{p}(J) = \widehat{p}(J)$.

Theorem 5.3 [16]. *Let A be a C*-algebra, S be a non-empty atomic orthocomplemented sup-lattice and $f : R(A) \to S$ be a strong homomorphism of quantales which is surjective on dual atoms. Then $f \cong \widetilde{p}$ for a non-zero irreducible representation $p : A \to B(H)$ iff f can be extended to an involutive quantale homomorphism $M(A) \to \mathcal{Q}(S)$.*

Remark 5.4 (1) By $f \cong \widetilde{p}$ we mean that there is a sup-lattice isomorphism $h : S \to L(H)$ such that $h \circ f = \widetilde{p}$.
(2) In the Theorem stated above, we have reformulated the main result of [16] in order to stress the role of the quantales $R(A)$. Mulvey and Pelletier [16] use pure states instead of dual atoms x in right-sided elements. They are related by

$$s = x \vee x^*.$$

The necessity in our formulation also follows from Proposition 3.3.

A consequence of Theorem 5.3 is that we can reconstruct a C*-algebra A from its involutive quantale $M(A)$.

6. Simple and spatial quantales

We have seen in section 2 that points of a frame are frame homomorphisms $L \to \mathbf{2}$. The frame $\mathbf{2}$ can be characterized among frames as being simple, that is, as having exactly two quotients **2** and **1**. This is the usual algebraic concept of simplicity. Geometrically, in the dual category of locales, it means that **2** has exactly two subspaces, the zero subspace and the space itself, which is the geometric meaning of a point. It is natural to apply the same approach to (involutive) quantales.

A non-trivial (involutive) quantale Q is said to be (*I-simple*) *simple* if any surjective homomorphism of (involutive) quantales is either an isomorphism or a constant homomorphism. Evidently, any simple involutive quantale is I-simple.

A quantale Q will be called *faithful* if, whenever

$$x \cdot a = y \cdot a \text{ and } b \cdot x = b \cdot y$$

for all $a \in R(Q)$ and $b \in L(Q)$, then

$$x = y$$

for all $x, y \in Q$.

We say that the multiplication of a quantale Q is *trivial* if $x \cdot y = 0$ for each $x, y \in Q$; equivalently $1 \cdot 1 = 0$. It is evident that the only simple quantale with a trivial multiplication is the quantale $0_2 = \{0, 1\}$. Analogously, the only I-simple quantale with a trivial multiplication is the quantale 0_2 with the identity involution.

Theorem 6.1 (i) *An (involutive) quantale with a non-trivial multiplication is (I-)simple iff it is a faithful discrete (I-)factor.*

(ii) *Any (I-)simple (involutive) quantale with a non-trivial multiplication is right-idempotent.*

Proof. (1) A quantale with a non-trivial multiplication is simple iff it is a faithful discrete factor (see [17]).
(2) Any faithful discrete I-factor is I-simple and, conversely, any right idempotent I-simple involutive quantale Q is a faithful discrete I-factor (see [19]).

It immediately follows from (1) and (2) that (ii) $\Longrightarrow$ (i).

Assume first that Q is a simple quantale with a non-trivial multiplication and $0 \neq b \in R(Q)$. Then, following (1) and [17] 3.3 and 2.2, we have that $b \cdot 1 = b$ and $1 \cdot b = 1$. Hence

$$b = b \cdot 1 = b \cdot 1 \cdot b = b \cdot b,$$

which means that Q is right-idempotent.

Now, let Q be an I-simple involutive quantale with a non-trivial multiplication which is not simple. Then Q is an I-factor (see [19]) and, by [17] 2.2, $1 \cdot x \cdot 1 = 1$ for all $0 \neq x \in H(Q)$. Let $a \in I(Q) - \{0, 1\}$. Then a must be non-hermitian, $a^* \in I(Q)$ and $a \vee a^*, a \cdot a^*, a^* \cdot a, a^* \wedge a \in I(Q) \cap H(Q)$. Hence, $a \vee a^* = 1$ and $a \cdot a^* = a^* \cdot a = a^* \wedge a = 0$.

