COVERS IN GENERALIZED FRAMES

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0. Introduction

The aim of this paper is to present a candidate for the so called 'partial frame' (see [10]) i.e. a meet-semilattice in which certain distinguished sets have joins distributing over finite meets. The basic reference for the present text is the classic book by Johnstone [5]. All basic definitions and conventions are taken from either in [5] or in [10]. A familiarity with [10] is assumed since we omit some of the proofs that we obtained as straightforward modifications of the corresponding proofs in that paper. For the motivating examples see the papers [1], [3] and [9]. For the theory of monads and basic facts concerning categories see [8].

The paper is divided in two parts. Section 1 introduces the notion of a partial frame. The monadicity of partial frames is established and a characterization of projective partial frames is given.

Section 2 contains the basic constructions of limits and colimits of partial frames. Especially, we give an explicit construction of a sum of partial frames.

Section 3 solves the problem of the preservation of some covering properties w.r. to sums of partial frames.

For F, G sets, F a finite subset of G we write $F \subseteq\subseteq G$. $\mathcal{M}eet$ denotes the category of meet-semilattices. Let $\mathcal{F}$ be a function with the domain $\mathcal{M}eet$ and the codomain $\mathcal{S}et$ such that $\mathcal{F}(L) \subseteq \mathcal{P}(L)$ (the system of all subsets of L). If $F \in \mathcal{F}(L)$, we shall speak about a set of a type $\mathcal{F}$ or an $\mathcal{F}$-set. If $X = \downarrow(F)$ for $F \in \mathcal{F}(L)$ then we call X to be an $\mathcal{F}$-generated set. The system of all nonempty $\mathcal{F}$-generated sets will be denoted by $\mathcal{F}_l(L)$. Recall that we have a meet-semilattice

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morphism $\downarrow$ from L to $\mathcal{F}_l(L)$ assigning to each element of L the principal ideal generated by it. For a subset m of L we define $\mathcal{P}_{\mathcal{F}}(m)= \{F \in \mathcal{F}(L) : F \subseteq m\}$.

1. Partial frames

In the sequel, let $\mathcal{F}$ be function with the domain $\mathcal{M}eet$ and the codomain $\mathcal{S}et$ satisfying the following conditions:

(A1) $L \in \mathcal{M}eet,\ t \in L$ implies $\{t\} \in \mathcal{F}(L),\ \emptyset \in \mathcal{F}(L) \subseteq \mathcal{P}(L)$,

(A2) $L \in \mathcal{M}eet, F, G \in \mathcal{F}(L)$ implies $F \wedge G \in \mathcal{F}(L)$,

(A3) $L \in \mathcal{M}eet,\ F \in \mathcal{F}(L),\ f \in F, f = \bigvee F_f$ for some $F_f \in \mathcal{F}(L)$ implies $\bigcup F_f \in \mathcal{F}(L)$,

(A4) $L, L' \in \mathcal{M}eet, h : L \to L'$ being a meet-semilattice morphism implies $h(\mathcal{F}(L)) = \mathcal{F}(h(L)) \subseteq \mathcal{F}(L')$,

(A5) $L \in \mathcal{M}eet, \emptyset \neq F \in \mathcal{F}(\mathcal{F}_l(L))$ implies $\bigcup\{f : f \in F\} \in \mathcal{F}_l(L)$.

Applying (A2) and (A3) we have that $\mathcal{F}_l(L)$ is a meet-semilattice closed under $\mathcal{F}$-joins and by (A4), we have that $F \in \mathcal{F}(L)$ implies $\downarrow(F) \in \mathcal{F}_l(L)$ ($\downarrow$ is a meet-semilattice morphism). Recall that the conditions **(A1)** , **(A2)** , **(A3)** , and **(A4)** are very close to the conditions (ii), (iii), (iv) and (v) in [4].

