CONTRIBUTIONS TO GENERAL ALGEBRA 15
Proceedings of the Klagenfurt Conference 2003 (AAA 66)
Verlag Johannes Heyn, Klagenfurt 2004

SCOTT-OPEN DISTRIBUTIVE FILTERS AND PRIME ELEMENTS OF QUANTALES

JAN PASEKA

Abstract. In 2002, D. Kruml introduced the notion of a distributive quantale and proved, assuming the Axiom of Choice, that any algebraic distributive quantale is spatial. In this paper we present a generalization of this result, namely that any continuous distributive quantale is spatial. Basics of the theory of radical nuclei are developed.

The theory of quantales [14] incorporates the theory of frames [9], the theory of one-sided (two-sided) ring ideals, the theory of closed subspaces (ideals) of a C*-algebra, etc. As discovered by Mulvey and Pelletier in [15], one may canonically associate to a C*-algebra A an involutive quantale $Q(A)$ such that $Q(A) \cong Q(B)$ as involutive quantales implies $A \cong B$ as C*-algebras.

An important notion introduced in the context of irreducible representations of C*-algebras is the notion of *prime element* of a quantale in [12]. In 2002, D. Kruml [13] introduced the notion of a distributive quantale and proved that any algebraic distributive quantale is spatial.

Motivated by this result, by the results of the present author from [16] and [17], by the paper [18] and by the work [4] of Banaschewski and Erné in the context of two-sided quantales, we shall generalize the corresponding ideas concerning prime elements and semiprime ideals for arbitrary quantales. In fact, it is precisely the lack of two-sidedness that causes some difficulties.

After preliminaries in Section 1, we develop in Section 2 basics of the theory of radical nuclei and we give a characterization of prime elements. In Section 3 we formulate a general principle — *General Separation Lemma for Quantales* (**GSLQ**), that any element outside some Scott-open distributive filter S in a non-trivial quantale is below a prime element outside S. Note that **GSLQ** was already known for two-sided cm-lattices, and that distributive filters have

2000 *Mathematics Subject Classification.* Primary 06B23, 06B99, 06F07, 06F99; secondary 54A05.

Key words and phrases. Quantale, cm-lattice, prime element, distributive filter, Scott-open filter.

Financial Support of the Grant Agency of the Czech Republic under the grant No. 201/02/0148 is gratefully acknowledged.

been investigated for quite some time under the name of *distributors* (see [7] for example and the references therein). We prove that this principle follows from the Axiom of Choice and we apply this principle and the Principle of Countable Dependent Choice to prove that any continuous distributive quantale is spatial. Finally, we look at the role of conjunctivity with respect to spatiality of a quantale.

All unexplained facts concerning frames and quantales can be found in [9], [19] and [4].

1. Preliminaries

1.1. Definition. By a *sup-lattice* we mean a complete lattice, a *sup-lattice morphism* is a mapping preserving arbitrary joins. The top element of a sup-lattice is denoted by 1, the bottom element by 0.

1.2. Definition. By a *cm-lattice* [11] we mean a sup-lattice Q equipped with a multiplication which preserves order:

$$a \leq b \text{ implies } a \cdot c \leq b \cdot c \text{ and } c \cdot a \leq c \cdot b$$

for all $a, b, c \in Q$. We say that a cm-lattice Q is *m-distributive* if for all $a, b, c \in Q$,

$$a \cdot (b \vee c) = a \cdot b \vee a \cdot c, \qquad (b \vee c) \cdot a = b \cdot a \vee c \cdot a.$$

By a *prequantale* we mean a cm-lattice Q such that multiplication distributes over arbitrary joins:

$$a \cdot \left(\bigvee a_i\right) = \bigvee (a \cdot a_i), \qquad \left(\bigvee a_i\right) \cdot a = \bigvee (a_i \cdot a)$$

for all $a, a_i \in Q$. Evidently, any prequantale is an m-distributive cm-lattice. A *quantale* is a prequantale with associative multiplication.

An element $a \in Q$ is called *right-sided* if $a \cdot 1 \leq a$, and *left-sided* if $1 \cdot a \leq a$. We write $a \in \mathcal{R}(Q)$, resp. $a \in \mathcal{L}(Q)$. Elements of $\mathcal{T}(Q) = \mathcal{R}(Q) \cap \mathcal{L}(Q)$ are called *two-sided*. A cm-lattice Q is called *right-sided* (*left-sided*, *two-sided*) if all its elements are right-sided (left-sided, two-sided). A cm-lattice Q is called *right-semiunital* (*right-semiidempotent*) if $r \cdot 1 = r$ for all $r \in \mathcal{R}(Q)$ ($r \cdot r = r$ for all $r \in \mathcal{R}(Q)$). Similarly, we may define the notion of a left-semiunital (left-semiidempotent) cm-lattice. A *semiunital* (*semiidempotent*) cm-lattice is a cm-lattice that is both right- and left-semiunital (right- and left-semiidempotent). Note that an associative m-distributive cm-lattice Q is semiunital if and only if $a \leq a \cdot 1 \wedge 1 \cdot a$ for all $a \in Q$ (see [12, Lemma 1.1]).