Let us define a relation $\sigma_a \subseteq Q \times Q$ by

$$x \sigma_a y \text{ iff } c \cdot x \cdot d = c \cdot y \cdot d \text{ and } c \cdot x^* \cdot d = c \cdot y^* \cdot d$$

for all $c \in L(Q), c \leq a$ and $d \in R(Q), d \leq a$. Evidently, σ_a is an equivalence relation closed under involution and arbitrary joins. Let $x_1 \sigma_a y_1$ and $x_2 \sigma_a y_2$. Then, for all $c \in L(Q), c \leq a$ and $d \in R(Q), d \leq a$, $c \cdot x_1 \cdot x_2 \cdot d = c \cdot x_1 \cdot (x_2 \cdot d) = c \cdot y_1 \cdot (x_2 \cdot d) = (c \cdot y_1) \cdot x_2 \cdot d = (c \cdot y_1) \cdot y_2 \cdot d = c \cdot y_1 \cdot y_2 \cdot d$ and, similarly, $c \cdot x_2^* \cdot x_1^* \cdot d = c \cdot y_2^* \cdot y_1^* \cdot d$ since $c \cdot y_1, c \cdot x_2^* \in L(Q)$ and $x_2 \cdot d, y_1^* \cdot d \in R(Q)$. Therefore, σ_a is an involutive quantale congruence on Q, i.e., either $\sigma_a = Q \times Q$ or $\sigma_a = \mathrm{id}_Q$. Since $a \cdot a \cdot a \vee (a \cdot a \cdot a)^* = (a \vee a^*)^3 = 1$, we have $a \cdot a \cdot a \neq 0$ and thus $(0, a) \notin \sigma_a$. Therefore $\sigma_a = \mathrm{id}_Q$.

Let $c \in L(Q), c \leq a$ and $d \in R(Q), d \leq a$. Then

$$\begin{aligned} c \cdot a \cdot d &= c \cdot (a \vee a^*) \cdot d = c \cdot 1 \cdot d = c \cdot 1 \cdot 1 \cdot d = c \cdot (a \vee a^*) \cdot 1 \cdot d \\ &= c \cdot a \cdot 1 \cdot d \vee c \cdot a^* \cdot 1 \cdot d = c \cdot a \cdot 1 \cdot d \end{aligned}$$

and

$$c \cdot a^* \cdot d = 0 = c \cdot (a \cdot 1)^* \cdot d.$$

Therefore $(a, a \cdot 1) \in \sigma_a$, i.e., $a = a \cdot 1$. Consequently,

$$a = a \cdot 1 = a \cdot (a \vee a^*) = a \cdot a.$$

This means that $I(Q)$ is an idempotent 2-sided quantale, i.e., a frame. Since a^* is a complement of a for any $a \in I(Q) - \{0, 1\}$, $I(Q)$ is a Boolean algebra. It follows from the proof of [19] 4.4. that $I(Q)$ is a four-element Boolean algebra. We will write $I(Q) = \{0, a, a^*, 1\}$.

Now, let $0 \neq b \in R(Q)$. Consider $c \in L(Q), c \leq a$ and $d \in R(Q), d \leq a$. Since $c \cdot b \in I(Q)$ and $c \cdot b \leq a$, we have either $c \cdot b = 0$ or $c \cdot b = a$. In the first case, $c \cdot b \cdot d = c \cdot b \cdot b \cdot d$. Assume that $c \cdot b = a$. Since $a \cdot b \in I(Q)$ and $a = c \cdot b \leq a \cdot b \leq a$, we have $a \cdot b = b$. Hence,

$$c \cdot b \cdot d = a \cdot d = a \cdot b \cdot d = c \cdot b \cdot b \cdot d.$$

We have proved that $c \cdot b \cdot d = c \cdot b \cdot b \cdot d$. At the same time, since $b^* \cdot d \in I(Q)$, we have

$$c \cdot b^* \cdot d = c \cdot b^* \cdot d \cdot b^* \cdot d \leq c \cdot b^* \cdot 1 \cdot b^* \cdot d \leq c \cdot b^* \cdot b^* \cdot d \leq c \cdot 1 \cdot b^* \cdot d \leq c \cdot b^* \cdot d.$$

Hence $c \cdot b^* \cdot d = c \cdot (b \cdot b)^* \cdot d$, which yields that $b \cdot b = b$. □

Remark 6.2 (1) The above characterization was first found for right-idempotent involutive quantales in [19]. Then it was extended (and strengthened) to quantales in [17].

(2) Any (I-)simple (involutive) quantale with a non-trivial multiplication is also left-idempotent. This is evident for involutive quantales and for quantales it follows from the proof of [17] 2.3.

