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INTERIOR TENSOR PRODUCT OF HILBERT MODULES

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Abstract. The interior tensor product in the context of the theory of Hilbert modules over involutive quantales is studied. Associativity, functoriality and representability via compact maps of the interior tensor product are established.

It is the purpose of this article to give an analog of the interior tensor product for Hilbert C*-modules in the context of involutive quantales.

This paper is closely related to the papers [2] and [3] (the same volume of Contributions to General Algebra as this paper) where the interested reader can find unexplained terms, notation and general properties concerning the Hilbert modules over involutive quantales. For facts concerning quantales and quantale modules in general we refer to [7]. We shall also assume some familiarity with the theory of tensor products of sup-semilattices. For motivation on Hilbert C*-modules and transfered notations to Hilbert modules over involutive quantales we recommend the first three chapters of [1].

The paper is organized as follows. First, we recall the notions not introduced in [2] and [3]. Then we recast the theory of the interior tensor product for Hilbert C*-modules (see [1]) in the quantale language of Hilbert A-modules over an involutive quantale A and deduce corresponding results here. We establish basic properties of the interior tensor product – associativity, functoriality and representability via compact maps.

In order to allow applications to involutive quantales, the interior tensor product plays a crucial role in proving the Eilenberg-Watts theorem and the fundamental Morita theorem for Hilbert modules (see [4]). Moreover, we were able to introduce in [5], using the interior tensor product, the Rieffel induction for involutive quantales (for motivation see [6]), i.e. we obtained a construction of a representation π_A of A from a representation π_B of B.

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Preliminaries

Let us begin by establishing the common symbols and notations in this paper.

Let A be a quantale. We shall denote by MOD_A (${}_AMOD$) the category of right (left) $A-$modules with the right (left) action $\diamond$ ($\bullet$) and right (left) module homomorphisms. For a right A-module X in MOD_A the submodule $\mathrm{ess}(X) = X \diamond A$ generated by the elements $x \diamond a$ is called the *essential part* of X. If $\mathrm{ess}(X) = X$ we say that X is (*right*) *essential.* The full subcategory of essential A-modules is denoted $\mathrm{ess} - MOD_A$. We shall say that A is *right separating* for the A-module M if $m \diamond (-) = n \diamond (-)$ implies $m = n$. M is called *right faithful* if $(-) \diamond a = (-) \diamond b$ implies that $a = b$.

Note that we may dually define the notion of a left (essential, separating, faithful) A-module with a left multiplication $\bullet$.

The theory of right (left) Hilbert A-modules (we refer the reader to [2] for details and examples) is a generalization of the theory of complete semilattices with a duality and it is the natural framework for the study of modules over an involutive quantale A endowed with A-valued inner products. Note only that for each right Hilbert A-module M there is a *conjugate* left Hilbert A-module M^*: as an sup-semilattice M^* is just M with a left module multiplication $a \bullet m = m \diamond a^*$ and a left A-valued inner product ${}_A\langle m, n\rangle = \langle m, n\rangle_A$ (see Lemma 1.5 in [2]). Recall that a Hilbert A-module M is said to be *full* if $\langle M, M\rangle = \{\bigvee_{j\in J} \langle x_j, y_j\rangle : x_j, y_j \in M, j \in J\} = A$. We shall denote by $HMOD_A$ the category of right Hilbert $A-$modules with morphisms the usual A-module maps. The full subcategory of essential right Hilbert A-modules is denoted $\mathrm{ess} - HMOD_A$. If we write $A \in \mathrm{ess} - HMOD_A$ we mean that A is an essential right Hilbert A-module with respect to the usual quantale multiplication. Then, evidently $1 \cdot 1 = 1$ in A and $a \cdot (-) = b \cdot (-)$ implies $a = b$.

Let $f : M \to N$ be a map between right (left) Hilbert A-modules. We say that a map $g : N \to M$ is a **-adjoint to* f and f is *adjointable* if $\langle f(m), n\rangle = \langle m, g(n)\rangle$ for all $m \in M$, $n \in N$. Note that the *-adjoint g to f is uniquely determined. Namely, if $\langle f(m), n\rangle = \langle m, g(n)\rangle = \langle m, h(n)\rangle$ then $g(n) = h(n)$ for all n i.e. $g = h$. We then put $f^* = g$. Evidently, any adjointable map is a right (left) module homomorphism (see [2]). The set of all adjointable maps from M to N is denoted by $\mathcal{A}_A(M, N)$, $\mathcal{A}_A(M) = \mathcal{A}_A(M, M)$. We shall denote by $Hilb_A$ ($\mathrm{ess} - Hilb_A$, $\mathrm{ess}_f - Hilb_A$) the category of (essential, essential full) right Hilbert A-modules and adjointable maps.

Let B be an involutive quantale such that $\varphi\colon B \to \mathcal{A}_A(M)$ is an involutive quantale homomorphism. Then φ is said to be *nondegenerate* if $M = \varphi(B)(M)$, i.e. for any $m \in M$ we have that $m = \bigvee_{j\in J} \varphi(b_j)(m_j)$ for suitable elements $m_j \in M$ and $b_j \in B$, $j \in J$.

We say that an adjointable map $f : M \to N$ is an *isometry* if, for all $m_1, m_2 \in M$, $\langle m_1, m_2\rangle = \langle f(m_1), f(m_2)\rangle$. This is equivalent to $f^* \circ f = \mathrm{id}_M$. Similarly, an adjointable map $f : M \to N$ is *unitary* if $f^* \circ f = \mathrm{id}_M$ and $f \circ f^* = \mathrm{id}_N$. Note that any surjective isometry is necessarily unitary.