1.1 Definition. *We shall say that a meet-semilattice $L \in \mathcal{M}eet$ is a partial $\mathcal{F}$-frame (shortly a partial frame) over $\mathcal{M}eet$ if every set of the type $\mathcal{F}$ has a join and which satisfies the $\mathcal{F}$-distributive law*

$$a \wedge \bigvee S = \bigvee\{a \wedge s : s \in S\} \tag{$\mathcal{F}$D}$$

for all $a \in L, S \in \mathcal{F}(L)$. Recall that, for each meet-semilattice $L \in \mathcal{M}eet$, the meet-semilattice $\mathcal{F}_l(L)$ is actually a partial frame over $\mathcal{S}$. A morphism $f : L \to L'$ of partial frames ($\mathcal{F}$-morphism) is a meet-semilattice morphism preserving $\mathcal{F}$ joins. The category of partial frames over $\mathcal{M}eet$ is denoted by $\mathcal{S}_{\mathcal{F}}$. It is easy to check (by applying (A4)) that $\mathcal{F}_l$ is a functorial construction.

A $\mathcal{F}$*-cover* C of a partial frame L is an $\mathcal{F}$-subset of L such that its join is the top element of L. We denote by $\mathcal{C}_{\mathcal{F}}(L)$ the set of all $\mathcal{F}$-covers of L.

An element a of a partial frame L is said to be *dense* if, for any $x \in L$, $x \wedge a = 0$ implies $x = 0$. *A dense cover* of L is an $\mathcal{F}$-subset $D \subseteq L$ such that $\bigvee U$ is dense in L. The system of all dense covers of the locale L is denoted by $\mathcal{D}(L)$. For any $a \in L$, we put $a^* = \{x \in L : x \wedge a = 0\}$. Then a is dense if and only if $a^* = \{0\}$.

A *partial subframe* L of a partial frame L' is a subset of L' such that the inclusion is a morphism of partial frames i.e. a meet-subsemilattice L such that $F \subseteq L, F \in \mathcal{F}(L)$ implies $F \in \mathcal{F}(L'), \bigvee F \in L$. A *congruence* on a partial frame L is an equivalence relation θ on L such that $a\theta b, c\theta d \Longrightarrow a \wedge c\theta b \wedge d$ and $F = \{f_i\}, G = \{g_i\} \in \mathcal{F}(L), f_i\theta g_i \Longrightarrow \bigvee F\theta \bigvee G$. Since the intersection of congruences is a congruence the set CL of all congruences on L is a complete lattice. Moreover CL is a frame. To see this we put, for all congruences θ, ϕ, $\theta \rightarrow \phi = \{(a, b) \in L \times L : (a, b) \wedge (c, d) \in \phi$ for all $(c, d) \in \theta\}$. Evidently, $\theta \rightarrow \phi$ is a congruence on L and $\alpha \subseteq \theta \rightarrow \phi$ if and only if $\alpha \cap \theta \subseteq \phi$ for any congruence α i.e. CL is a frame.

Examples.

Let $L \in \mathcal{M}eet$ and put $\mathcal{N}(L) = \{\{a\} : a \in L\} \cup \{\emptyset\}$.

(1) Put $\mathcal{F}(L)= \mathcal{N}(L)$. Then $\mathcal{F}_l(L)$ is isomorphic to L and $\mathcal{S}_{\mathcal{F}}= \mathcal{M}eet$.

(2) Put $\mathcal{F}(L)= \{F : F \subseteq\subseteq L\}$. Then $\mathcal{F}_l(L)$ is a distributive lattice and $\mathcal{S}_{\mathcal{F}} = \mathcal{DL}at$.

(3) Put $\mathcal{F}(L)= \{F : F \subseteq L, cardF \in \kappa\}, \kappa$ being a regular cardinal. Then $\mathcal{F}_l(L)$ is a κ-frame (see [10]) and $\mathcal{S}_{\mathcal{F}} = \kappa - \mathcal{F}rm$.

(4) Put $\mathcal{F}(L)= \mathcal{P}(L)$. Then $\mathcal{F}_l(L)$ is a frame and $\mathcal{S}_{\mathcal{F}} = \mathcal{F}rm$.

