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MULTIPLIER ALGEBRAS OF INVOLUTIVE QUANTALES

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ABSTRACT. The aim of this note is to establish a description of the (semi-) multiplier algebra of an involutive quantale.

Investigations in the theory of C*-algebras make often use of Hilbert C*-modules as a tool for proving. Research fields benefiting from it include operator K-theory, index theory for operator-valued conditional expectations, group representation theory and others. Beside this, the theory of Hilbert C*-modules is very interesting on its own right. Our main purpose here is to expose some more connections between the theory of Hilbert C*-modules and their lattice analogs, Hilbert Q-modules. Of course the main motivation for the machinery developed here, is that certain problems concerning (involutive) quantales should be solvable by transferring the methods of Hilbert C*-modules (see [6]) into the Hilbert Q-modules framework. Similarly, as for C*-algebras (see [7]) we shall introduce the notion of a multiplier of an involutive quantale and we show that any involutive quantale with a maximal essential extension over sup-lattices with a duality has a unique maximal essential extension to its multiplier.

The results and definitions follow a natural logical sequence, so we begin without further delay. We refer to [4] for additional information and complementary results, and to [1], [5] and [8] for background information on Hilbert C*-modules and quantales.

In what follows, a complete lattice will be called *sup-lattice*. *Sup-lattice homomorphisms* are maps between sup-lattices preserving arbitrary joins. We shall denote, for sup-lattices S and T, by $SUP(S,T)$ the sup-lattice of all sup-lattice homomorphisms from S to T, with the supremum given by the pointwise ordering of mappings. Recall that a *quantale* is a sup-lattice Q with

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an associative binary multiplication satisfying

$$x \cdot \bigvee_{i \in I} x_i = \bigvee_{i \in I} x \cdot x_i \text{ and } (\bigvee_{i \in I} x_i) \cdot x = \bigvee_{i \in I} x_i \cdot x$$

for all $x, x_i \in Q$, $i \in I$ (I is a set). 1 denotes the greatest element of Q, 0 is the smallest element of Q. A quantale Q is said to be *unital* if there is an element $e \in Q$ such that $e \cdot a = a = a \cdot e$ for all $a \in Q$. A *subquantale* Q' of a quantale Q is a subset of Q closed under $\bigvee$ and $\cdot$. Since the operators $a \cdot -$ and $- \cdot b : Q \to Q$, $a, b \in Q$ preserve arbitrary joins, they have right adjoints. Explicitly, they are given by

$$a \to_r c = \bigvee\{s \in Q | a \cdot s \leq c\} \quad \text{and} \quad b \to_l d = \bigvee\{t \in Q | t \cdot b \leq d\}$$

respectively.

An *involution* on a sup-lattice S is a unary operation such that

$$\begin{aligned} a^{**} &= a, \\ (\bigvee a_i)^* &= \bigvee a_i^* \end{aligned}$$

for all $a, a_i \in S$. An *involution* on a quantale Q is an involution on the sup-lattice Q such that

$$(a \cdot b)^* = b^* \cdot a^*,$$

for all $a, b \in Q$. A sup-lattice (quantale) with an involution is said to be *involutive.*

By a *morphism of (involutive) quantales* will be meant a $\bigvee$- (*-) and $\cdot$-preserving mapping $f : Q \to Q'$. If a morphism preserves the unital element we say that it is *unital.*

Let A be a subset of a quantale Q. We shall denote by $\mathrm{comm}(A)$ the *commutant* of A in Q (see [3]) defined as

$$\mathrm{comm}(A) = \{b \in Q : (\forall a \in A)(a \cdot b = b \cdot a)\}.$$

Similarly, we write $\mathrm{bicomm}(A)$ for the *bicommutant* of A in Q defined as

$$\mathrm{bicomm}(A) = \mathrm{comm}(\mathrm{comm}(A)).$$

By the *quantale $\mathscr{Q}(M) = SUP(M, M)$ of endomorphisms of the sup-lattice M* (see [2]) will be meant the unital quantale of sup-preserving mappings from M to itself, with the multiplication given by the composition of mappings, and with the unit given by the identity mapping. Note that $\mathscr{Q}(M) \cong (M \otimes M^{op})^{op}$; here $\otimes$ is a tensor product of sup-lattices.

In [3], there was introduced the idea of the Cayley representation $\mathscr{C}(Q) = \mathscr{Q}(Q[e] \times Q[e]^{op})$ of an arbitrary involutive quantale Q; here the involutive quantale embedding $\alpha_Q : Q \to \mathscr{C}(Q)$ is defined by $\alpha_Q(a) = (a \cdot -, a^* \to_r -)$, for all $a \in Q$, and $Q[e]$ is the unital involutive quantale obtained by adding the unit e to Q, putting $e = 0 \vee e$ and freely generating it by the set $Q \cup \{e\}$ if Q is non-unital (see [3]), otherwise we define $Q[e] = Q$.