Recall that from [2] we know that, for any right Hilbert A-module M and for all $m \in M$, the map $m^\sim : A \to M$ defined by $a \mapsto m \diamond a$ has a *-adjoint $m^\star : M \to A$ defined by $n \mapsto \langle m, n\rangle$. The maps of the form

$$T = \bigvee_{j\in J} \Theta_{n_j, m_j},$$

here $\Theta_{n,m} = n \diamond \langle m, -\rangle = n^\sim \circ m^\star$, $m \in M, n \in N$ are said to be *compact maps* and the set of all compact maps will be denoted $\mathcal{K}_A(M, N)$ or just $\mathcal{K}_A(M)$ in case $M = N$. Note that any compact map is an adjointable map and that a composition of a compact map with an adjointable one is again compact, i.e. $\mathcal{K}_A(N, P) \circ \mathcal{A}_A(M, N) \subseteq \mathcal{K}_A(M, P)$ and $\mathcal{A}_A(N, P) \circ \mathcal{K}_A(M, N) \subseteq \mathcal{K}_A(M, P)$. An expository treatment of compact maps on Hilbert A-modules may be found in [2].

1. Interior tensor product of Hilbert modules

Definition 1.1. *Let A and B be involutive quantales, and let F and X be right Hilbert B-modules. We say that F is a* right Hilbert A–B bimodule *if it is a left A-module satisfying*

$$a \bullet (x \diamond b) = (a \bullet x) \diamond b \quad \textit{and} \quad \langle a \bullet x, y\rangle_B = \langle x, a^* \bullet y\rangle_B$$

for all $a \in A$, $x, y \in F$, and $b \in B$.

We say that a (full essential) right Hilbert $A - B$ bimodule X is an (imprimitivity) $A - B$ bimodule *if X is also a (full essential) left Hilbert A-module in such a way that*

$${}_A\langle x, y\rangle \bullet z = x \diamond \langle y, z\rangle_B \text{ and } {}_A\langle x \diamond b, y\rangle = {}_A\langle x, y \diamond b^*\rangle .$$

Recall that the property $a \bullet (x \diamond b) = (a \bullet x) \diamond b$ is redundant — it follows from the property $\langle a \bullet x, y\rangle_B = \langle x, a^* \bullet y\rangle_B$ (see Lemma 1.2) — but we include it to emphasize that F is a balanced bimodule.

Lemma 1.2. *Let A and B be involutive quantales, and let F be a right Hilbert B-module. Then F is a right Hilbert A–B bimodule if and only if we have an involutive quantale homomorphism $\varphi\colon A \to \mathcal{A}_B(F)$ such that $a \bullet x = \varphi(a)(x)$.*

Proof. Let F be a right Hilbert A–B bimodule. We put $\varphi(a)(x) = a \bullet x$. Let us show that $\varphi(a) \in \mathcal{A}_B(F)$. We have

$$\langle \varphi(a)(x), y\rangle = \langle a \bullet x, y\rangle = \langle x, a^* \bullet x\rangle = \langle x, \varphi(a^*)(y)\rangle,$$

i.e. the adjointable map to $\varphi(a)$ is $\varphi(a^*)$. Hence $\varphi(a)^* = \varphi(a^*)$. Evidently, φ preserves arbitrary joins and multiplication since F is a left A-module.

Conversely, let $\varphi\colon A \to \mathcal{A}_B(F)$ be an involutive quantale homomorphism and let us put $a \bullet x = \varphi(a)(x)$. Then F is a left A-module. Let $a \in A, b \in B$, $x, y \in F$. Then $a \bullet (x \diamond b) = \varphi(a)(x \diamond b) = \varphi(a)(x) \diamond b = (a \bullet x) \diamond b$ and $\langle a \bullet x, y\rangle = \langle \varphi(a)(x), y\rangle = \langle x, \varphi(a^*)(y)\rangle = \langle x, a^* \bullet x\rangle$. $\square$

Example 1.3. 1. A sup-semilattice S with a duality is an imprimitivity $\mathcal{K}_{\mathbf{2}}(S) - \mathbf{2}$ bimodule with ${}_{\mathcal{K}_{\mathbf{2}}(S)}\langle s, t\rangle = \Theta_{s,t}$ (see [2], Lemma 2.12).
2. If B is an involutive quantale, $B \in HMOD_B$ then B becomes a right-Hilbert B–B bimodule by putting

$$a \bullet b = a \cdot b, b \diamond c = b \cdot c \quad \text{and} \quad \langle a, b\rangle_B = a^* b$$

for $a, b, c \in B$. More generally, if $\varphi\colon A \to \mathcal{A}_B(B)$ is an involutive quantale homomorphism, then B becomes a right Hilbert A–B bimodule with a left action given by

$$a \bullet b = \varphi(a)(b).$$

3. Let X be a right Hilbert B-module. Then we have an involutive quantale embedding $\iota_X : \mathcal{K}_B(X) \to \mathcal{A}_B(X)$ and the identity map $\mathrm{id}_{\mathcal{A}_B(X)} : \mathcal{A}_B(X) \to \mathcal{A}_B(X)$. Hence X is both a right Hilbert $\mathcal{K}_B(X) - B$ bimodule and a right Hilbert $\mathcal{A}_B(X) - B$ bimodule.

Moreover, if X is essential and full as a right Hilbert B-module then X is essential as a left $\mathcal{K}_B(X)(\mathcal{A}_B(X))$-module, i.e. ι_X ($\mathrm{id}_{\mathcal{A}_B(X)}$) are nondegenerate involutive quantale homomorphisms. In fact, X is then an imprimitivity $\mathcal{K}_B(X) - B$ bimodule; here ${}_{\mathcal{K}_B(X)}\langle x, y\rangle = \Theta_{x,y}$.
4. Let X, Y be full essential right Hilbert B-modules. Then $\mathcal{K}_B(Y, X)$ may be regarded as an imprimitivity $\mathcal{K}_B(X) - \mathcal{K}_B(Y)$ bimodule.