A *morphism of cm-lattices* is a sup-lattice morphism preserving multiplication.

A closure operator $k : Q \to Q$ on a cm-lattice Q such that $k(a) \cdot k(b) \leq k(a \cdot b)$ is called a *multiplicative nucleus*. Recall that closure operators on Q are naturally ordered by the pointwise order.

2. Prime and distributive elements

Let us recall from [12] and [13] the following definitions.

2.1. Definition. An element $p \neq 1$ of a cm-lattice Q is said to be *prime* if

$$r \cdot l \leq p \Rightarrow r \leq p \text{ or } l \leq p$$

for all $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$. Let $\mathcal{P}(Q)$ denote the set of all primes of Q. Q is called *spatial* if each of its elements is a meet of primes. Any spatial (associative) cm-lattice is semiunital (semiidempotent). A *point* of a cm-lattice Q is a non-zero sup-preserving map $f : Q \to \mathbf{2}$ such that $f(r \cdot l) = f(r) \wedge f(l)$ for all $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$; here $\mathbf{2} = \{0, 1\}$ is the 2-element Boolean algebra.

Put

$$\mathbf{p}(x) = \bigwedge \{p \in \mathcal{P}(Q) : x \leq p\}$$

for any $x \in Q$. Then $\mathbf{p} : Q \to Q$ is a closure operator. Let $Q_{\mathbf{p}}$ be the complete lattice of all $\mathbf{p}$-fixed elements x, i.e. $\mathbf{p}(x) = x$. We shall speak about the *spatialization* of Q.

2.2. Definition. An element a of a cm-lattice Q is said to be *distributive* (see [13]) if

$$(r \vee a) \wedge (l \vee a) = r \cdot l \vee a$$

for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$. Note that in any cm-lattice whose least element is distributive we have $r \wedge l = r \cdot l$ for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$, in particular any two-sided quantale whose least element is distributive is a frame. Q is called *distributive* if all of its elements are distributive. Any two-sided distributive cm-lattice is a distributive complete lattice and vice versa. A non-trivial upper subset $S \subseteq Q$ is called a *distributive filter* of Q if $r \vee a$, $l \vee a \in S$, $a \in Q$, $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$ together imply $r \cdot l \vee a \in S$. Let $\Lambda(Q)$ denote the complete cm-lattice of all distributive filters of Q; here the multiplication coincides with the intersection. A cm-lattice Q is called a *wcm-lattice* [16] if $r \vee c = 1 = l \vee c$ implies $r \cdot l \vee c = 1$ for all $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$, $c \in Q$, i.e. $\{1\}$ is a distributive filter. The notion of a distributive filter was motivated also by the notion of a semiprime filter from [18].

2.3. Lemma. *Let Q be a semiunital right-sided (left-sided, two-sided) m-distributive cm-lattice. Then Q is a wcm-lattice. Moreover, any wcm-lattice has an idempotent top element.*

Proof. Let $r \vee c = 1 = l \vee c$, $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$, $c \in Q = \mathcal{R}(Q)$. Then $1 = l \vee c = 1 \cdot l \vee c = (r \vee c) \cdot l \vee c = r \cdot l \vee (c \cdot l \vee c) = r \cdot l \vee c$. The rest is evident. $\square$

Recall that by the same considerations as in [13] we have that any spatial cm-lattice is distributive and evidently any (associative m-distributive) distributive cm-lattice is a semiunital (semiidempotent) wcm-lattice. Moreover, a dual atom of a cm-lattice is prime iff it is distributive. Clearly, any two-sided dual atom m of an m-distributive cm-lattice with either an idempotent top element or $1 \cdot 1 \not\leq m$ is prime. In fact, we have

2.4. Lemma. *Let Q be a non-trivial wcm-lattice. Then any dual atom in Q is prime.*

Proof. Let $p \in Q$ be a dual atom of Q, $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$, $r \cdot l \leq p$. Assume $r \not\leq p$ and $l \not\leq p$, i.e. $r \vee p = 1 = l \vee p$. Then $p = r \cdot l \vee p = 1$, a contradiction. □

2.5. Definition. Let Q be a cm-lattice. Then Q is said to be *dually atomic* if for any $a \neq 1$, $a \in Q$ there is a dual atom $m \in Q$ such that $a \leq m$.