Let Q be a quantale. A *left module over* Q (shortly a left Q-module) is a sup-lattice M, together with a *module action* $\_\bullet\_ : Q \times M \to M$ satisfying

$$\begin{array}{lcl}(a \cdot b) \bullet m & = & a \bullet (b \bullet m) \\ (\bigvee S) \bullet m & = & \bigvee\{s \bullet m : s \in S\} \\ a \bullet \bigvee X & = & \bigvee\{a \bullet x : x \in X\}\end{array}$$

for all $a, b \in Q$, $m \in M$, $S \subseteq Q$, $X \subseteq M$. Let M and N be modules over Q and let $f : M \to N$ be a sup-lattice homomorphism. f is a *module homomorphism* if $f(a \bullet m) = a \bullet f(m)$ for all $a \in Q, m \in M$.

Note first that if M is a sup-lattice then M is a left $\mathscr{Q}(M)$-module such that $f \bullet m = f(m)$ for all $f \in \mathscr{Q}(M)$ and all $m \in M$. Secondly, we may dually define the notion of a right Q-module. Moreover, all propositions stated for left Q-modules are valid in a dualized form for right Q-modules.

The theory of Hilbert Q-modules (we refer the reader to [4] for details and examples) is a generalization of the theory of complete semilattices with a duality and it is the natural framework for the study of modules over an involutive quantale Q endowed with Q-valued inner products.

Let Q be an involutive quantale, M a right (left) Q-module with a right (left) module action $\diamond$ ($\bullet$). We say that M is a *right* (*left*) *Hilbert Q-module* (*right* (*left*) *strict Hilbert Q-module*) if M is equipped with a map $\langle -, - \rangle : M \times M \to Q$, called the *inner product*, such that for all $a \in Q$, $m, n \in M$ and $m_i \in M$, where $i \in I$, the conditions (1)–(5) ((1)–(6)) are satisfied:

(1) $\langle m, n\rangle \cdot a = \langle m, n \diamond a\rangle \quad (a \cdot \langle m, n\rangle = \langle a \bullet m, n\rangle)$;

(2) $\bigvee_{i\in I} \langle m_i, n\rangle = \langle \bigvee_{i\in I} m_i, n\rangle$;

(3) $\bigvee_{i\in I} \langle m, m_i\rangle = \langle m, \bigvee_{i\in I} m_i\rangle$;

(4) $\langle m, n\rangle^* = \langle n, m\rangle$;

(5) $\langle -, m\rangle = \langle -, n\rangle$ ($\langle m, -\rangle = \langle n, -\rangle$) implies $m = n$;

(6) $\langle m, m\rangle = 0$ implies $m = 0$.

Let $f : N \to M$ be a map between right (left) Hilbert Q-modules. We say that a map $f^* : M \to N$ is a **-adjoint to f* and f is *adjointable* if

$$\langle f(n), m\rangle = \langle n, f^*(m)\rangle$$

for all $m \in M$, $n \in N$. Automatically, f is then a Q-module homomorphism.

The $\bigvee$-semilattice of all adjointable maps from N to M is denoted by $\mathscr{A}_Q(N, M)$. For $m \in M$ and $n \in N$, define a *fr-operator* $\Theta_{m,n} : N \to M$ by $\Theta_{m,n}(p) = m \diamond \langle n, p\rangle = \langle p, n\rangle \bullet m$ for all $p \in N$. We shall denote by $\mathscr{K}_Q(N, M)$

the sub-$\bigvee$-semilattice of $\mathscr{A}_Q(N, M)$ generated by the set $\{\Theta_{m,n} : m \in M, n \in N\}$ and the elements of $\mathscr{K}_Q(N, M)$ we will call *compact operators.*

Note that $\mathscr{A}_Q(M)$, the $\bigvee$-semilattice of all adjointable maps from M to itself, is an involutive quantale. Moreover, if $Q = \mathbf{2}$, the 2-element Boolean algebra, then we have $\mathscr{Q}(M) = \mathscr{A}_{\mathbf{2}}(M)$. Namely, $\bigvee$-semilattices with a duality (complete ortholattices) are exactly the Hilbert (strict) **2**-modules.

Multiplier algebras

Definition 1. *Let Q be an involutive quantale, M an involutive $\bigvee$-semilattice with an involution $^{\#}$, and let M be both a left Q-module with action $\bullet$ and a right Q-module with action $\diamond$ such that*

$$\begin{aligned} (a \bullet m) \diamond b &= a \bullet (m \diamond b), \\ (a \bullet m)^{\#} &= m^{\#} \diamond a^{*} \end{aligned}$$

for all $a, b \in Q$, $m \in M$. Then we shall say that M is an involutive bimodule *over Q. An* involutive bimodule homomorphism *is both a left- and right-module homomorphism preserving involution. Moreover, if Q is an involutive $\bigvee$-subsemilattice of M we shall say that Q is a* semiideal *of M and M is a* semiextension *of Q, if*

$$a \bullet b = a \cdot b = a \diamond b \text{ and } a \bullet m, m \diamond b \in Q$$

for all $a, b \in Q$ and all $m \in M$. If moreover M is an involutive quantale, Q an involutive subquantale and a semiideal of M we call Q an ideal *of M and M an* extension *of Q. We call a semiideal (ideal) Q* essential *if for all $m, n \in M$,*

$$a \bullet m = a \bullet n \text{ and } m \diamond b = n \diamond b \text{ for all } a, b \in Q \implies m = n.$$