Lemma 1.4. *Let A and B be involutive quantales, A essential, and let X be a right essential full Hilbert A–B bimodule. Then X is an imprimitivity A–B bimodule if and only if we have an involutive quantale isomorphism $\varphi\colon A \to \mathcal{K}_B(X)$ such that $a \bullet x = \varphi(a)(x)$.*

Proof. Let X be an imprimitivity A–B bimodule, $x, y \in X$. Then $\varphi({}_A\langle x, y\rangle) = {}_A\langle x, y\rangle \bullet (-) = x \diamond \langle y, -\rangle_B = \Theta_{x,y}$. Since A is essential and any element from $\mathcal{K}_B(X)$ is a join of elements of the form $\Theta_{x,y}$, φ is a surjective involutive quantale homomorphism. Let $\varphi(a_1) = \varphi(a_2)$, $a_1 = \bigvee_{j\in J} {}_A\left\langle x_j^1, y_j^1\right\rangle$, $a_2 = \bigvee_{k\in K} {}_A\langle x_k^2, y_k^2\rangle$. Then, for all $z, w \in X$, ${}_A\left\langle \bigvee_{j\in J} \left\langle x_j^1, y_j^1\right\rangle \bullet z, w\right\rangle = {}_A\langle \bigvee_{k\in K} \langle x_k^2, y_k^2\rangle \bullet z, w\rangle$. Hence $\bigvee_{j\in J} {}_A\left\langle x_j^1, y_j^1\right\rangle \cdot {}_A\langle z, w\rangle = \bigvee_{k\in K} {}_A\langle x_k^2, y_k^2\rangle \cdot$

${}_A\langle z, w\rangle$, i.e. $a_1 = a_2$. Thus φ is an isomorphism. The converse follows from the preceding example, part 3. $\square$

Proposition 1.5. *Let A, B be involutive quantales. Suppose that E is a left Hilbert A-module and F is a right Hilbert $A - B$ bimodule. Then the sup-semilattice of left A-compact maps ${}_A\mathcal{K}(E, F)$ generated by the elements ${}^L\Theta_{y,x} = {}_A\langle -, x\rangle \bullet y$, $x \in E$, $y \in F$ and equipped with the natural right B-module action $(f \diamond b)(m) = f(m) \diamond b$, for all $m \in E$ and all $b \in B$, and with the inner product given by*

$$\langle \Theta_{y_1,x_1}, \Theta_{y_2,x_2}\rangle_{\mathcal{K}_A(E,F)} = \langle y_1, \langle x_1, x_2\rangle \bullet y_2\rangle$$

for all $x_1, x_2 \in E$ and $y_1, y_2 \in F$ is a right Hilbert B-module.

Proof. Since E is a left Hilbert A module we shall work with the maps ${}^L\Theta_{y,x} = \langle -, x\rangle \bullet y$ and ${}_A\mathcal{K}(E, F)$ is a sup-subsemilattice of the sup-lattice ${}_AMOD(E, F)$ of homomorphisms between left modules E and F generated by elements of the form $\Theta_{y,x}$. Evidently, ${}_A\mathcal{K}(E, F)$ is a B-module. The rest proceeds similarly as in the Proposition 1.20 in [2]. $\square$

Proposition 1.6. *Let A, B be involutive quantales. Suppose that E is a right Hilbert A-module, F is a right Hilbert $A-B$ bimodule. Then the sup-semilattice tensor product $E \otimes F$ is a pre-Hilbert B-module under the pre-inner product given on simple tensors by*

$$\langle x_1 \otimes y_1, x_2 \otimes y_2\rangle_{E\otimes F} = \langle y_1, \langle x_1, x_2\rangle \bullet y_2\rangle$$

for all $x_1, x_2 \in E$ and $y_1, y_2 \in F$.

Proof. Let us recall that from the definition of $\langle -, -\rangle_{E\otimes F}$ on generators we get a map from $(E\otimes F)\times(E\otimes F)$ to B that preserves joins in each variable. The action of B is given by $(x\otimes y) \diamond b = x\otimes(y \diamond b)$, for all $b \in B$, $x \in E$ and $y \in F$. Let $u \in E$ and $v \in F$. Then, we have $\langle u \otimes v, (x \otimes y) \diamond b\rangle = \langle u \otimes v, x \otimes (y \diamond b)\rangle = \langle v, \langle u, x\rangle \bullet (y \diamond b)\rangle = \langle v, (\langle u, x\rangle \bullet y) \diamond b\rangle = \langle v, \langle u, x\rangle \bullet y\rangle \cdot b = \langle u \otimes v, x \otimes y\rangle \cdot b$ and similarly, $\langle u \otimes v, x \otimes y\rangle^* = \langle v, \langle u, x\rangle \bullet y\rangle^* = \langle\langle u, x\rangle \bullet y, v\rangle = \langle y, \langle u, x\rangle^* \bullet v\rangle = \langle y, \langle x, u\rangle \bullet v\rangle = \langle x \otimes y, u \otimes v\rangle$. $\square$

$(E \otimes F)_{R_H}$ is by definition the quotient of $E \otimes F$, the elements of which are the top elements of equivalence classes of the relation $R_H = \{(U, V) \in (E \otimes F) \times (E \otimes F) : \langle U, W\rangle = \langle V, W\rangle \text{ for all } W \in E \otimes F\}$. We shall denote by $j_{R_H} : E \otimes F \to (E \otimes F)_{R_H}$ the corresponding module nucleus; $j_{R_H}(U) = \bigvee\{V \in E \otimes F : (U, V) \in R_H\}$.

Definition 1.7. *Let A, B be involutive quantales. Suppose that E is a right Hilbert A-module, F is a right Hilbert $A - B$ bimodule. Then the Hilbert B-module $(E \otimes F)_{R_H}$ will be denoted by $E\dot{\otimes}_A F$ and we shall speak about the* interior module tensor product over A.