Suppose $\mathcal{F}$, $\mathcal{G}$ satisfying (A1)-(A5) such that $L \in \mathcal{S}$ implies $\mathcal{F}(L) \subseteq \mathcal{G}(L)$. As in [10], there is a forgetful functor $\mathcal{U}_{\mathcal{F},\mathcal{G}}$ from $\mathcal{S}_{\mathcal{G}}$ to $\mathcal{S}_{\mathcal{F}}$. We say that $\mathcal{F}$ is *well behaved wrt.* $\mathcal{G}$ and write $\mathcal{F} \sqsubseteq \mathcal{G}$ if the following conditions are satisfied:

(W1) for all $L \in \mathcal{S}_{\mathcal{F}}$ there is an idempotent mapping $\widehat{\ }: \mathcal{G}(L) \to \mathcal{G}(L)$ such that $\{\downarrow(\bigvee F) : F \prec G, F \in \mathcal{F}(L)\} = \{\downarrow(\bigvee F) : F \prec \widehat{G}, F \in \mathcal{F}(L)\} = \downarrow(\widehat{G})$ for all $G \in \mathcal{G}(L)$,

(W2) $L, L' \in \mathcal{S}_{\mathcal{F}}$, $h : L \to L'$ being an $\mathcal{F}$-morphism implies $h(\widehat{G}_L) = \widehat{h(G)}_{L'}$ for all $G \in \mathcal{G}(L)$.

Particularly, $\downarrow(\widehat{G}) = \widehat{\downarrow(G)}$.

In the following, we shall always assume that $\mathcal{F} \sqsubseteq \mathcal{G}$. Recall that, in the case of κ-frames, we may put $\widehat{G} = \{(\bigvee F) : F \subseteq G, card(F) \in \lambda\}$ for all $G \subseteq L, card(G) \in \kappa$.

We define an $\mathcal{FG}$-ideal of a partial $\mathcal{F}$-frame to be the $\mathcal{G}$-generated set closed under $\mathcal{F}$-joins. We denote by $\mathcal{Q}_{\mathcal{F},\mathcal{G}}(L)$ the system of all $\mathcal{F}\mathcal{G}$-ideals. It is easy to check that $\mathcal{Q}_{\mathcal{F},\mathcal{G}}(L)$ is a $\mathcal{G}$-frame, $\downarrow\, : L \to \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L)$ is an $\mathcal{F}$-morphism (using (A5) and the well behavedness) and that $\mathcal{Q}_{\mathcal{F},\mathcal{G}}$ is a functorial construction (factorizing through $\mathcal{G}_l(L)$). Moreover, $\mathcal{Q}_{\mathcal{F},\mathcal{G}}$ preserves injections and surjections (look for principal ideals). The reader should compare our $\mathcal{Q}_{\mathcal{F},\mathcal{G}}$-construction with the F^{λ}_{κ}-construction in [10].

1.2 Theorem. *$\mathcal{U}_{\mathcal{F},\mathcal{G}}$ has as a left adjoint $\mathcal{Q}_{\mathcal{F},\mathcal{G}}$ from $\mathcal{S}_{\mathcal{F}}$ to $\mathcal{S}_{\mathcal{G}}$ with adjunction morphisms $\eta_{\mathcal{F},\mathcal{G}}: \ id \to \mathcal{U}_{\mathcal{F},\mathcal{G}}\mathcal{Q}_{\mathcal{F},\mathcal{G}}$ and $\epsilon_{\mathcal{F},\mathcal{G}} : \mathcal{Q}_{\mathcal{F},\mathcal{G}}\mathcal{U}_{\mathcal{F},\mathcal{G}} \to id$.*

Proof. First, we put $\eta_{\mathcal{F},\mathcal{G}}L(x) = \downarrow(\{x\})$ and $\epsilon_{\mathcal{F},\mathcal{G}}K([F]) = \bigvee F$ for all $L \in \mathcal{S}_{\mathcal{F}}$ and all $K \in \mathcal{S}_{\mathcal{G}}$. It is easy to see that both prescriptions are correct and the corresponding mappings are morphisms in $\mathcal{S}_{\mathcal{F}}$, $\mathcal{S}_{\mathcal{G}}$ resp. To check the naturality condition is a routine verification.