We shall say that a semiextension (extension) M of Q is maximal *if any other such semiextension (extension) M' of Q can be embedded in it.*

Recall that Q is essential in M iff

for all $m, n \in M$, $a \bullet m = a \bullet n$ for all $a \in Q \implies m = n$

iff

for all $m, n \in M$, $m \diamond b = n \diamond b$ for all $b \in Q \implies m = n$.

Moreover, Q is essential in Q iff Q is a Hilbert Q-module.

Let Q be an involutive quantale, M a (left) Hilbert Q-module and $M^{\circledast}$ the $\bigvee$-semilattice of all Q-module homomorphisms from M to Q. Then we know that $M^{\circledast}$ is a left Q-module with the module multiplication $(c \bullet f)(m) = f(m) \cdot c^{*}$, for all $c \in Q$, $m \in M$, $f \in M^{\circledast}$. We have a canonical embedding $\iota_M : M \to M^{\circledast}$ defined by $m \mapsto \langle -, m \rangle$, $m \in M$. Namely, let $a \in Q$, $m, n \in M$. Then

$$\begin{aligned} \iota_M(a \bullet m)(n) &= \langle n, a \bullet m \rangle = \langle a \bullet m, n \rangle^{*} = (a \cdot \langle m, n \rangle)^{*} = \\ &= \langle n, m \rangle \cdot a^{*} = (a \bullet \langle -, m \rangle)(n) = (a \bullet \iota_M(m))(n). \end{aligned}$$

Note that for $T \in \mathrm{Hom}_Q(M, M)$ we have a Q-module homomorphism $T^{\circledast} : M^{\circledast} \to M^{\circledast}$ defined by the prescription $(T^{\circledast}(f))(m) = f(T(m))$, $f \in M^{\circledast}$, $m \in M$. Namely, we have

$$(a \bullet T^{\circledast}(f))(m) = T^{\circledast}(f)(m) \cdot a^* = f(T(m)) \cdot a^* = \\ = (a \bullet f)(T(m)) = (T^{\circledast}(a \bullet f))(m).$$

Evidently, $(T_1 \circ T_2)^{\circledast} = T_2^{\circledast} \circ T_1^{\circledast}$. Similarly, for any $T \in \mathrm{Hom}_Q(M, M^{\circledast})$ we have a $\bigvee$-preserving map $T^{\#} : M \to M^{\circledast}$ defined by $(T^{\#}(m))(n) = (T(n)(m))^*$ for all $m, n \in M$. Evidently, the map $^{\#}$ is an involution on the $\bigvee$-semilattice $\mathrm{Hom}_Q(M, M^{\circledast})$ since $^{\#}$ preserves arbitrary joins and any $T^{\#}$ is a module homomorphism by

$$(a \bullet T^{\#}(m))(n) = T^{\#}(m)(n) \cdot a^* = (T(n)(m))^* \cdot a^* = \\ (a \cdot T(n)(m))^* = (T(n)(a \bullet m))^* = T^{\#}(a \bullet m)(n).$$

Lemma 2. *Let Q and Q' be involutive quantales, M a Hilbert Q-module. Then an involutive quantale homomorphism $\rho : Q' \to \mathscr{A}_Q(M)$ determines a structure of a $\rho(Q')$ bimodule on* $\mathrm{Hom}_Q(M, M^{\circledast})$ *by the following prescription:*

$$\rho(a) \bullet T = \rho(a^*)^{\circledast} \circ T, \quad T \diamond \rho(a) = T \circ \rho(a)$$

for all $a \in Q'$, $T \in$ $\mathrm{Hom}_Q(M, M^{\circledast})$.

Proof. Let us check the left module axioms. Let $a, b \in Q', T \in \mathrm{Hom}_Q(M, M^{\circledast})$. Then

$$\begin{aligned}(\rho(a) \circ \rho(b)) \bullet T &= \rho(a \cdot b) \bullet T = \rho((a \cdot b)^*)^{\circledast} \circ T = \rho(b^* \cdot a^*)^{\circledast} \circ T \\ &= ((\rho(a^*)^{\circledast} \circ \rho(b^*)^{\circledast}) \circ T = \rho(a^*)^{\circledast} \circ (\rho(b^*)^{\circledast} \circ T) \\ &= \rho(a^*)^{\circledast} \circ (\rho(b) \bullet T) = \rho(a) \bullet (\rho(b) \bullet T).\end{aligned}$$