We shall sometimes write $x\dot{\otimes}_A y$ for the element $j_{R_H}(x\otimes y)$. Recall that, for all $a\in A$ and all $x\in E, y\in F$, we have that $(x\diamond a)\dot{\otimes}_A y = x\dot{\otimes}_A a\bullet y$. Namely, we have that, for all $u\in E$ and $v\in F$,

$$\begin{aligned}\langle u\otimes v, (x\diamond a)\otimes y\rangle &= \langle v, \langle u, x\diamond a\rangle \bullet y\rangle = \langle v, (\langle u, x\rangle\cdot a)\bullet y\rangle\\ &= \langle v, \langle u, x\rangle \bullet (a\bullet y)\rangle = \langle u\otimes v, x\otimes a\bullet y\rangle\,,\end{aligned}$$

i.e. $((x\diamond a)\otimes y, x\otimes a\bullet y)\in R_H$.

Similarly, for all $b\in B$ and all $x\in E, y\in F$, we have that $x\dot{\otimes}_A(y\diamond b) = (x\dot{\otimes}_A y)\diamond b$. This follows from the fact that we have a B-module action $\diamond_{E\otimes F}: (E\otimes F)\times B\to E\otimes F$ defined by $(x\otimes y)\diamond_{E\otimes F} b = x\otimes(y\diamond b)$ and that j_{R_H} is a B-module map.

Let us point out that, for $F\in \mathrm{ess}-HMOD_B$, we have that $E\dot{\otimes}_A F\in \mathrm{ess}-HMOD_B$. Similarly, $E\dot{\otimes}_A F$ is full whenever E and F are full and F is left essential.

Moreover, for the interior tensor product construction we have

Proposition 1.8. *Let A, B be involutive quantales. Suppose further that E_1, E_2 are right Hilbert A-modules, F_1, F_2 are right Hilbert $A-B$ bimodules, that $f: E_1\to E_2$ is an adjointable right A-module map, $g: F_1\to F_2$ is an adjointable right B-module map and a left A-module map. Then $f\dot{\otimes}_A g: E_1\dot{\otimes}_A F_1\to E_2\dot{\otimes}_A F_2$ is an adjointable right B-module map.*

Proof. Let us assume that f and g are both adjointable. We define $f\dot{\otimes}_A g = j^2_{R_H}\circ(f\otimes g)\circ {j^1_{R_H}}^{\vdash}$; here ${j^1_{R_H}}^{\vdash}$ is the right adjoint to $j^1_{R_H}$ in posets, i.e. $(f\dot{\otimes}_A g)(x) = j^2_{R_H}(f\otimes g)(x)$, for all $x\in E_1\dot{\otimes}_A F_1\subseteq E_1\otimes F_1$. We shall prove that $(f\dot{\otimes}_A g)$ is adjointable. We have

$$\begin{aligned}&\langle (f\otimes g)(p\otimes q), u\otimes v\rangle = \langle f(p)\otimes g(q)), u\otimes v\rangle = \langle g(q), \langle f(p), u\rangle \bullet v\rangle\\ &= \langle \langle f(p), u\rangle^* \bullet g(q), v\rangle = \langle \langle u, f(p)\rangle \bullet g(q), v\rangle = \langle g(\langle u, f(p)\rangle \bullet q), v\rangle\\ &= \langle \langle f^*(u), p\rangle \bullet q, g^*(v)\rangle = \langle q, \langle p, f^*(u)\rangle \bullet g^*(v)\rangle = \langle p\otimes q, (f^*\otimes g^*)(u\otimes v)\rangle\,.\end{aligned}$$

Applying the above computation we obtain

$$\begin{aligned}&\langle (f\dot{\otimes}_A g)(s\dot{\otimes}_A t), m\dot{\otimes}_A n\rangle = \bigvee\nolimits_{p\otimes q\le s\dot{\otimes}_A t, u\otimes v\le m\dot{\otimes}_A n} \langle (f\otimes g)(p\otimes q), u\otimes v\rangle\\ &= \bigvee\nolimits_{p\otimes q\le s\dot{\otimes}_A t, u\otimes v\le m\dot{\otimes}_A n} \langle p\otimes q, (f^*\otimes g^*)(u\otimes v)\rangle = \left\langle s\dot{\otimes}_A t, (f^*\dot{\otimes}_A g^*)(m\dot{\otimes}_A n)\right\rangle,\end{aligned}$$

i.e. the *-adjoint to $f\dot{\otimes}_A g$ is the map $f^*\dot{\otimes}_A g^*$. In fact, from the above computation we obtain that for adjointable f and g we have $(f\dot{\otimes}_A g)(s\dot{\otimes}_A t) = f(s)\dot{\otimes}_A g(t)$. □

Definition 1.9. *Let A, B be involutive quantales. We say that $\Psi: Hilb_A\to Hilb_B$ is a SUP-functor if $\Psi(\bigvee_{i\in I} T_i) = \bigvee_{i\in I}\Psi(T_i)$ for all $\{T_i : i\in I\}\subseteq \mathcal{A}_A(M, N)$, $M, N\in Hilb_A$. A SUP-functor Ψ is a* *-functor *if $\Psi(T^*) = \Psi(T)^*$ for all adjointable maps T.*

Theorem 1.10. *Let A, B be involutive quantales. Suppose that F is a right Hilbert $A-B$ bimodule. Then the function $(-)\dot{\otimes}_A F: Hilb_A\to Hilb_B$ which* *assigns to each right Hilbert A-module E the inner tensor product $E\dot{\otimes}_A F$ and to each adjointable map f between right Hilbert A-modules the adjointable map $f\dot{\otimes}_A \mathrm{id}_F$ between right Hilbert B-modules is a *-functor. If moreover F is left faithful, i.e. the induced map $\varphi : A \to \mathcal{A}_B(F)$ is an injective homomorphism of involutive quantales then $(-)\dot{\otimes}_A F$ is faithful on adjointable mappings.*

Proof. Note that evidently $(g \circ f)\dot{\otimes}_A \mathrm{id}_F = (g\dot{\otimes}_A \mathrm{id}_F) \circ (f\dot{\otimes}_A \mathrm{id}_F)$. Since we have $(\mathrm{id}_E\dot{\otimes}_A \mathrm{id}_F)(s\dot{\otimes}_A t) = j_{R_H}(s\dot{\otimes}_A t) = s\dot{\otimes}_A t$, $(-)\dot{\otimes}_A F$ is a functor.