2.6. Proposition. *Let Q be a dually atomic cm-lattice. Then the following are equivalent:*

1. *Q is a wcm-lattice.*
2. *Every dual atom of Q is prime.*

Proof. (1) $\Longrightarrow$ (2): This follows from 2.4.
(2) $\Longrightarrow$ (1): Let $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$, $c \in Q$ such that $r \vee c = 1 = l \vee c$. Assume $r \cdot l \vee c < 1$. Then there exists a dual atom $p \in Q$ such that $r \cdot l \vee c \leq p$. Since p is prime we have that $1 = r \vee c \leq p \vee c = p$ or $1 = l \vee c \leq p \vee c = p$, a contradiction. □

Recall that assuming the Axiom of Choice we have that any compact sup-lattice is dually atomic and therefore any non-trivial compact wcm-lattice has a prime element.

Motivated by the above concepts, we shall introduce a generalization of the localic nucleus [19] and multiplicative nucleus [22].

2.7. Definition. Let Q be a cm-lattice, $k : Q \to Q$ a closure operator on Q. Then k is called a *radical nucleus* on Q if $k(r \cdot l) = k(r) \wedge k(l)$ for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$. We say that k is a *spatial nucleus* on Q if Q_k is a spatial cm-lattice.

Let us mention that $k(r \cdot l) \leq k(r) \wedge k(l)$ for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$ holds for all monotone maps on a cm-lattice Q.

Note that for any such k,

$$Q_k = \{a \in Q : k(a) = a\}$$

is a cm-lattice equipped with a binary operation $\bullet$ such that

$$\bigvee_{Q_k} x_i = k(\bigvee x_i), \quad \bigwedge_{Q_k} x_i = k(\bigwedge x_i) = \bigwedge x_i \quad \text{and} \quad x \bullet y = k(x \cdot y).$$

Let us recall that any spatial multiplicative nucleus k on a cm-lattice Q is a radical nucleus. Namely, for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$, we have $k(r) \bullet k(l) \leq k(r \cdot l)$, $k(r) \in \mathcal{R}(Q_k)$, $k(l) \in \mathcal{L}(Q_k)$. Let $p \in \mathcal{P}(Q_k)$, $p \geq k(r) \bullet k(l)$. Then $p \geq k(r)$ or $p \geq k(l)$. In particular $p \geq k(r) \wedge k(l)$, i.e. $k(r \cdot l) \geq k(r) \wedge k(l)$.

If Q is two-sided, the multiplication in Q_k coincides with the meet operation, in particular for any radical nucleus k on a two-sided quantale Q, Q_k is a frame. This easily follows from the inequalities

$$k(x) \wedge k(y) = k(x \cdot y) \leq k(k(x) \cdot k(y)) \leq k(k(x)) \wedge k(k(y)) = k(x) \wedge k(y)$$

for all $x, y \in Q$.

Moreover, if $r_1, r_2 \in \mathcal{R}(Q)$ and $l_1,\ l_2 \in \mathcal{L}(Q)$ such that $k(r_1) = k(r_2)$ and $k(l_1) = k(l_2)$ we have that $k(r_1 \cdot l_1) = k(r_2 \cdot l_2)$. Evidently, a closure operator $k : Q \to Q$ on a distributive cm-lattice Q is a radical nucleus iff

$$k(r \wedge l) = k(r) \wedge k(l) \qquad (*)$$

for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$.

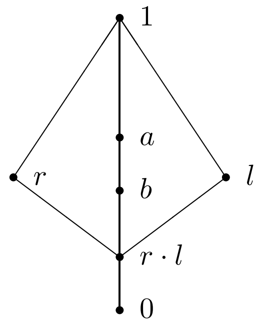
If j is a radical nucleus on Q satisfying $(*)$ and k is an arbitrary closure operator on Q satisfying $(*)$ with $j \leq k$, then k is also radical since

$$k(r) \wedge k(l) = k(r \wedge l) \leq k(j(r \wedge l)) = k(j(r \cdot l)) \leq k(k(r \cdot l)) = k(r \cdot l) \leq k(r) \wedge k(l)$$

for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$.

2.8. Example. There is a non-spatial semiidempotent quantale and a nontrivial radical nucleus on it such that its range is a non-distributive non-spatial semiidempotent quantale.