Now, let $f : \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L) \to K$ be a morphism in $\mathcal{S}_{\mathcal{G}}$, $L \in \mathcal{S}_{\mathcal{F}}$, $K \in \mathcal{S}_{\mathcal{G}}$. To conclude that the required adjunction holds, we have to show that there is a unique $\phi : L \to K$ such that $f = \epsilon_{\mathcal{F},\mathcal{G}}K \circ \mathcal{Q}_{\mathcal{F},\mathcal{G}}(\phi)$. We put, for $a \in L$, $\phi(a) = f(\downarrow(\{a\}))$. This determines ϕ uniquely and since $\phi = f \circ \downarrow$ it is a morphism in $\mathcal{S}_{\mathcal{F}}$. $\square$

1.3 Corollary. *The triple $\mathbf{T} = (\mathcal{U}_{\mathcal{F},\mathcal{G}}\mathcal{Q}_{\mathcal{F},\mathcal{G}}, \eta, \mathcal{U}_{\mathcal{F},\mathcal{G}}\epsilon\mathcal{Q}_{\mathcal{F},\mathcal{G}})$ is a monad on $\mathcal{S}_{\mathcal{F}}$.*

1.4 Corollary. *$\mathcal{S}_{\mathcal{F}}$ is monadic.*

Proof. This follows from the fact that $\mathcal{M}eet$ is a finitary algebraic category (this implies monadicity over $\mathcal{S}et$) and every $\mathcal{S}_{\mathcal{F}}$ is monadic over $\mathcal{M}eet$ by 1.3 ($\mathcal{M}eet$ is well behaved wrt. each $\mathcal{S}_{\mathcal{F}}$). $\square$

1.5 Definition. *Let L be a partial $\mathcal{G}$-frame, $a, b \in L$. We say that a is $\mathcal{F}\mathcal{G}$-below b and write $a \ll_{\mathcal{F},\mathcal{G}} b$ if, for any $G \in \mathcal{G}(L), b \leq \bigvee G$ there is $g \in \widehat{G}$ such that $a \leq h$. We say that an element $c \in L$ is $\mathcal{F}\mathcal{G}$-compact if $c \ll_{\mathcal{F},\mathcal{G}} c$. Recall that a join of an $\mathcal{F}$-set of $\mathcal{F}\mathcal{G}$-compact elements is an $\mathcal{F}\mathcal{G}$-compact element (using (A3)). The set of all $\mathcal{F}\mathcal{G}$-compact elements is denoted $K_{\mathcal{F}\mathcal{G}}(L)$. L is called $\mathcal{F}\mathcal{G}$-coherent if each element of L is a join of a $\mathcal{G}$-subset $G, G = \widehat{G}$, $G \subseteq K_{\mathcal{F}\mathcal{G}}(L)$ and $K_{\mathcal{F}\mathcal{G}}(L)$ is closed under finitary meets (including the empty meet). For a $\mathcal{F}\mathcal{G}$-coherent partial frame L, $K_{\mathcal{F}\mathcal{G}}(L)$ is an $\mathcal{F}$-frame.*

We say that L is stably $\mathcal{F}\mathcal{G}$-contimuous if, for each $a \in L$, there is a subset $k(a) \subseteq \{x : x \ll_{\mathcal{F},\mathcal{G}} a\}, k(a) \in \mathcal{Q}_{\mathcal{F},\mathcal{G}}$ such that $a = \bigvee\{x : x \in k(a)\}$, 1 is $\mathcal{F}\mathcal{G}$-compact and $a \in k(b), c \in k(d)$

implies $a \wedge c \in k(b \wedge d)$.

Recall that, for a stably $\mathcal{FG}$-contimuous partial $\mathcal{G}$-frame L, we have $k(a) = \{x : x \ll_{\mathcal{F},\mathcal{G}} a\}$. One can verify that $k : L \to \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L)$ is a $\mathcal{G}$-morphism and that $\bigvee \circ k = id_L$.