Evidently, since $(-)\dot{\otimes}_A F$ preserves arbitrary joins it is a SUP-functor. Since interior tensor product preserves adjointability it is a *-functor. Suppose that F is left faithful. Let $t_1, t_2 \in \mathcal{A}_A(E_1, E_2)$ such that $t_1\dot{\otimes}_A \mathrm{id}_F = t_2\dot{\otimes}_A \mathrm{id}_F$. Then, for all $x \in E_1, u \in E_2$ and all $y, v \in F$, we have that $\langle y, \langle t_1(x), u\rangle \bullet v\rangle = \big\langle t_1(x)\dot{\otimes}_A y, u\dot{\otimes}_A v\big\rangle = \big\langle (t_1\dot{\otimes}_A \mathrm{id}_F)(x\dot{\otimes}_A y), u\dot{\otimes}_A v\big\rangle = \big\langle (t_2\dot{\otimes}_A \mathrm{id}_F)(x\dot{\otimes}_A y), u\dot{\otimes}_A v\big\rangle = \big\langle t_2(x)\dot{\otimes}_A y, u\dot{\otimes}_A v\big\rangle = \langle y, \langle t_2(x), u\rangle \bullet v\rangle$, i.e. $\langle t_1(x), u\rangle \bullet v = \langle t_2(x), u\rangle \bullet v$. Then $\langle t_1(x), u\rangle = \langle t_2(x), u\rangle$, i.e. $t_1(x) = t_2(x)$. Hence $t_1 = t_2$. $\square$

Corollary 1.11. *Let A, B be involutive quantales. Suppose that E is a right Hilbert A-module, F is a right Hilbert $A - B$ bimodule with the induced homomorphism $\varphi : A \to \mathcal{A}_B(F)$ of involutive quantales. Then, for all $t \in \mathcal{A}_A(E)$, we have a map $t\dot{\otimes}_A \mathrm{id}_F : E\dot{\otimes}_A F \to E\dot{\otimes}_A F$ defined on generators by $(t\dot{\otimes}_A \mathrm{id}_F)(x\dot{\otimes}_A y) = t(x)\dot{\otimes}_A y$. This gives us a unital involutive quantale homomorphism $\varphi_* : \mathcal{A}_A(E) \to \mathcal{A}_B(E\dot{\otimes}_A F)$, $t \mapsto t\dot{\otimes}_A \mathrm{id}_F$. If φ is injective then so is φ_*.*

Proposition 1.12. *Let A, B be involutive quantales. Suppose that E_1 is a right Hilbert A-module, E_2 is a right essential Hilbert A-module, F is a right Hilbert $A - B$ bimodule with the induced involutive quantale homomorphism $\varphi : A \to \mathcal{K}_B(F)$. Then we have the restriction $(-)^{\mathcal{K}}\dot{\otimes}_A \mathrm{id}_F : \mathcal{K}_A(E_1, E_2) \to \mathcal{K}_B(E_1\dot{\otimes}_A F, E_2\dot{\otimes}_A F)$. If φ is surjective (injective) then so is $(-)^{\mathcal{K}}\dot{\otimes}_A \mathrm{id}_F$.*

Proof. Let us recall that φ is also a map from A to $\mathcal{A}_B(F)$. Applying the same procedure as for C*-modules in [1], Lemma 4.6 and Proposition 4.7, we may define, for any element $x \in E_j$, $1 \leq j \leq 2$ an adjointable map $\Theta_x : F \to E_j\dot{\otimes}_A F$, $f \mapsto x\dot{\otimes}_A f$. Its adjoint is then the map $\Theta_x^* : E_j\dot{\otimes}_A F \to F$, $u\dot{\otimes}_A v \mapsto \varphi(\langle x, u\rangle)(v)$. One can easily check that $(\Theta_{x \diamond a, y})\dot{\otimes}_A \mathrm{id}_F = \Theta_x \circ \varphi(a) \circ \Theta_y^*$ and that this element is in $\mathcal{K}_B(E_1\dot{\otimes}_A F, E_2\dot{\otimes}_A F)$ since $\varphi(a) \in \mathcal{K}_B(F)$. For completeness, we sketch the arguments of [1]. Let $a \in A$, $y, u \in E_1, x \in E_2$ and $v \in F$. Then

$$\begin{aligned}(\Theta_x \circ \varphi(a) \circ \Theta_y^*)(u\dot{\otimes}_A v) &= \Theta_x(\varphi(a)(\varphi(\langle y, u\rangle)(v))) = x\dot{\otimes}_A(\varphi(a)(\varphi(\langle y, u\rangle)(v))) \\ &= x \diamond (a \cdot \langle y, u\rangle)\dot{\otimes}_A v = \Theta_{x \diamond a, y}(u)\dot{\otimes}_A v = (\Theta_{x \diamond a, y}\dot{\otimes}_A \mathrm{id}_F)(u\dot{\otimes} v).\end{aligned}$$

By $E_2 \diamond A = E_2$, we have that any $\Theta_{x,y}\dot{\otimes}_A \mathrm{id}_F \in \mathcal{K}_B(E_1\dot{\otimes}_A F, E_2\dot{\otimes}_A F)$.

Suppose that φ is surjective. Let $y \in E_1$, $x \in E_2$ and $z, w \in F$. Then we can find an element $a \in A$ such that $\varphi(a) = \Theta_{z,w}$. Let $u \in E_1$ and $v \in F$.