Proof. Let $Q = \{0, 1, r, l, r \cdot l, a, b\}$ be a quantale with the order and the multiplication defined by the following table:



$\cdot$	0	$r \cdot l,\ l$	$a, b, r, 1$
0	0	0	0
$r \cdot l,\ r$	0	$r \cdot l$	r
$l, a, b, 1$	0	l	1

Then Q is a non-spatial semiidempotent quantale, $\mathcal{P}(Q) = \{r, l, 0\}$, $\mathcal{R}(Q) = \{1, r, 0\}$, $\mathcal{L}(Q) = \{1, l, 0\}$. The map $\kappa : Q \to Q$ given by $\kappa(x) = x$ for all $x \in Q - \{b\}$, $\kappa(b) = a$ is evidently a radical nucleus on Q, $Q_\kappa = Q - \{b\}$ is a non-spatial semiidempotent subquantale of Q, $\mathcal{P}(Q_\kappa) = \{r, l, 0\}$, $\mathcal{R}(Q_\kappa) = \{1, r, 0\}$, $\mathcal{L}(Q_\kappa) = \{1, l, 0\}$. Since $r \vee a = l \vee a = 1$, $r \cdot l \vee a = a$, we have that $(r \vee a) \wedge (l \vee a) = 1 \neq r \cdot l \vee a$. In particular, Q and Q_κ are non-distributive quantales. $\square$

2.9. Lemma. *Let Q be a cm-lattice, $k : Q \to Q$ a radical nucleus on Q such that $k(\mathcal{R}(Q)) \subseteq \mathcal{R}(Q)$, $k(\mathcal{L}(Q)) \subseteq \mathcal{L}(Q)$ and $p \in Q_k$ be a fixed element of k. Then p is a prime element of Q_k if and only if it is a prime element of Q.*

Proof. Let p be a prime element of Q_k, $r \cdot l \leq p$, $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$. Then $k(r) \bullet k(l) = k(k(r) \cdot k(l)) = k(r) \wedge k(l) = k(r \cdot l) \leq k(p) = p$, i.e. $r \leq k(r) \leq p$ or $l \leq k(l) \leq p$. Conversely, let p be a prime element of Q, $r \bullet l \leq p$, $r = k(r) \in \mathcal{R}(Q_k)$, $l = k(l) \in \mathcal{L}(Q_k)$. Then $r \cdot 1 \leq k(r \cdot 1) = r \bullet 1 \leq r$, i.e. $r \in \mathcal{R}(Q)$ and similarly $l \in \mathcal{L}(Q)$. We have $r \cdot l \leq r \bullet l \leq p$, i.e. $r \leq p$ or $l \leq p$. $\square$

A connection between prime elements, completely prime distributive filters and radical nuclei is furnished by the following observation:

2.10. Proposition. *Let Q be a cm-lattice. Then there is a one-to-one correspondence between the following sets:*

i. *prime elements of Q,*
ii. *radical nuclei k on Q such that $Q_k \cong \mathbf{2}$,*
iii. *points of Q,*
iv. *completely prime distributive filters of Q.*

More precisely, assigning to each prime p the map k_p as follows

$$k_p(x) = \begin{cases} p & \text{if } x \leq p, \\ 1 & \text{otherwise} \end{cases}$$

yields a one-to-one correspondence between $\mathcal{P}(Q)$ and (ii)*, assigning to each radical nucleus k on Q such that $Q_k \cong \mathbf{2}$ a map $f_k : Q \to \mathbf{2}$ given by $f_k(x) = 1$ iff $k(x) = 1$ yields a one-to-one correspondence between* (ii) *and* (iii)*, assigning to each point f of Q the upper set $S_f = \{x \in Q : f(x) = 1\}$ yields a one-to-one correspondence between* (iii) *and* (iv) *and assigning to each completely prime distributive filter S of Q an element $p_S \in Q$ by $p_S = \bigvee\{x \in Q : x \notin S\} \notin S$ yields a one-to-one correspondence between* (iv) *and* (i)*.*

Proof. Let $p \in \mathcal{P}(Q)$. Then k_p is a closure operator on Q such that $Q_{k_p} = \{p, 1\} \cong \mathbf{2}$. Since p is prime we have that $k_p(r \cdot l) \leq p$ iff $r \cdot l \leq p$ iff ($r \leq p$ or $l \leq p$) iff ($k_p(r) \leq p$ or $k_p(l) \leq p$) iff $k_p(r) \wedge k_p(l) \leq p$ for all $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$.

The one-to-one correspondence between (ii) and (iii) is evident.

Let $f : Q \to \{0, 1\}$ be a point of Q. Then S_f is a completely prime upper set. Assume $r \vee a$, $l \vee a \in S_f$, $a \in Q$, $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$. If $f(a) = 1$ we have $f(r \cdot l \vee a) = 1$, i.e. $r \cdot l \vee a \in S_f$. Assume $f(a) = 0$. Then $f(l) = f(r) = l$. Therefore $f(r \cdot l) = 1$ and we also have $f(r \cdot l \vee a) = 1$.

Let S be a completely prime distributive filter. Then $S = Q - {\downarrow}(p_S)$. Let us show that p_S is a prime element. Assume $r \cdot l \leq p_S$, $r \not\leq p_S$, $l \not\leq p_S$, $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$. This implies that $r \vee 0$, $l \vee 0 \in S$, i.e. $r \cdot l \in S$, a contradiction. $\square$

We turn now to some properties of the spatialization.