As in frames, where we speak about coherent and stably continuous frames, $\mathcal{FG}$-coherence implies stable $\mathcal{FG}$-continuity. Moreover, the functor $\mathcal{Q}_{\mathcal{F},\mathcal{G}}$: $\mathcal{S}_{\mathcal{F}} \to \mathcal{S}_{\mathcal{G}}$ induces an equivalence between $\mathcal{S}_{\mathcal{F}}$ and the category $\mathcal{FGC}oh\mathcal{S}_{\mathcal{G}}$ of all $\mathcal{FG}$-coherent partial $\mathcal{G}$-frames and those $\mathcal{G}$-morphisms which preserve $\mathcal{FG}$-compact elements. It is easy to check (see 1.6) that $K_{\mathcal{FG}}$ is functorial, going from $\mathcal{FG}$-coherent partial $\mathcal{G}$-frames into partial $\mathcal{F}$-frames. The corresponding natural isomorphisms are $\downarrow : M \to K_{\mathcal{FG}}(\mathcal{Q}_{\mathcal{F},\mathcal{G}}(M))$, $\bigvee : \mathcal{Q}_{\mathcal{F},\mathcal{G}}(K_{\mathcal{FG}}(L)) \to L$, $M \in \mathcal{S}_{\mathcal{F}}, L \in \mathcal{FGC}oh\mathcal{S}_{\mathcal{G}}$.

Note also that our definitions differ formally from that in [10], but one easily sees that they coincide for κ-frames.

As in [10] for κ-frames, one may prove the following.

1.6 Proposition. *A partial $\mathcal{G}$-frame L is $\mathcal{F}$-free i.e. isomorphic to $\mathcal{Q}_{\mathcal{F},\mathcal{G}}(M)$ for some partial $\mathcal{F}$-frame M if and only if it is $\mathcal{FG}$-coherent. Moreover, we have :*

(1) *for any partial $\mathcal{F}$-frame M, $\mathcal{Q}_{\mathcal{F},\mathcal{G}}(L)$ is $\mathcal{FG}$-coherent and $K_{\mathcal{FG}}(\mathcal{Q}_{\mathcal{F},\mathcal{G}}(M)) = \downarrow(M) = \{\downarrow(x) : x \in M\}$,*

(2) *for any $\mathcal{FG}$-coherent partial $\mathcal{G}$-frame L, we have an isomorphismus of partial $\mathcal{G}$-frames $\bigvee : \mathcal{Q}_{\mathcal{F},\mathcal{G}}(K_{\mathcal{FG}}(L)) \to L$.*

Proof. (1) Let $\downarrow(G) \in K_{\mathcal{FG}}(\mathcal{Q}_{\mathcal{F},\mathcal{G}}(M)), G = \widehat{G} \in \mathcal{G}(M)$. Then by (A5) $\downarrow(G) \subseteq \downarrow(g)$ for some $g \in G$ i.e. $K_{\mathcal{FG}}(\mathcal{Q}_{\mathcal{F},\mathcal{G}}(M)) \subseteq \downarrow(M)$. The reverse inclusion is evident. The rest from the fact that every element of $\mathcal{Q}_{\mathcal{F},\mathcal{G}}(M)$ is a 'good' $\mathcal{G}$-join of principal ideals i.e. it has the form $\bigvee\{\downarrow(g) : g \in G\}, G = \widehat{G} \in \mathcal{G}(L)$.

(2) The proof copies 1.4(b) from [10]. Recall only that, for for any $\mathcal{FG}$-coherent partial $\mathcal{G}$-frame L, one may easily check that $a = \bigvee \widehat{G}, \widehat{G} \in \mathcal{G}(L), \widehat{G} \subseteq K_{\mathcal{FG}}(L)$ implies $K_{\mathcal{FG}}(L) \cap \downarrow(\widehat{G}) = K_{\mathcal{FG}}(L) \cap \downarrow(a)$ and $k(a) = \downarrow(\widehat{G})$. $\square$

Following Madden [10], we define an $\mathcal{F}$-*section* for a $\mathcal{G}$-morphism $q : K \to L$ to be an $\mathcal{F}$-morphism $s : L \to K$ such that $q \circ s = id_L$. As in [10], a $\mathcal{G}$-morphism with an $\mathcal{F}$-section is surjective and the $\mathcal{F}$-section injective. The leading example of a $\mathcal{G}$-morphism with an $\mathcal{F}$-section are our $\epsilon_{\mathcal{F},\mathcal{G}}L = \bigvee_L$ and $\eta_{\mathcal{F},\mathcal{G}}L = \downarrow$ from the theorem 1.2.

1.7 Definition. *A $\mathcal{G}$-frame P is said to be $\mathcal{F}$-projective if, whenever we are given a diagram of the form*

$$\begin{array}{ccc} & & P \\ & & \big\downarrow{\scriptstyle f} \\ K & \xrightarrow{q} & L \end{array}$$

with q having an $\mathcal{F}$-section, there exists a $\mathcal{G}$-morphism $g : P \to K$ such that $q \circ g = f$.