Then

$$\begin{aligned}\Theta_{x\dot{\otimes}_A z, y\dot{\otimes}_A w}(u\dot{\otimes}_A v) &= (x\dot{\otimes}_A z) \diamond \left\langle y\dot{\otimes}_A w, u\dot{\otimes}_A v\right\rangle = (x\dot{\otimes}_A z) \diamond \langle w, \varphi(\langle y, u\rangle)(v)\rangle \\ &= x\dot{\otimes}_A(z \diamond \langle w, \varphi(\langle y, u\rangle)(v)\rangle) = x\dot{\otimes}_A(\Theta_{z,w}(\varphi(\langle y, u\rangle)(v)) \\ &= x\dot{\otimes}_A(\varphi(a)(\varphi(\langle y, u\rangle)(v)) = x\dot{\otimes}_A(\varphi(a \cdot \langle y, u\rangle)(v)) \\ &= (\Theta_{x \,\diamond\, a, y}\dot{\otimes}_A \mathrm{id}_F)(u\dot{\otimes}_A v).\end{aligned}$$

Since $\mathcal{K}_B(E_1\dot{\otimes}_A F, E_2\dot{\otimes}_A F)$ is generated by the elements of the form $\Theta_{x\dot{\otimes}_A z, y\dot{\otimes}_A w}$ it follows that $(-)^{\mathcal{K}}\dot{\otimes}_A \mathrm{id}_F$ is surjective. The injectivity follows from the Proposition 1.10. □

Corollary 1.13. *Let A, B be involutive quantales. Suppose that E is a right essential full Hilbert A-module, F is a right Hilbert $A-B$ bimodule and $\varphi : A \to \mathcal{K}_B(F)$ is the induced nondegenerate involutive quantale homomorphism. Then the restriction $\varphi_*^{\mathcal{K}} : \mathcal{K}_A(E) \to \mathcal{K}_B(E\dot{\otimes}_A F)$ of φ_* is a unital nondegenerate involutive quantale homomorphism. If φ is surjective (injective) then so is $\varphi_*^{\mathcal{K}}$.*

Proof. Evidently, $\varphi_*^{\mathcal{K}}$ is a unital involutive quantale homomorphism. Let us show that it is nondegenerate. Let $x, y, u \in E$, $v \in F$. Then we have $\varphi_*(\Theta_{x,y})(u\dot{\otimes}_A v) = \Theta_{x,y}(u)\dot{\otimes}_A v = x \diamond \langle y, u\rangle \dot{\otimes}_A v = x\dot{\otimes}_A \varphi(\langle y, u\rangle)(v)$. Since E is full and φ is nondegenerate then any element of $E\dot{\otimes}_A F$ is a join of elements of this form. □

Recall that the cartesian product $\prod_{j\in J} E_j$ of Hilbert modules coincides as usual for quantale modules with their sum $\coprod_{j\in J} E_j$. The respective projections π_j and injections i_j are defined by $\pi_j((x_k)_{k\in J}) = x_j$, $\pi_k(i_j(x_j)) = x_j$ if $j = k$ and $\pi_k(i_j(x_j)) = 0$ if $j \neq k$.

Lemma 1.14. *Let A, B be involutive quantales. Suppose that $E_j, j \in J$ are Hilbert A-modules, F is a Hilbert $A-B$ bimodule. Then we have a unitary isomorphism $\coprod_{j\in J}(E_j\dot{\otimes}_A F) \cong \left(\coprod_{j\in J} E_j\right)\dot{\otimes}_A F$.*

Proof. Note that we have unitary maps $h : \left(\coprod_{j\in J} E_j\right)\dot{\otimes}_A F \to \coprod_{j\in J}(E_j\dot{\otimes}_A F)$ defined by $(m_j)_{j\in J}\dot{\otimes}_A n \mapsto (m_j\dot{\otimes}_A n)_{j\in J}$, $\overline{h} : \coprod_{j\in J}(E_j\dot{\otimes}_A F) \to \left(\coprod_{j\in J} E_j\right)\dot{\otimes}_A F$ defined by $(m_j\dot{\otimes}_A n_j)_{j\in J} \mapsto \bigvee_{j\in J} i_j(m_j)\dot{\otimes}_A n_j$. Evidently, h and $\overline{h}$ are inverse of each other. So $\overline{h} = h^*$. □

Lemma 1.15. *Let A, B, C be involutive quantales. Suppose that F is a right Hilbert A–B bimodule and G is a right Hilbert B–C bimodule. Then the interior tensor product $F \otimes_B G$ is a right Hilbert A–C bimodule under the left action given on simple tensors by $a \bullet (n\dot{\otimes}_B o) = (a \bullet n)\dot{\otimes}_B o$ for all $m \in F$ and $n \in G$.*

Proof. First, let us check that our definition is correct. Let $\bigvee_{j\in J}(n_j\dot{\otimes}_B o_j) = \bigvee_{k\in I}(p_k\dot{\otimes}_B r_k)$. Then

$$\begin{aligned}\langle\bigvee_{j\in J}(a\bullet n_j\dot{\otimes}_B o_j), u\dot{\otimes}_B v\rangle &= \bigvee_{j\in J}\langle o_j, \langle a\bullet n_j, u\rangle \bullet v\rangle\\ &= \bigvee_{j\in J}\langle o_j, \langle n_j, a^*\bullet u\rangle \bullet v\rangle = \left\langle\bigvee_{j\in J} n_j\dot{\otimes}_B o_j, (a^*\bullet u)\dot{\otimes}_B v\right\rangle\\ &= \left\langle\bigvee_{k\in I} p_k\dot{\otimes}_B r_k, (a^*\bullet u)\dot{\otimes}_B v\right\rangle = \bigvee_{k\in I}\langle r_k, \langle p_k, a^*\bullet u\rangle \bullet v\rangle\\ &= \bigvee_{k\in I}\langle r_k, \langle a\bullet p_k, u\rangle \bullet v\rangle = \left\langle\bigvee_{k\in I}(a\bullet p_k\dot{\otimes}_B r_k), u\dot{\otimes}_B v\right\rangle.\end{aligned}$$