2.11. Proposition. *Let Q be a cm-lattice. Then the spatialization $\mathbf{p} : Q \to Q_{\mathbf{p}}$ is a radical nucleus on Q. Moreover, if $\mathbf{p}(\mathcal{R}(Q)) \subseteq \mathcal{R}(Q)$ and $\mathbf{p}(\mathcal{L}(Q)) \subseteq \mathcal{L}(Q)$ then $\mathbf{p} : Q \to Q_{\mathbf{p}}$ is the smallest spatial radical nucleus k on Q such that $k(\mathcal{R}(Q)) \subseteq \mathcal{R}(Q)$, $k(\mathcal{L}(Q)) \subseteq \mathcal{L}(Q)$.*

Proof. Let us prove that $\mathbf{p}$ is a radical nucleus. Let $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$. Let $p \in \mathcal{P}(Q)$, $p \geq r \cdot l$. Then $p \geq r$ or $p \geq l$. Therefore $p \geq \mathbf{p}(r)$ or $p \geq \mathbf{p}(l)$. In particular $p \geq \mathbf{p}(r) \wedge \mathbf{p}(l)$, i.e. $\mathbf{p}(r \cdot l) \geq \mathbf{p}(r) \wedge \mathbf{p}(l)$.

Let $\mathbf{p}(\mathcal{R}(Q)) \subseteq \mathcal{R}(Q)$ and $\mathbf{p}(\mathcal{L}(Q)) \subseteq \mathcal{L}(Q)$. Evidently, $Q_{\mathbf{p}}$ is a spatial cm-lattice since $\mathcal{P}(Q_{\mathbf{p}}) = \mathcal{P}(Q)$ by Lemma 2.9, and any element of $Q_{\mathbf{p}}$ is a meet of primes in Q.

Let $k : Q \to Q_k$ be a spatial radical nucleus on Q such that $k(\mathcal{R}(Q)) \subseteq \mathcal{R}(Q)$, $k(\mathcal{L}(Q)) \subseteq \mathcal{L}(Q)$. Then, again by the Lemma 2.9, $\mathcal{P}(Q_k) \subseteq \mathcal{P}(Q) = \mathcal{P}(Q_{\mathbf{p}})$. In particular, $k(x) \geq \mathbf{p}(x)$ for all $x \in Q$, and $Q_k \subseteq Q_{\mathbf{p}}$. ☐

2.12. Corollary. *Let Q be a quantale. Then the spatialization $\mathbf{p} : Q \to Q_{\mathbf{p}}$ is the smallest spatial multiplicative nucleus on Q.*

Proof. Let us prove that $\mathbf{p}$ is a multiplicative nucleus. It is enough to check that $\mathbf{p}(x) \cdot \mathbf{p}(y) \leq \mathbf{p}(x \cdot y)$. Let $x \cdot y \leq p$, $p \in \mathcal{P}(Q)$. Then from [12, Lemma 2.8] we know that $x \leq \bigvee\{z \in Q : z \cdot y \leq p\} \in \mathcal{P}(Q)$. Therefore $\mathbf{p}(x) \leq \bigvee\{z \in Q : z \cdot y \leq p\}$, i.e. $\mathbf{p}(x) \cdot y \leq p$. Similarly, $\mathbf{p}(x) \cdot \mathbf{p}(y) \leq p$ for all $p \in \mathcal{P}(Q)$, in particular $\mathbf{p}(x) \cdot \mathbf{p}(y) \leq \mathbf{p}(x \cdot y)$. Since any multiplicative nucleus preserves both right-sided and left-sided elements of Q and any spatial muliplicative nucleus is radical we are done. ☐

3. Scott-open distributive filters and prime elements

Obviously, by the same manner as in [4] we construct, for any subset S of a cm-lattice Q, a map $s_S : Q \to Q$ defined by

$$s_S(a) = \bigvee\{x \in Q : x \vee y \in S \text{ implies } a \vee y \in S, \text{ for all } y \in Q\}.$$

We define an element of Q to be *almost S-maximal* (compare [10]) if it is both prime and s_S-fixed.

We then have the following crucial lemma. Recall first that an upper set S is *Scott-open* if any up-directed set whose join belongs to S is eventually in S. Then the singleton $\{1\}$ is a Scott-open distributive filter of a cm-lattice Q iff Q is a compact wcm-lattice.

3.1. Lemma. *Let Q be a cm-lattice, $S \subseteq Q$ a Scott-open distributive filter on Q. Then the map s_S is a radical nucleus such that $s_S(a) = 1$ iff $a \in S$, and Q_{s_S} is compact lattice, that is, a complete lattice with a compact top element. Moreover, any maximal element of $Q - S$ is almost S-maximal.*

Proof. Following [4], for each $a \in Q$, let

$$S_a = \{x \in Q : x \vee y \in S \text{ implies } a \vee y \in S, \text{ for all } y \in Q\}.$$

Evidently, $0 \in S_a$, S_a is a lower set closed under finite and directed joins, i.e. $s_S(a) \in S_a$. This gives us that s_S is a closure operator on Q. The compactness of Q_{s_S} and $s_S(a) = 1$ iff $a \in S$ follow from [4].