1.8 Proposition. *A $\mathcal{G}$-frame P is $\mathcal{F}$-projective if and only if it is a retract of an $\mathcal{F}$-free $\mathcal{G}$-frame.*

1.9 Lemma. *A retract of a stably $\mathcal{FG}$-continuous $\mathcal{G}$-frame is stably $\mathcal{FG}$-continuous.*

Proof. Let $f : L \to K, g : K \to L$ be $\mathcal{G}$-morphisms such that $f \circ g = id_K$, L be stably $\mathcal{FG}$-continuous. Recall that $a \ll_{\mathcal{F},\mathcal{G}} g(b)$ implies $f(a) \ll_{\mathcal{F},\mathcal{G}} b$. Namely, $\widehat{G} \in \mathcal{G}(K), b \leq \bigvee \widehat{G} \implies g(\widehat{G}) \in \mathcal{G}(L), g(b) \leq \bigvee g(\widehat{G}) \implies$ there is by (W2) $h \in \widehat{G}$ such that $a \leq g(h) \implies$ there is $h \in \widehat{G}$ such that $f(a) \leq h$. The rest follows immediately the corresponding part of the proof of 2.4 [10]. □

1.10 Theorem. *A $\mathcal{G}$-frame P is $\mathcal{F}$-projective if and only if it is a stably $\mathcal{FG}$-continuous frame.*

Proof. One proves the proposition in the same manner as Madden did for κ-frames using 1.9. We note only that one has to use (A3) first time to count that an $\mathcal{F}$-join of elements $\mathcal{FG}$-below some element is $\mathcal{FG}$-below it and second time to verify the existence of a $\mathcal{G}$-morphism needed for projectiveness. □

1.11 Corollary. *The categories $\mathcal{FGC}oh\mathcal{S}_\mathcal{G}$ of $\mathcal{FG}$-coherent $\mathcal{G}$-frames and $\mathcal{FGS}t\mathcal{C}ont\mathcal{S}_\mathcal{G}$ of stably $\mathcal{FG}$-continuous $\mathcal{G}$-frames are both coreflective in the category $\mathcal{S}_\mathcal{G}$ of $\mathcal{G}$-frames, with the coreflection functor $\mathcal{Q}_{\mathcal{F},\mathcal{G}}$ and the coreflection map $\epsilon_{\mathcal{F},\mathcal{G}}L = \bigvee_L : \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L) \to L$.*

Proof. The proof is a reformulating of Prop. 10 in [2]. Namely, by the same arguments it is sufficient to prove that any $\mathcal{G}$-morphism $h : M \to L, M$ stably $\mathcal{FG}$-continuous, uniquely factors through $\bigvee : \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L) \to L$. Let us consider the commutative diagram

$$\begin{array}{ccccc} \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L) & \xrightarrow{\bigvee_L} & L & \xleftarrow[\bigvee_L]{} & \mathcal{Q}_{\mathcal{F},\mathcal{G}}(L) \\ \big\uparrow{\scriptstyle \mathcal{Q}_{\mathcal{F},\mathcal{G}}(h)} & & \big\uparrow{\scriptstyle h} & & \big\uparrow{\scriptstyle \mathcal{Q}_{\mathcal{F},\mathcal{G}}(h)} \\ \mathcal{Q}_{\mathcal{F},\mathcal{G}}(M) & \xrightarrow{\bigvee_M} & M & \xrightarrow{k} & \mathcal{Q}_{\mathcal{F},\mathcal{G}}(M) \end{array}$$

where k is given by $k(a) = \{x : x \ll_{\mathcal{F},\mathcal{G}} a\}$. The rest follows [2]. $\square$

2. Limits and colimits in partial frames

First of all we note that, when constructing limits in partial frames, they are the same as for meet-semilatticess (in fact sets). This implies completeness of $\mathcal{S}_{\mathcal{F}}$. To prove cocompletness it is enough to investigate the coequalizers and sums. By (A4), the coequalizers in partial quantales coincide with coequalizers in meet-semilattices. A sum of partial frames can be constructed by taking their sum in $\mathcal{M}eet$, lifting it in partial frames and factorizing the result through the least congruence such that the superposition of all injections and the lifting map will be morphisms of partial frames. In the sequel we will discuss this construction more explicitly.