Similarly, we have

$$\begin{aligned}(\textstyle\bigvee_{i \in I} \rho(a_i)) \bullet T &= \rho(\textstyle\bigvee_{i \in I} a_i) \bullet T = \rho(\textstyle\bigvee_{i \in I} a_i^*)^{\circledast} \circ T \\ &= \textstyle\bigvee_{i \in I} (\rho(a_i^*)^{\circledast} \circ T) = \textstyle\bigvee_{i \in I} (\rho(a_i) \bullet T)\end{aligned}$$

and

$$\rho(a) \bullet (\textstyle\bigvee_{i \in I} T_i) = \rho(a^*)^{\circledast} \circ (\textstyle\bigvee_{i \in I} T_i) = \textstyle\bigvee_{i \in I} \rho(a^*)^{\circledast} \circ T_i = \textstyle\bigvee_{i \in I} \rho(a) \bullet T_i.$$

By the same arguments we can prove that $\mathrm{Hom}_Q(M, M^{\circledast})$ is a right module. Let us check that it is an involutive bimodule. We have, for all $a, b \in Q'$ and all $T \in \mathrm{Hom}_Q(M, M^{\circledast})$,

$$\begin{aligned}(\rho(a) \bullet T) \diamond \rho(b) &= (\rho(a^*)^{\circledast} \circ T) \circ \rho(b) \\ &= \rho(a^*)^{\circledast} \circ (T \circ \rho(b)) = \rho(a) \bullet (T \diamond \rho(b))\end{aligned}$$

and

$$\begin{aligned}(\rho(a) \bullet T)^{\#}(m)(n) &= (\rho(a^*)^{\circledast} \circ T)^{\#}(m)(n) = \left[\left(\rho(a^*)^{\circledast}(T(n))\right)(m)\right]^* \\ &= [T(n)(\rho(a^*)(m))]^* = T^{\#}(\rho(a)^*(m))(n) \\ &= (T^{\#} \circ \rho(a)^*)(m)(n) = (T^{\#} \diamond \rho(a)^*)(m)(n). \quad \square\end{aligned}$$

Note that we have a $\bigvee$- and involution-preserving embedding defined by $j_M : \mathscr{A}_Q(M) \to \mathrm{Hom}_Q(M, M^{\circledast})$, $T \mapsto i_M \circ T$, i.e. $j_M(T)(m) = \langle -, T(m)\rangle$ for all $m \in M$. Namely, we have $j_M(T)^{\#} = j_M(T^*)$, $j_M(\bigvee_{i\in I} T_i) = \bigvee_{i\in I} j_M(T_i)$ and $T^{\circledast} \circ i_M = T^*$.

Definition 3. *Let Q and Q' be involutive quantales, M a Hilbert Q-module. We say that an involutive quantale homomorphism $\alpha : Q' \to \mathscr{A}_Q(M)$ is* non-degenerate *if $M = \{\bigvee_{i\in I} \alpha(a_i)(m_i) : a_i \in Q', m_i \in M\}$. We shall say that Q' is* non-degenerate *if it is a Hilbert Q'-module and the involutive quantale homomorphism $\alpha_{Q'} : Q' \to \mathscr{Q}(Q' \times Q'^{op})$ defined by $\alpha_{Q'}(a) = (a \cdot -, a^* \to_r -)$, for all $a \in Q'$, is non-degenerate.*

It may be remarked that Q' is non-degenerate iff it is essential in itself, $Q' \cdot Q' = Q'$ and $Q' \to_r Q' = Q'$. Evidently, any unital involutive quantale is non-degenerate.

Lemma 4. *Let Q and Q' be involutive quantales, M a Hilbert Q-module, Q' a semiideal of an involutive Q' bimodule N, $\alpha : Q' \to \mathscr{A}_Q(M)$ a non-degenerate involutive quantale homomorphism. Then α extends uniquely to an involutive bimodule homomorphism $\overline{\alpha} : N \to \mathrm{Hom}_Q(M, M^{\circledast})$, i.e.*

$$\overline{\alpha}(a) = \iota_M(\alpha(a)),\ \overline{\alpha}(a \bullet n) = \alpha(a) \bullet \overline{\alpha}(n),\ \overline{\alpha}(n \diamond b) = \overline{\alpha}(n) \diamond \alpha(b),$$

for all $a, b \in Q'$, $n \in N$. If α is injective and Q' essential in N then $\overline{\alpha}$ is also injective.