From the definition of the left action and of the interior tensor product it is evident that $F\otimes_B G$ is a left A-module and a right Hilbert C-module. Let us check the following:

$$\begin{aligned}\left\langle a\bullet(n\dot{\otimes}_B o), u\dot{\otimes}_B v\right\rangle &= \left\langle (a\bullet n)\dot{\otimes}_B o, u\dot{\otimes}_B v\right\rangle = \left\langle n\dot{\otimes}_B o, (a^*\bullet u)\dot{\otimes}_B v\right\rangle\\ &= \left\langle n\dot{\otimes}_B o, a^*\bullet(u\dot{\otimes}_B v)\right\rangle.\end{aligned}$$

Then $F\otimes_B G$ is a right Hilbert A–C bimodule. $\square$

Lemma 1.16. *Let A, B be involutive quantales, $A\in HMOD_A$, F a right Hilbert $A-B$ bimodule such that $F\in \mathrm{ess}-{}_AMOD$. Then the canonical map $\chi_F : A\dot{\otimes}_A F\to F$ given by $a\dot{\otimes}_A n\mapsto a\bullet n$ is a unitary isomorphism of Hilbert B-modules and a bimodule isomorphism of right Hilbert $A-B$ bimodules.*

Proof. Let us show that the definition of χ_F is correct. Assume $\bigvee_{j\in J} a_j\dot{\otimes}_A y_j = \bigvee_{k\in I} b_k\dot{\otimes}_A z_k$. Since $F\in \mathrm{ess}-{}_AMOD$ any element of F is a join of elements of the form $c\bullet u$. We have

$$\begin{aligned}\left\langle\bigvee_{j\in J} a_j\bullet y_j, c\bullet u\right\rangle &= \bigvee_{j\in J}\langle a_j\bullet y_j, c\bullet u\rangle = \bigvee_{j\in J}\left\langle y_j, (a_j^*\cdot c)\bullet u\right\rangle\\ &= \bigvee_{j\in J}\langle y_j, \langle a_j, c\rangle \bullet u\rangle = \left\langle\bigvee_{j\in J} a_j\dot{\otimes}_A y_j, c\dot{\otimes}_A u\right\rangle\\ &= \left\langle\bigvee_{k\in I} b_k\dot{\otimes}_A z_k, c\dot{\otimes}_A u\right\rangle = \left\langle\bigvee_{k\in I} b_k\bullet z_k, c\bullet u\right\rangle.\end{aligned}$$

We claim that χ_F is an adjointable map. Let us define a map $\kappa_F : F\to A\dot{\otimes}F$ by $\kappa_F(y) = \bigvee\{a\dot{\otimes}_A z : a\bullet z\leq y\}$. Then we have

$$\begin{aligned}&\langle\chi_F(\bigvee_{j\in J} a_j\dot{\otimes}_A y_j), y\rangle = \left\langle\bigvee_{j\in J} a_j\bullet y_j, \bigvee\{a\bullet z : a\bullet z\leq y\}\right\rangle\\ &= \bigvee\{\langle a_j\bullet y_j, a\bullet z\rangle : j\in J, a\bullet z\leq y\} = \bigvee\{\left\langle a_j\dot{\otimes}_A y_j, a\dot{\otimes}_A z\right\rangle : j\in J, a\diamond z\leq y\}\\ &= \left\langle\bigvee_{j\in J} a_j\dot{\otimes}_A y_j, \kappa_F(y)\right\rangle.\end{aligned}$$

To show that χ_F is a unitary isomorphism it is enough to note that it is a surjective inner-product preserving adjointable map. We have $a\bullet\chi_F(c\dot{\otimes}_A n) = a\bullet(c\bullet n) = (a\cdot c)\bullet n) = \chi_F(a\cdot c\dot{\otimes}_A n)$. Hence χ_F and also κ_F are bimodule isomorphisms. $\square$

We have also the following two important lemmas.

Lemma 1.17. *Let A be an involutive quantale, $A, E\in ess-HMOD_A$. Then the canonical map $\pi_E : E\dot{\otimes}_A A\to E$ given by $m\dot{\otimes}_A a\mapsto m\diamond a$ is a unitary isomorphism of Hilbert A-modules.*

Proof. First, let us show that the definition of π_E is correct. Assume $\bigvee_{j\in J} y_j\dot{\otimes}_A a_j = \bigvee_{k\in I} z_k\dot{\otimes}_A b_k$. Since $E \in \mathrm{ess} - HMOD_A$ any element of E is a join of elements of the form $u \diamond c$. We have

$$\begin{aligned}\left\langle \bigvee_{j\in J} y_j \diamond a_j, u\diamond c\right\rangle &= \bigvee_{j\in J} \langle y_j \diamond a_j, u \diamond c\rangle = \bigvee_{j\in J} a_j^* \cdot \langle y_j, u\rangle \cdot c\\ &= \bigvee_{j\in J} \langle a_j, \langle y_j, u\rangle \bullet c\rangle = \left\langle \bigvee_{j\in J} y_j\dot{\otimes}_A a_j, u\dot{\otimes}_A c\right\rangle\\ &= \langle \bigvee_{k\in I} z_k\dot{\otimes}_A b_k, u\dot{\otimes}_A c\rangle = \langle \bigvee_{k\in I} z_k \diamond b_k, u\diamond c\rangle .\end{aligned}$$

Now, we shall check that π_E is an adjointable map. Let us define a map $\iota_E : E \to E\dot{\otimes} A$ by $\iota_E(y) = \bigvee\{z\dot{\otimes}_A a : z\diamond a \leq y\}$. Then we have

$$\begin{aligned}&\langle \pi_E(\bigvee_{j\in J} y_j\dot{\otimes}_A a_j), y\rangle = \left\langle \bigvee_{j\in J} y_j \diamond a_j, \bigvee\{z\diamond a : z\diamond a \leq y\}\right\rangle\\ &= \bigvee\{\langle y_j \diamond a_j, z\diamond a\rangle : j\in J, z\diamond a\leq y\} = \bigvee\{\langle y_j\dot{\otimes}_A a_j, z\dot{\otimes}_A a\rangle : j \in J, z\diamond a \leq y\}\\ &= \left\langle \bigvee_{j\in J} y_j\dot{\otimes}_A a_j, \iota_E(y)\right\rangle .\end{aligned}$$