Let us check that s_S is a radical nucleus. Let $r \in \mathcal{R}(Q)$ and $l \in \mathcal{L}(Q)$. Since s_S is monotone we have $s_S(r \cdot l) \leq s_S(r) \wedge s_S(l)$. Conversely, consider any $x \leq s_S(r) \wedge s_S(l)$. Then $x \vee y \in S$ implies $s_S(r) \vee y \in S$ and in particular $r \vee y \in S$. Similarly $l \vee y \in S$. Hence, applying distributivity of S, we have $r \cdot l \vee y \in S$. Therefore $s_S(r) \wedge s_S(l) \in S_{r \cdot l}$ and $S_r \cap S_l = S_{r \cdot l}$. It follows that $s_S(r) \wedge s_S(l) = s_S(r \cdot l)$.

Let $m \in Q - S$ be maximal in $Q - S$. Then $m \leq s_S(m) \in Q - S$ and therefore $m = s_S(m) \in Q_{s_S}$. Let us check that m is prime in Q. Assume $r \in \mathcal{R}(Q)$, $l \in \mathcal{L}(Q)$ such that $r \not\leq m$ and $l \not\leq m$. Then $r \vee m$, $l \vee m \in S$. This gives us that $r \cdot l \vee m \in S$, i.e. $r \cdot l \not\leq m$. □

3.2. Corollary. *Let Q be a two-sided cm-lattice, $S \subseteq Q$ a Scott-open distributive filter on Q. Then the multiplication in Q_{s_S} coincides with the meet operation, and Q_{s_S} is a compact two-sided wcm-lattice.*

Proof. The first part follows from the note after Definition 2.7. Let $r, l, x \in Q_{s_S}$, $s_S(r \vee x) = 1 = s_S(l \vee x)$. Then $r \vee x$, $l \vee x \in S$. Therefore also $r \cdot l \vee x \in S$. In particular $r \bullet l \vee x \in S$, i.e. $s_S(r \wedge l \vee x) = 1$. □

We introduce the following *General Separation Lemma for Quantales*:

GSLQ. *Any element outside some Scott-open distributive filter S in a non-trivial quantale is below a prime element outside S.*

Furthermore, we consider the *General Separation Lemma for Complete Multiplicative Lattices*:

GSLC. *Any element outside some Scott-open distributive filter S in a non-trivial cm-lattice is below a prime element outside S.*

Clearly, **GSLC** implies **GSLQ**. The following proposition is obvious. Recall only that a completely prime upper set is a complement of a principal ideal.

3.3. Proposition.

(i) **GSLC** *holds if and only if every Scott-open distributive filter of a non-trivial cm-lattice is an intersection of completely prime distributive filters.*

(ii) **GSLQ** *holds if and only if every Scott-open distributive filter of a non-trivial quantale is an intersection of completely prime distributive filters.*

Let us recall the following well-known fact.

3.4. Lemma. *Let Q be a two-sided m-distributive cm-lattice, $F \subseteq Q$. Then the following conditions are equivalent:*

1. *F is a filter, i.e. it is an upper set such that $x, y \in F$ implies $x \cdot y \in F$,*
2. *F is a distributive filter.*

The preceding lemma implies that, in the case of m-distributive cm-lattices, the *Separation Lemma for Complete Multiplicative Lattices* (**SLC**) is a special instance of our *General Separation Lemma for Complete Multiplicative Lattices* since this lemma was formulated in [7] for two-sided cm-lattices. Note also that in this case the cm-lattice $\Lambda(Q)$ of distributive filters of an m-distributive two-sided cm-lattice is a distributive lattice.

The first basic result of this section is now the following one (see [4, Proposition 1] for comparison).

3.5. Proposition. *The Axiom of Choice implies both* **GSLQ** *and* **GSLC**. *Moreover, in the presence of the Axiom of Choice an element of a cm-lattice Q is s_S-fixed if and only if it is a meet of maximal elements of $Q - S$ containing it.*

Proof. Let s_S be the radical nucleus on the cm-lattice Q associated with S as in Lemma 3.1. Then, for any $a \in Q - S$, the set $\{x \in Q - S : x \geq a\}$ is closed under up-directed joins and by Zorn's lemma there is a maximal element $m \in Q - S$ such that $a \leq m$. Then m is almost S-maximal and therefore it is prime.