Proof. Let us define $\overline{\alpha}(n)(\bigvee_{i\in I} \alpha(a_i)(m_i)) = \bigvee_{i\in I} (\iota_M \circ \alpha(n \diamond a_i))(m_i)$ for all $a_i \in Q', m_i \in M$. First, let us show the correctness of our definition. Let $\bigvee_{i\in I} \alpha(a_i)(m_i) = \bigvee_{j\in J} \alpha(b_j)(r_j)$ and let $y = \bigvee_{k\in K} \alpha(c_k)(q_k) \in M$. Then

$$\begin{aligned}
\overline{\alpha}(n)(\textstyle\bigvee_{i\in I} \alpha(a_i)(m_i))(y) &= \textstyle\bigvee_{i\in I} (\iota_M \circ \alpha(n \diamond a_i))(m_i)(y)\\
&= \textstyle\bigvee_{i\in I} \left\langle \bigvee_{k\in K} \alpha(c_k)(q_k), \alpha(n \diamond a_i)(m_i)\right\rangle\\
&= \textstyle\bigvee_{i\in I, k\in K} \langle (\alpha(n \diamond a_i)^* \circ \alpha(c_k))(q_k), m_i\rangle\\
&= \textstyle\bigvee_{i\in I, k\in K} \langle (\alpha(a_i^* \bullet n^* \diamond c_k)(q_k), m_i\rangle\\
&= \textstyle\bigvee_{i\in I, k\in K} \langle (\alpha(n^* \diamond c_k)(q_k), \alpha(a_i)(m_i)\rangle\\
&= \textstyle\bigvee_{k\in K} \left\langle (\alpha(n^* \diamond c_k)(q_k), \bigvee_{i\in I} \alpha(a_i)(m_i)\right\rangle\\
&= \textstyle\bigvee_{k\in K} \left\langle (\alpha(n^* \diamond c_k)(q_k), \bigvee_{j\in J} \alpha(b_j)(r_j)\right\rangle\\
&= \overline{\alpha}(n)(\textstyle\bigvee_{j\in J} \alpha(b_j)(r_j))(y).
\end{aligned}$$

Now, let us check that $\overline{\alpha}$ preserves arbitrary joins and involution. Assume $n_k \in N, k \in K$. Then we have

$$\begin{aligned}
\overline{\alpha}(\textstyle\bigvee_{k\in K} n_k)(\bigvee_{i\in I} \alpha(a_i)(m_i)) &= \textstyle\bigvee_{i\in I} (\iota_M \circ \alpha((\bigvee_{k\in K} n_k) \diamond a_i))(m_i)\\
\textstyle\bigvee_{i\in I, k\in K} (\iota_M \circ \alpha(n_k \diamond a_i))(m_i) &= \textstyle\bigvee_{k\in K} \overline{\alpha}(n_k)(\bigvee_{i\in I} \alpha(a_i)(m_i))
\end{aligned}$$

and, for $y = \bigvee_{j\in J} \alpha(b_j)(r_j)$,

$$\begin{array}{rcl} \overline{\alpha}(n)^{\#}(\bigvee_{i\in I} \alpha(a_i)(m_i))(y) & = & \overline{\alpha}(n)(y)(\bigvee_{i\in I} \alpha(a_i)(m_i))^* = \\ \left\langle \bigvee_{i\in I} \alpha(a_i)(m_i), \bigvee_{j\in J} \alpha(n\diamond b_j)(r_j)\right\rangle^* & = & \left\langle \bigvee_{j\in J} \alpha(n\diamond b_j)(r_j), \bigvee_{i\in I} \alpha(a_i)(m_i)\right\rangle \\ = \bigvee_{i\in I, j\in J} \langle \alpha(n\diamond b_j)(r_j), \alpha(a_i)(m_i)\rangle & = & \bigvee_{i\in I, j\in J} \langle \alpha(a_i^* \bullet n \diamond b_j)(r_j), m_i\rangle \\ = \bigvee_{i\in I, j\in J} \langle \alpha(b_j)(r_j), \alpha(n^*\diamond a_i)(m_i)\rangle & = & \overline{\alpha}(n^*)(\bigvee_{i\in I} \alpha(a_i)(m_i))(y). \end{array}$$

Let us show that $\overline{\alpha}(a\bullet n) = \alpha(a)\bullet\overline{\alpha}(n)$. We have

$$\begin{array}{rcl} \overline{\alpha}(a\bullet n)(\bigvee_{i\in I} \alpha(a_i)(m_i))(y) & = & \bigvee_{i\in I} (\iota_M \circ \alpha(a\bullet n\diamond a_i))(m_i)(y) = \\ \bigvee_{i\in I} \langle y, \alpha(a\bullet n\diamond a_i)(m_i)\rangle & = & \bigvee_{i\in I} \langle \alpha(a)^*(y), \alpha(n\diamond a_i)(m_i)\rangle = \\ \langle \alpha(a^*)(y), \bigvee_{i\in I} \alpha(n\diamond a_i)(m_i)\rangle & = & \iota_M(\bigvee_{i\in I} \alpha(n\diamond a_i)(m_i))(\alpha(a^*)(y)) = \\ \overline{\alpha}(n)(\bigvee_{i\in I} \alpha(a_i)(m_i))(\alpha(a^*)(y)) & = & \alpha(a^*)^{\circledast}(\overline{\alpha}(n)(\bigvee_{i\in I} \alpha(a_i)(m_i)))(y) = \\ (\alpha(a^*)^{\circledast} \circ \overline{\alpha}(n))(\bigvee_{i\in I} \alpha(a_i)(m_i))(y) & = & (\alpha(a)\bullet\overline{\alpha}(n))(\bigvee_{i\in I} \alpha(a_i)(m_i))(y). \end{array}$$