Corollary 6.3 [19, 17] *Let S be a sup-lattice (with a duality). Then the (involutive) quantale $\mathcal{Q}(S)$ is simple.*

Corollary 6.4 *Let Q be a simple (involutive) quantale which is not isomorphic to 0_2. Then Q is isomorphic to a strong (involutive) subquantale of $\mathcal{Q}(S)$ for some sup-lattice S (with a duality).*

Proof. Consider $\mu : Q \to \mathcal{Q}(R(Q))$ from 4.6. Then, since Q is a factor, μ is strong. Moreover, if Q is involutive and discrete we have that $R(Q)$ is equipped with a duality and μ is an involutive quantale homomorphism. The injectivity of μ follows from the faithfulness of Q. □

Remark 6.5 (1) A four-element Boolean algebra is an example of an I-simple involutive quantale which is not simple (see [19]). More generally, any involutive quantale $\mathcal{Q}(S) \times \mathcal{Q}(S^{op})$, where S is a sup-lattice, is I-simple but not simple.
(2) Any I-simple involutive quantale is isomorphic to a strong involutive subquantale of $\mathcal{Q}(S) \times \mathcal{Q}(S^{op})$ for some sup-lattice S (see [13]).

Theorem 6.6 [18] *Any (involutive) quantale has an (involutive) embedding into an (I-)simple (involutive) quantale.*

Hence, the definition of (I-)spatiality by means of enough (involutive) quantale homomorphisms into (I-)simple quantales is too weak. We use strong homomorphisms for defining points, i.e., we work in the category of (involutive) quantales and strong (involutive) homomorphisms (note that this does not change the concept of (I-)simplicity because any surjective (involutive) quantale homomorphism of (involutive) quantales is strong). This is also supported by Lemma 5.1 – any non-zero irreducible representation of a C*-algebra should yield a point of an associated quantale. Another concept of point is introduced in [16] (see Remark 6.17 at the end of our paper).

An (involutive) quantale Q is called *(I)-spatial* if it has enough points. This means that for any two distinct elements $x, y \in Q$ there exists a strong (involutive) quantale homomorphism $f : Q \to P$ into an (I)-simple quantale P such that $f(x) \neq f(y)$.

Proposition 6.7 *Any (I-)spatial (involutive) quantale satisfying $1 \cdot 1 = 1$ is both right- and left-idempotent.*

Proof follows from Theorem 6.1 and Remark 6.2.

Proposition 6.8 [13] (i) *A quantale satisfying $1 \cdot 1 = 1$ is spatial iff it has enough strong quantale homomorphisms from Q into quantales $\mathcal{Q}(S)$.*
(ii) *An involutive quantale satisfying $1 \cdot 1 = 1$ is I-spatial iff it has enough strong quantale homomorphisms from Q into quantales $\mathcal{Q}(S) \times \mathcal{Q}(S^{op})$.*

Proof follows from 6.4 and 6.5.

Remark 6.9 More generally, any (I-)spatial (involutive) quantale is isomorphic to an (involutive) strong subquantale of a product of quantales of the form $\mathcal{Q}(S)$ ($\mathcal{Q}(S) \times \mathcal{Q}(S^{op})$) and of copies of 0_2.

Proposition 6.10 [13] *Any I-spatial involutive quantale Q is spatial and, conversely, any spatial involutive quantale satisfying* $1 \cdot 1 = 1$ *is I-spatial.*

Proof. Assume first that Q is an I-spatial involutive quantale and $x, y \in Q$, $x \neq y$. Then there exists a strong involutive quantale homomorphism $f : Q \to P$ into an I-simple involutive quantale P such that $f(x) \neq f(y)$. Hence, either $P = 0_2$ or P is a strong subquantale of $\mathcal{Q}(S) \times \mathcal{Q}(S^{op})$ by Remark 6.5. In the first case, $f : Q \to P$ is a strong quantale homomorphism separating x and y. In the second case, the projections $p_1 : \mathcal{Q}(S) \times \mathcal{Q}(S^{op}) \to \mathcal{Q}(S)$ and $p_2 : \mathcal{Q}(S) \times \mathcal{Q}(S^{op}) \to \mathcal{Q}(S^{op})$ are strong quantale homomorphisms and either $p_1 \circ f$ or $p_2 \circ f$ separates x and y.