Since π_E is a surjective inner-product preserving adjointable map it is a unitary isomorphism. $\square$

Lemma 1.18. *Let A, B be involutive quantales. Suppose that E is a right Hilbert A-module and F is a right Hilbert $A-B$ bimodule. Then $\mathcal{K}_A(E^*, F) \cong E\dot{\otimes}_A F$ as right Hilbert B-modules.*

Proof. We shall prove that the natural maps $\delta_{E,F} : \mathcal{K}_A(E^*, F) \to E\dot{\otimes}_A F$ given by ${}^L\Theta_{f,e} \mapsto e\dot{\otimes}_A f$ and $\gamma_{E,F} : E\dot{\otimes}_A F \to \mathcal{K}_A(E^*, F)$ defined as $e\dot{\otimes}_A f \mapsto {}^L\Theta_{f,e}$ are unitary isomorphisms. First, let us check that they are correctly defined. Assume that $\bigvee_i \Theta_{f_i,e_i} = \bigvee_j \Theta_{v_j,u_j}$. We have

$$\begin{aligned}&\langle \delta_{E,F}(\bigvee_i \Theta_{f_i,e_i}), e\dot{\otimes}_A f\rangle = \langle \bigvee_i e_i\dot{\otimes}_A f_i, e\dot{\otimes}_A f\rangle = \bigvee_i \langle e_i\dot{\otimes}_A f_i, e\dot{\otimes}_A f\rangle\\ &= \langle \bigvee_i \langle e, e_i\rangle \bullet f_i, f\rangle = \left\langle \bigvee_j \langle e, u_j\rangle \bullet v_j, f\right\rangle = \left\langle \delta_{E,F}(\bigvee_j \Theta_{v_j,u_j}), e\dot{\otimes}_A f\right\rangle .\end{aligned}$$

Evidently, $\delta_{E,F}$ is inner-product preserving and surjective, i.e. it is a unitary map. The proof for $\gamma_{E,F}$ goes the same way. $\square$

Proposition 1.19. *Let A, B, C be involutive quantales. Suppose that E is a right Hilbert A-module, F is a right Hilbert A–B bimodule and G is a right Hilbert B–C bimodule. Then $(E\dot{\otimes}_A F)\dot{\otimes}_B G \cong E\dot{\otimes}_A(F\dot{\otimes}_B G)$, i.e. the interior tensor product is associative.*

Proof. Let us define a map $h : (E\dot{\otimes}_A F)\dot{\otimes}_B G \to E\dot{\otimes}_A(F\dot{\otimes}_B G)$ on simple tensors by $(m\dot{\otimes}_A n)\dot{\otimes}_B o \mapsto m\dot{\otimes}_A(n\dot{\otimes}_B o)$. First, let us check the correctness. Assume that $\bigvee_{j\in J}(m_j\dot{\otimes}_A n_j)\dot{\otimes}_B o_j = \bigvee_{k\in I}(r_k\dot{\otimes}_A s_k)\dot{\otimes}_B t_k$. We have then

$$\begin{aligned}&\langle m_j\dot{\otimes}_A(n_j\dot{\otimes}_B o_j), u \otimes_A (v\dot{\otimes}_B w)\rangle = \langle n_j\dot{\otimes}_B o_j, \langle m_j, u\rangle \bullet (v\dot{\otimes}_B w)\rangle\\ &= \langle n_j\dot{\otimes}_B o_j, (\langle m_j, u\rangle \bullet v)\dot{\otimes}_B w\rangle = \langle o_j, \langle n_j, \langle m_j, u\rangle \bullet v\rangle \bullet w\rangle\\ &= \langle o_j, \langle m_j\dot{\otimes}_A n_j, u\dot{\otimes}_A v\rangle \bullet w\rangle = \langle (m_j\dot{\otimes}_A n_j)\dot{\otimes}_B o_j, (u \otimes_A v)\dot{\otimes}_B w\rangle\end{aligned}$$

and similarly

$$\left\langle r_k\dot{\otimes}_A(s_k\dot{\otimes}_B t_k), u\otimes_A (v\dot{\otimes}_B w)\right\rangle = \left\langle (r_k\dot{\otimes}_A s_k)\dot{\otimes}_B t_k, (u\otimes_A v)\dot{\otimes}_B w\right\rangle.$$

This gives us that $h(\bigvee_{j\in J}(m_j\dot{\otimes}_A n_j)\dot{\otimes}_B o_j) = h(\bigvee_{k\in I}(r_k\dot{\otimes}_A s_k)\dot{\otimes}_B t_k)$. Similarly, we may correctly define a map $g : E\dot{\otimes}_A(F\dot{\otimes}_B G) \to (E\dot{\otimes}_A F)\dot{\otimes}_B G$ on simple tensors by $m\dot{\otimes}_A(n\dot{\otimes}_B o) \mapsto (m\dot{\otimes}_A n)\dot{\otimes}_B o$ and the maps are inverse of each other. They are also adjointable since we have

$$\begin{aligned}&\left\langle h((m\dot{\otimes}_A n)\dot{\otimes}_B o), u\otimes_A (v\dot{\otimes}_B w)\right\rangle = \left\langle m\dot{\otimes}_A(n\dot{\otimes}_B o), u\otimes_A (v\dot{\otimes}_B w)\right\rangle\\ &= \left\langle (m\dot{\otimes}_A n)\dot{\otimes}_B o, (u\otimes_A v)\dot{\otimes}_B w\right\rangle = \left\langle (m\dot{\otimes}_A n)\dot{\otimes}_B o, g(u\otimes_A (v\dot{\otimes}_B w))\right\rangle.\end{aligned}$$

So we have a unitary isomorphism between $(E\dot{\otimes}_A F)\dot{\otimes}_B G$ and $E\dot{\otimes}_A(F\dot{\otimes}_B G)$. $\square$

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