Similarly, we have

$$\begin{array}{rcl} \overline{\alpha}(n\diamond b)(\bigvee_{i\in I} \alpha(a_i)(m_i)) & = & \bigvee_{i\in I} \alpha(n\diamond b\diamond a_i)(m_i) = \\ \overline{\alpha}(n)(\bigvee_{i\in I} \alpha(b\diamond a_i)(m_i)) & = & \overline{\alpha}(n)(\alpha(b)(\bigvee_{i\in I} \alpha(a_i)(m_i))) \\ & = & (\overline{\alpha}(n)\diamond\alpha(b))(\bigvee_{i\in I} \alpha(a_i)(m_i)). \end{array}$$

Now, let $\beta : N \to \mathrm{Hom}_Q(M, M^{\circledast})$ be an involutive bimodule homomorphism such that $\beta(a) = \iota_M(\alpha(a))$, $\beta(a\bullet n) = \alpha(a)\bullet\beta(n)$, $\beta(n\diamond b) = \beta(n)\diamond\alpha(b)$. Then

$$\begin{array}{rcl} \beta(n)(\bigvee_{i\in I} \alpha(a_i)(m_i)) & = & \bigvee_{i\in I} (\beta(n)\circ\alpha(a_i))(m_i) \\ & = & \bigvee_{i\in I} \beta(n\diamond a_i)(m_i) \\ & = & \bigvee_{i\in I} (\iota_M\circ\alpha(n\diamond a_i))(m_i) \\ & = & \overline{\alpha}(n)(\bigvee_{i\in I} \alpha(a_i)(m_i)). \end{array}$$

Finally, if α is injective, we have, for all $n_1, n_2 \in N$ and all $a \in Q'$ such that $\overline{\alpha}(n_1) = \overline{\alpha}(n_2)$, the following:

$$\iota_M \circ \alpha(n_1\diamond a) = \overline{\alpha}(n_1\diamond a) = (\overline{\alpha}(n_1)\circ\alpha)(a) = (\overline{\alpha}(n_2)\circ\alpha)(a) = \iota_M\circ\alpha(n_2\diamond a).$$

This gives us that $\alpha(n_1\diamond a) = \alpha(n_2\diamond a)$ for all $a \in Q'$, i.e. $n_1\diamond a = n_2\diamond a$, i.e. $n_1 = n_2$. □

Corollary 5. *Let Q and Q' be involutive quantales, M a Hilbert Q-module, Q' an involutive subquantale and an ideal of an involutive quantale Q'', $\alpha : Q' \to \mathscr{A}_Q(M)$ a non-degenerate involutive quantale homomorphism. Then α extends uniquely to an involutive quantale homomorphism $\overline{\overline{\alpha}} : Q'' \to A_Q(M)$. If α is injective and Q' essential in Q'', then $\overline{\overline{\alpha}}$ is also injective.*

Proof. Let us define

$$\overline{\overline{\alpha}}(n)(\bigvee_{i\in I} \alpha(a_i)(m_i)) = \bigvee_{i\in I} \alpha(n\cdot a_i)(m_i)$$

for all $n \in Q'$. Then $\overline{\alpha} = \iota_M \circ \overline{\overline{\alpha}}$, i.e. $\overline{\overline{\alpha}}$ is correctly defined, by the injectivity of ι_M. It is easy to verify that $\overline{\alpha}(n_1 \cdot n_2) = \iota_M \circ \overline{\overline{\alpha}}(n_1) \circ \overline{\overline{\alpha}}(n_2)$, i.e. $\overline{\overline{\alpha}}(n_1 \cdot n_2) = \overline{\overline{\alpha}}(n_1) \circ \overline{\overline{\alpha}}(n_2)$. Evidently, $\overline{\overline{\alpha}}$ preserves arbitrary joins. Note that

$$\begin{array}{rcl}\left\langle y, \overline{\overline{\alpha}}(n^*)(\bigvee_{i\in I} \alpha(a_i)(m_i))\right\rangle & = & \left\langle y, \bigvee_{i\in I} \alpha(n^* \cdot a_i)(m_i)\right\rangle = \\ \left\langle \alpha(n)(y), \bigvee_{i\in I} \alpha(a_i)(m_i)\right\rangle & = & \left\langle \overline{\overline{\alpha}}(n)(y), \bigvee_{i\in I} \alpha(a_i)(m_i)\right\rangle\end{array}$$

i.e. $\overline{\overline{\alpha}}$ preserves involution. If α is injective we have (by the injectivity of $\overline{\alpha}$) that $\overline{\overline{\alpha}}$ is injective. □