There are different ways how to construct sums of frames such as in the papers [4], [5] or [6]. For partial frames seems not to be so elegant description. However, we will give an analogous description to that of [6].

Let $L_\gamma, \gamma \in \Gamma$ be a family of partial frames and write B for the sum in $\mathcal{M}eet$ of L_γ. B consists in fact of formal reduced meets $\bigwedge_{\gamma \in \Gamma_0} x_\gamma$, $\Gamma_0 = \{\gamma_i : i = 1, \ldots, n\}$, over the disjoint union $\amalg L_\gamma$ of the system L_γ. Evidently, if x is an nonzero element of B 'containing' some element x_γ of L_γ then we may write $x = x_\gamma \wedge \bigwedge X_p$, here $X_p = \{x_\delta : \delta \neq \gamma, \delta \in \Gamma_0\} \cup \{1_\delta : \delta \neq \gamma, \delta \notin \Gamma_0\}$ is a subset of $\amalg L_\gamma$.

We shall call an $\mathcal{F}$-connector a system $c = \big(m, (R^{\uparrow}_{\gamma}, R^{\downarrow}_{\gamma}, \gamma \in \Gamma), m_1, m_2\big)$, where $m \subseteq B, m_1, m_2 \in \mathcal{F}_l(B)$, $m_1, m_2 \subseteq m$, $R^{\downarrow}_{\gamma} \subseteq \mathcal{P}_{\mathcal{F}}(m) \times m$, $R^{\downarrow}_{\gamma} \subseteq m \times \mathcal{P}_{\mathcal{F}}(m)$ such that the following conditions hold:

Let $k \subseteq m$. Whenever

$$(m_1 \subseteq k) \ \& \ [(nR^{\uparrow}_{\gamma}x) \ \& \ (n \subseteq k) \Longrightarrow x \in k] \ \& \ [(xR^{\downarrow}_{\gamma}n) \ \& \ (x \in k) \Longrightarrow n \subseteq k] \tag{1}$$

or

$$(m_2 \subseteq k) \ \& \ [(nR^{\uparrow}_{\gamma}x) \ \& \ (x \in k) \Longrightarrow n \subseteq k] \ \& \ [(xR^{\downarrow}_{\gamma}n) \ \& \ (n \subseteq k) \Longrightarrow x \in k] \tag{2}$$

holds, it is $k = m$.

Elements $m_1, m_2 \in \mathcal{F}_l(B)$ are said to be $C_{\mathcal{F}}$-equivalent ($m_1 \sim_{C_{\mathcal{F}}} m_2$), if either $m_1 = \downarrow(0) = m_2$

or there exists an $\mathcal{F}$-connector $c = \big(m, (R^{\uparrow}_{\gamma},\, R^{\downarrow}_{\gamma},\, \gamma \in \Gamma), m_1, m_2\big)$ such that it holds:

$$
\begin{aligned}
&(xR^{\uparrow}_{\gamma}m') \text{ or } (m'R^{\downarrow}_{\gamma}x) \implies (x \neq 0)\\
&\&\Big(x_\gamma = \bigvee m'_\gamma\Big) \;\&\; \Big(x = x_\gamma \wedge \bigwedge X_p, m' = m'_\gamma \wedge \bigwedge X_p\Big).
\end{aligned}
\tag{3}
$$

2.1 Proposition. *" $\sim_{C_{\mathcal{F}}}$" is an equivalence relation on $\mathcal{F}_l(B)$.*

Proof. Let $m \subseteq B$. We put $xS^{\downarrow}_{\gamma}m',\ m'S^{\uparrow}_{\gamma}x$ if and only if $(x \neq 0)\&\big(x_\gamma = \bigvee m'_\gamma\big)$ & $(x = x_\gamma \wedge \bigwedge X_p)$ & $(m' = m'_\gamma \wedge \bigwedge X_p)$ for each $x \in m,\ m' \in \mathcal{P}_{\mathcal{F}}(m)$. Such relations $S^{\uparrow}_{\gamma}$, $S^{\downarrow}_{\gamma}$ will be called standard, an $\mathcal{F}$-connector with standard relations $S^{\uparrow}_{\gamma}$ and $S^{\downarrow}_{\gamma}$ is said to be standard.