Since any maximal element of $Q - S$ is s_S-fixed so is any meet of such maximal elements. Conversely, let $a \in Q$ be s_S-fixed and let $b = \bigwedge\{m \in Q - S : m \text{ maximal in } Q - S,\, a \leq m\}$. Suppose $b \not\leq a$. Then there is an element $c \in Q$ such that $c \vee b \in S$, $c \vee a \notin S$. This yields a maximal element m in $Q - S$ such that $a \leq c \vee a \leq m$, $c \vee b \not\leq m$, i.e. $b \not\leq m$, a contradiction. $\square$

The core of the preceding result is that any maximal element out of a Scott-open distributive filter is prime. A special case is Lemma 2.4.

3.6. Remark. Recall from [4] that the *Separation Lemma for Quantales* is equivalent to the Prime Ideal Theorem (**PIT**). The proof of this equivalence is based on the fact that, for a Scott-open distributive filter S of a non-trivial two-sided quantale Q, the cm-lattice Q_{s_S} is a non-trivial compact frame. In particular it has by the **PIT** a prime element out of S that is in turn prime in Q by the Lemma 2.9.

We then have the following series of theorems concerning spatiality or existence of points.

3.7. Theorem. *Given* **GSLQ**, *every non-trivial compact w-quantale Q has a prime element.*

Proof. We put $S = \{1\}$. Then S is a Scott-open distributive filter such that $0 \notin S$, i.e. there is $p \in \mathcal{P}(Q) - S$ such that $0 \leq p$. $\square$

3.8. Remark. We note that, in any distributive quantale Q, each Scott-open filter is a Scott-open distributive filter. This gives us the following desired propositions.

3.9. Theorem. *Given the* **Principle of Countable Dependent Choice** (**CDC**) *and* **GSLQ***, every continuous distributive quantale is spatial.*

Proof. Let $a \not\leq b$, $a, b \in Q$. By continuity, there exists $c \in Q$ such that $c \ll a$, $c \not\leq b$. Using **CDC**, we obtain a sequence $(a_n : n \in \omega)$ in Q for which

$$a_0 = a, \quad a_{n+1} \ll a_n \quad \text{and} \quad a_n \not\leq b \qquad (\text{all } n \in \omega).$$

Hence $S = \{x \in Q : a_n \leq x \text{ for some } n \in \omega\}$ is a Scott-open filter such that $b \notin S$ and using **GSLQ** there is $p \in \mathcal{P}(Q) - S$ such that $b \leq p$ and consequently $a \not\leq p$. $\square$

Assuming stable continuity or algebraicity of Q, **CDC** is redundant.

3.10. Proposition. *Given* **GSLQ***, every stably continuous distributive quantale is spatial.*

Proof. Let $a \not\leq b$, $a, b \in Q$. By continuity, there exists $c \in Q$ such that $c \ll a$, $c \not\leq b$. We put $S = \{x \in Q : c \ll x\}$. Then S is a Scott-open filter such that $b \notin S$ and using **GSLQ** as before there is $p \in \mathcal{P}(Q) - S$ such that $b \leq p$, $a \not\leq p$. $\square$

3.11. Proposition. *Given* **GSLQ***, every algebraic distributive quantale is spatial.*

Proof. Let $a \not\leq b$, $a, b \in Q$. By algebraicity, there exists $c \in Q$ compact such that $c \leq a$, $c \not\leq b$. We put $S = \{x \in Q : c \leq x\}$. Then S is a Scott-open filter such that $b \notin S$, i.e. there is $p \in \mathcal{P}(Q) - S$ such that $b \leq p$, $a \not\leq p$. $\square$

Using the Axiom of Choice, the preceding proposition was proved in [13].

3.12. Remark. Recall that a frame L is said to be *conjunctive* if for each two elements $a, b \in L$, $a \not\leq b$ there is an element $c \in L$ such that $a \vee c = 1$, $b \vee c \neq 1$. Such frames were studied by Isbell [8] who called them *subfit.* They were renamed by Simmons [21] because the defining property is dual to *Wallman's disjunctivity.* For further details concerning conjunctivity in frames see papers [1], [2], [3], [5], [6] and [10].

3.13. Definition. A quantale Q is said to be *conjunctive* if for each two elements $a, b \in Q$, $a \not\leq b$ there is an element $c \in Q$ such that $a \vee c = 1$, $b \vee c \neq 1$. This is equivalent to the condition that $s_{\{1\}}$ is the identity map on Q.

3.14. Proposition. *Given* **GSLQ***, any compact conjunctive w-quantale is spatial.*

Proof. Let $a, b \in Q$, $a \nleq b$. Then there is an element $c \in Q$ such that $a \vee c = 1$, $b \vee c \neq 1$. Since $b \vee c \notin \{1\}$ there is a prime element $p \in \mathcal{P}(Q)$ such that $b \vee c \leq p \notin \{1\}$, i.e. $a \nleq p$, $b \leq p$. □

We conclude the paper with a generalization of the above proposition and some comments.