Definition 6. *Let Q, Q' be involutive quantales and let M be a Hilbert Q-module. Suppose that $\alpha : Q' \to \mathscr{A}_Q(M)$ is a non-degenerate injective involutive quantale homomorphism. We say that the set*

$$\mathscr{S}\mathscr{M}_M(Q') = \{T \in \mathrm{Hom}_Q(M, M^{\circledast}) : T \diamond \alpha(Q') \subseteq \overline{\alpha}(Q') \text{ and } \alpha(Q') \bullet T \subseteq \overline{\alpha}(Q')\}$$

is an (M, α)-semimultiplier *of the involutive quantale Q'. Similarly, we say that the set*

$$\mathscr{M}_M(Q') = \{T \in \mathscr{A}_Q(M) : T \circ \alpha(Q') \subseteq \alpha(Q') \text{ and } \alpha(Q') \circ T \subseteq \alpha(Q')\}$$

is an (M, α)-multiplier *of the involutive quantale Q'.*

Evidently, from the proof of Lemma 4 (Corollary 5) we easily see that $\mathscr{S}\mathscr{M}_M(Q')$ is a semiextension of $\alpha(Q')$ ($\mathscr{M}_M(Q')$ is an extension of $\alpha(Q')$). Applying the non-degeneracy we have that, for all T_1, T_2, $T_1 \circ \alpha(a) = T_2 \circ \alpha(a)$ for all $a \in Q'$ implies $T_1 = T_2$, i.e. our semiextension (extension) is essential. Moreover, for any non-degenerate involutive quantale Q', we have a $(Q' \times Q'^{op}, \alpha_{Q'})$-semimultiplier (multiplier) over **2**. We then have the following:

Lemma 7. *Let Q, Q' be involutive quantales and let M be a Hilbert Q-module. Suppose that $\alpha : Q' \to \mathscr{A}_Q(M)$ is a non-degenerate injective involutive quantale homomorphism. Then $\mathscr{S}\mathscr{M}_M(Q')$ ($\mathscr{M}_M(Q')$) is a maximal essential semiextension (extension) of $\alpha(Q')$.*

Proof. Let X be any essential semiextension of $\alpha(Q')$. Then we have an injective embedding $\widetilde{\alpha} : X \to \mathrm{Hom}_Q(M, M^{\circledast})$ such that $\widetilde{\alpha}(a) = \iota_M(\alpha(a))$, $\widetilde{\alpha}(a \bullet n) = \alpha(a) \bullet \widetilde{\alpha}(n) \in \overline{\alpha}(Q')$, $\widetilde{\alpha}(n \diamond b) = \widetilde{\alpha}(n) \diamond \alpha(b) \in \overline{\alpha}(Q')$ for all $a, b \in Q'$ and all $n \in X$, i.e. $\widetilde{\alpha}(n) \in \mathscr{S}\mathscr{M}_M(Q')$. The extension part of the proof uses the same ideas. □

Proposition 8. *Let Q' be an involutive quantale with a maximal essential semiextension (extension) over* **2**. *Then Q' has a unique (up to isomorphism) maximal essential semiextension (extension), identical on Q'. In particular, all (M, α)-semimultipliers ((M, α)-multipliers) are isomorphic.*

Proof. Let $\alpha_0 : Q' \to \mathscr{Q}(\overline{M})$ be a maximal semiextension (extension) over **2** and let X be a maximal essential extension of Q', $\alpha : Q' \to X$. We shall show that X is isomorphic to the $(\overline{M}, \alpha_0)$-semiextension $\mathscr{S}\mathscr{M}_{\overline{M}}(Q')$. By the maximality of our semiextension there are involutive bimodule monomorphisms β :

$X \to \mathscr{S}\mathscr{M}_{\overline{M}}(Q')$ and $\gamma : \mathscr{S}\mathscr{M}_{\overline{M}}(Q') \to X$ such that $\beta_{/\alpha(Q')} \circ \gamma_{/\alpha_0(Q')} = \mathrm{id}_{\alpha_0(Q')}$ and $\gamma_{/\alpha_0(Q')} \circ \beta_{/\alpha(Q')} = \mathrm{id}_{\alpha(Q')}$. Then $\beta \circ \gamma : \mathscr{S}\mathscr{M}_{\overline{M}}(Q') \to \mathscr{S}\mathscr{M}_{\overline{M}}(Q') \subseteq \mathrm{Hom}_{\mathbf{2}}(\overline{M}, \overline{M}^{\circledast}) \simeq \mathscr{A}_{\mathbf{2}}(\overline{M}, \overline{M}^{op}) \simeq \mathscr{Q}(\overline{M})$ and $\beta \circ \gamma_{/\alpha_0(Q')} = \mathrm{id}_{\alpha_0(Q')}$. By Lemma 4, $\mathrm{id}_{\alpha_0(Q')}$ extends both to $\beta \circ \gamma$ and to $\mathrm{id}_{\mathscr{S}\mathscr{M}_{\overline{M}}(Q')}$, i.e. from the uniqueness we have $\beta \circ \gamma = \mathrm{id}_{\mathscr{S}\mathscr{M}_{\overline{M}}(Q')}$. The multiplier part is evident. □