Conversely, let Q be a spatial involutive quantale satisfying $1 \cdot 1 = 1$. By Proposition 6.8 we have enough strong quantale homomorphisms from Q into quantales $\mathcal{Q}(\mathcal{S})$. Any strong quantale homomorphism $f : Q \to \mathcal{Q}(S)$ induces a strong involutive quantale homomorphism $g : Q \to \mathcal{Q}(S) \times \mathcal{Q}(S^{op})$ defined by $g(a) = (f(a), f(a^*)_*)$, for all $a \in Q$. Hence Q is an I-spatial involutive quantale.□

Example 6.11 A spatial involutive quantale which is not I-spatial:

Consider the quantale $Q = 0_2 \times 0_2$ with involution

$$(a, b)^* = (b, a).$$

Then Q is evidently spatial. Assume that there is a strong involutive quantale homomorphism $f : Q \to P$ into an I-simple involutive quantale P satisfying $f(0, 1) \neq f(1, 0)$. Since Q has a trivial multiplication, $P = 0_2$. Let, for instance, $f(0, 1) = 0$ and $f(1, 0) = 1$. Then

$$1 = f(1, 0) = f(0, 1)^* = 0^*,$$

which is a contradiction.

D. Kruml [13] has found a characterization of spatial quantales by introducing an appropriate concept of a prime element in quantales.

Definition 6.12 *An element $p \neq 1$ of a quantale Q is called* prime *if*

$$r \cdot l \le p \quad \Longrightarrow \quad r \le p \text{ or } l \le p$$

for all $r \in R(Q), l \in L(Q)$.

It is clear that the prime elements of a frame coincide with those from section 2.

Theorem 6.13 [13] *A quantale Q satisfying* $1 \cdot 1 = 1$ *is spatial iff each of its elements is a meet of primes.*

Remark 6.14 (i) If Q is an idempotent right-sided quantale then $1 \neq p \in Q$ is a prime iff

$$x \cdot y \le p \quad \text{implies} \quad x \le p \text{ or } y \le p$$

for all $x, y \in Q$. In fact, whenever p is a prime and $x \cdot y \leq p$ then

$$x \cdot (1 \cdot y) = (x \cdot 1) \cdot y = x \cdot y \leq p$$

and therefore

$$x \leq p \text{ or } y = y \cdot y \leq 1 \cdot y \leq p.$$

It has been shown in [22] that, in an idempotent right-sided quantale Q, any element is an intersection of primes iff Q has enough strong homomorphisms into idempotent right-sided factor quantales.
(ii) The elements $\rho_s \vee \lambda_t$ from 4.4 are primes in $\mathcal{Q}(S)$.

Proposition 6.15 *Let A be a C*-algebra and J_1, J_2 be two distinct right-sided elements of $M(A)$. Then there is a strong involutive quantale homomorphism $f : M(A) \to \mathcal{Q}(L(H))$, where H is a Hilbert space, such that $f(J_1) \neq f(J_2)$.*

The proof follows from Proposition 3.7 and Remark 5.2. Hence $M(A)$ is "spatial on right-sided elements". But $M(A)$ is not necessarily spatial, as the following example shows.

Example 6.16 (D. Kruml) Let $A = B(\mathbb{C}^2)$ be the C*-algebra of complex 2×2-matrices. As a linear space, $A \cong \mathbb{C}^4$. Let

$$X = \left\{ \left(\begin{array}{cc} z & x \\ y & x+y \end{array} \right) \;\middle|\; x, y, z \in \mathbb{C} \right\}.$$

Then X is a dual atom in $M(A)$. We will show that X is not prime, which implies that it cannot be an intersection of primes. Consider

$$X_1 = \left\{ \left(\begin{array}{cc} a & b \\ 0 & 0 \end{array} \right) \;\middle|\; a, b \in \mathbb{C} \right\} \quad \text{and} \quad X_2 = \left\{ \left(\begin{array}{cc} c & 0 \\ d & 0 \end{array} \right) \;\middle|\; c, d \in \mathbb{C} \right\}.$$

Then $X_1 \in R(A)$, $X_2 \in L(A)$, $X_1 \not\subseteq X$, $X_2 \not\subseteq X$ and $X_1 \cdot X_2 \subseteq X$ because

$$\left(\begin{array}{cc} a & b \\ 0 & 0 \end{array} \right) \left(\begin{array}{cc} c & 0 \\ d & 0 \end{array} \right) = \left(\begin{array}{cc} ac+bd & 0 \\ 0 & 0 \end{array} \right) \in X.$$

Remark 6.17 Another definition of points of a (Gelfand) quantale is introduced in [16]. They are defined as involutive quantale homomorphisms $f : Q \to \mathcal{Q}(S)$ where S is an atomic orthocomplemented lattice and, for each atom $s \in S$, we have

$$\{f(q)(s) | q \in Q\} = S.$$

Evidently, such homomorphisms are strong. If A is a C*-algebra and $p : A \to B(H)$ a non-zero irreducible representation then $\widehat{p} : M(A) \to \mathcal{Q}(L(H))$ is a point in this sense (see [16]).

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