3.15. Proposition. *Given* **GSLQ***, any conjunctive quantale such that* $\{1\}$ *coincides with an intersection of Scott-open distributive filters is spatial.*

Proof. Let $a, b \in Q$, $a \nleq b$. Then there is an element $c \in Q$ such that $a \vee c = 1$, $b \vee c \neq 1$. Since $b \vee c \notin \{1\}$ there is a Scott-open distributive filter S such that $b \vee c \notin S$. From 3.3 there is a prime element $p \in \mathcal{P}(Q)$ and a completely prime distributive filter S_p such that $b \vee c \notin S_p$. Then $b \vee c \leq p \notin \{1\}$, i.e. $a \nleq p$, $b \leq p$. □

3.16. Remark. If we replace — beginning in Theorem 3.7 and ending with Proposition 3.15 — the words "quantale" and "**GSLQ**" by the words "cm-lattice" and "**GSLC**", all of our results in this setting remain valid (compare the results of Keimel [11], Rav [18] and Rosický [20]).

3.17. Remark. We hope that it will be possible to prove that the Prime Ideal Theorem implies **GSLC** (**GSLQ**) at least for some nice classes of cm-lattices (quantales). One of the obstacles, that makes this task more complicated than it was in the case of two-sided m-distributive cm-lattices (quantales), is that we do not know whether the Scott-open distributive filters form a distributive lattice. Also, it would be helpful to have an explicit description of the Scott-open distributive filter generated by a set.

Acknowledgment

I want to thank here the anonymous referee for many valuable comments that allowed to highly improve the final version of this paper.

References

[1] B. Banaschewski, The power of the ultrafilter theorem, *J. London Math. Soc.*, (2) **50** (1983), 193–202.
[2] B. Banaschewski and R. Harting, Lattice aspects of radical ideals and choice principles, *Proc. London Math. Soc.*, **50** (1985), 385–404.
[3] B. Banaschewski and R. Harting, $\bigvee$-rings, Preprint, Hamilton (1985).
[4] B. Banaschewski and M. Erné, On Krull's separation lemma, *Order*, (3) **10** (1993), 253–260.
[5] M. Erné, Distributors and Wallmanufacture, Preprint Series University of Hannover **204**, 1986, and *J. Pure and Appl. Alg.*, **68** (1990), 109–125.
[6] M. Erné, Closure operators on cm-lattices, Preprint, Hannover, 1996.

[7] M. Erné, Prime ideal theory for general algebras, *Applied Categorical Structures*, **8** (2000), 115–144.
[8] J. R. Isbell, Atomless parts of spaces, *Math. Scand.*, **31** (1972), 5–32.
[9] P. T. Johnstone, *Stone spaces*, Cambridge Univ. Press, Cambridge (1982).
[10] P. T. Johnstone, Almost maximal ideals, *Fund. Math.*, **123** (1984), 201–206.
[11] K. Keimel, A unified theory of minimal prime ideals, *Acta Math. Acad. Sci. Hung.*, **23** (1972), 51–69.
[12] D. Kruml, Spatial quantales, *Applied Categorical Structures*, **10** (2002), 49–62.
[13] D. Kruml, Distributive quantales, *Applied Categorical Structures*, **11** (2003), 561–566.
[14] C. J. Mulvey, &, *Suppl. Rend. Circ. Mat. Palermo*, ser. II **12** (1986), 99–104.
[15] C. J. Mulvey and J. Wick Pelletier, On the quantisation of points, *Journal of Pure and Applied Algebra*, **175** (2002), 289–325.
[16] J. Paseka, Conjunctivity in quantales, *Archivum Mathematicum (Brno)*, (4) **24** (1988), 173–180.
[17] J. Paseka, A note on the prime ideal theorem, *Acta Univ. Carolinae — Mathematica et Physica*, (2) **30** (1989), 131–136.
[18] Y. Rav, Semiprime ideals in general lattices, *Journal of Pure and Applied Algebra*, **56** (1989), 105–118.
[19] K. I. Rosenthal, *Quantales and their applications*, Pitman Research Notes in Mathematics Series 234, Longman Scientific & Technical (1990).
[20] J. Rosický, Multiplicative lattices and frames, *Acta Math. Hung.*, **49** (1987), 391–395.
[21] H. Simmons, The lattice theoretical part of topological separation properties, *Proc. Edinburgh Math. Soc.*, (2) **21** (1978), 41–48.
[22] H. Simmons, Two sided multiplicative lattices and ring ideals, *unpublished manuscript* (1986), 1–21.

Department of Mathematics, Masaryk University,
Janáčkovo nám. 2a, 602 00 Brno
E-mail address: paseka@math.muni.cz