Theorem 9. *Let Q' be an involutive quantale with a maximal essential semiextension (extension) $\alpha_0 : Q' \to \mathscr{Q}(\overline{M})$ over $\mathbf{2}$. Then $\mathscr{M}_{\overline{M}}(Q') \simeq \mathscr{S}\mathscr{M}_{\overline{M}}(Q')$ $\simeq \{T \in \mathrm{bicomm}(\alpha_0(Q')) : T \circ \alpha_0(Q') \subseteq \alpha_0(Q')$ and $\alpha_0(Q') \circ T \subseteq \alpha_0(Q')\}$, i.e. any maximal essential semimultiplier is isomorphic to any maximal essential multiplier and both are isomorphic to the involutive quantale $\mathscr{M}(Q') =$*

$$\{T \in \mathrm{bicomm}(\alpha_0(Q')) : T \circ \alpha_0(Q') \subseteq \alpha_0(Q') \text{ and } \alpha_0(Q') \circ T \subseteq \alpha_0(Q')\}.$$

Proof. Note that we have a one-to-one correspondence between $\alpha_0(a)^{\circledast}$ and $\alpha_0(a^*)$, for all $a \in Q'$, since $\alpha_0(a)^{\circledast} = \alpha_0(b)^{\circledast}$ implies $\alpha_0(a) = \alpha_0(b)$, i.e. $a = b$. Any element $S \in \mathrm{Hom}_{\mathbf{2}}(\overline{M}, \overline{M}^{\circledast})$ is of the form $S = \iota_{\overline{M}} \circ T$, $T \in \mathscr{Q}(\overline{M})$, and the same holds for the module actions. So we have $\mathscr{S}\mathscr{M}_{\overline{M}}(Q') \simeq \mathscr{M}_{\overline{M}}(Q')$. Evidently, $\mathscr{M}_{\overline{M}}(Q') \supseteq \{T \in \mathrm{bicomm}(\alpha_0(Q')) : T \circ \alpha_0(Q') \subseteq \alpha_0(Q')$ and $\alpha_0(Q') \circ T \subseteq \alpha_0(Q')\}$. Let us show that $\mathscr{M}_{\overline{M}}(Q') \subseteq \mathrm{bicomm}(\alpha_0(Q'))$. Assume $T \in \mathscr{M}_{\overline{M}}(Q')$, $R \in \mathrm{comm}(\alpha_0(Q'))$, $a, b \in Q'$. Then

$$\begin{array}{rcccl} T \circ R \circ \alpha_0(b) & = & T \circ \alpha_0(b) \circ R & = & R \circ T \circ \alpha_0(b), \\ \alpha_0(a) \circ T \circ R & = & R \circ \alpha_0(a) \circ T & = & \alpha_0(a) \circ R \circ T \end{array}$$

i.e. $T \circ R = R \circ T$, i.e. $T \in \mathrm{bicomm}(\alpha_0(Q'))$. □

Theorem 10. *Let Q be an involutive quantale, M a Hilbert Q-module such that $M = \{\bigvee_{i \in I} \langle m_i, p_i \rangle \bullet r_i : m_i, p_i, r_i \in M\}$. Then $\mathscr{A}_Q(M) \simeq \mathscr{M}(\mathscr{K}_Q(M))$ and $\mathrm{Hom}_Q(M, M^{\circledast}) \simeq \mathscr{S}\mathscr{M}(\mathscr{K}_Q(M))$.*

Proof. We have an embedding (inclusion) $\rho_M : \mathscr{K}_Q(M) \to \mathscr{A}_Q(M)$ and evidently this embedding is non-degenerate. So we have

$$\begin{aligned} \mathscr{M}(\mathscr{K}_Q(M)) \simeq \{T \in \mathrm{Hom}_Q(M, M^{\circledast}) : T \circ \iota_M \circ \mathscr{K}_Q(M) \subseteq \iota_M \circ \mathscr{K}_Q(M) \text{ and} \\ (\mathscr{K}_Q(M))^{\circledast} \circ T \subseteq \iota_M \circ \mathscr{K}_Q(M)\} = \mathrm{Hom}_Q(M, M^{\circledast}) \end{aligned}$$

and

$$\begin{aligned} \mathscr{M}(\mathscr{K}_Q(M)) \simeq \{\, T \in \mathscr{A}_Q(M) : T \circ \mathscr{K}_Q(M) \subseteq \mathscr{K}_Q(M) \text{ and} \\ \mathscr{K}_Q(M) \circ T \subseteq \mathscr{K}_Q(M)\} = \mathscr{A}_Q(M). \end{aligned}$$ □

Corollary 11. *Let Q be a non-degenerate involutive quantale. Then $\mathscr{A}_Q(Q) \simeq \mathscr{M}(\mathscr{K}_Q(Q))$.*

Proof. We have that Q is a Hilbert Q-module with the standard multiplication satisfying the assumptions of Theorem 10. The rest is evident. □

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