Let $m_1 \in \mathcal{F}_l(B)$. We put $m = m_1 = m_2$ and let c be the corresponding standard $\mathcal{F}$-connector. Then $m_1 {\sim_{C_{\mathcal{F}}}} m_1$ i.e. $\sim_{C_{\mathcal{F}}}$ is reflexive.

Let $m_1 {\sim_{C_{\mathcal{F}}}} m_2$, c be the corresponding $\mathcal{F}$-connector. Clearly, if we interchange m_1 with m_2 and define the "new" $R^{\uparrow}_{\gamma}$ as an inverse relation to the "old" $R^{\downarrow}_{\gamma}$ and similarly for the "new" $R^{\downarrow}_{\gamma}$, we obtain an $\mathcal{F}$-connector c' i.e. $m_2 {\sim_{C_{\mathcal{F}}}} m_1$. This yields the symmetry of $\sim_{C_{\mathcal{F}}}$.

Let $m_1 {\sim_{C_{\mathcal{F}}}} m_2$, $m_2 {\sim_{C_{\mathcal{F}}}} m_3$. Then there exist $\mathcal{F}$-connectors

$$a = \big(m(a), (R^{\uparrow}_{\gamma}(a), R^{\downarrow}_{\gamma}(a), \gamma \in \Gamma), m_1, m_2\big), b = \big(m(b), (R^{\uparrow}_{\gamma}(b), R^{\downarrow}_{\gamma}(b), \gamma \in \Gamma), m_2, m_3\big)$$

such that (3) holds. We put $m = m(a) \cup m(b)$, $R^{\uparrow}_{\gamma}{=}R^{\uparrow}_{\gamma}(a){\cup}R^{\uparrow}_{\gamma}(b){\subseteq}\ S^{\uparrow}_{\gamma}(m)$, $R^{\downarrow}_{\gamma}{=}R^{\downarrow}_{\gamma}(a){\cup}R^{\downarrow}_{\gamma}(b){\subseteq}$ $S^{\downarrow}_{\gamma}(m)$ and $c = \big(m, (R^{\uparrow}_{\gamma}, R^{\downarrow}_{\gamma}, \gamma \in \Gamma), m_1, m_3\big)$. Then c is clearly an $\mathcal{F}$-connector satisfying (3).

We shall check the condition (1).

Let $(m_1 \subseteq k)$ & $[(nR^{\uparrow}_{\gamma}x)$ & $(n \subseteq k) \implies x \in k]$ & $[(xR^{\downarrow}_{\gamma}n)$ & $(x \in k) \implies n \subseteq k]$. We put $k(a) = k \cap m(a)$ and $k(b) = k \cap m(b)$. Then $(m_1 \subseteq k(a))$ & $[\big(nR^{\uparrow}_{\gamma}(a)\ x\big)$ & $(n \subseteq k(a)) \implies x \in k(a)]$ & $[\big(xR^{\downarrow}_{\gamma}(a)\ n\big)$ & $(x \in k(a)) \implies n \subseteq k(a)]$. Then $k(a) = m(a)$, so we have $m_2 \subseteq k$. Now, let $(m_2 \subseteq k(b))$ & $[\big(nR^{\uparrow}_{\gamma}(b)\ x\big)$ & $(n \subseteq k(b)) \implies x \in k(b)]$ & $[\big(xR^{\downarrow}_{\gamma}(b)\ n\big)$ & $(x \in k(b)) \implies n \subseteq k(b)]$. Again, $k(b) = m(b)$ i.e. $k = m$.

The same argument may be used for checking (2). Since the condition (3) is trivially preserved by the union of relations, we are ready. $\square$

2.2 Proposition. *" $\sim_{C_{\mathcal{F}}}$" is an $\mathcal{F}$-congruence on $\mathcal{F}_l(B)$ i.e. $\mathcal{F}_l(B)/_{\sim_{C_{\mathcal{F}}}}$ is a partial frame.*

Proof. Let $m_1(a) {\sim_{C_{\mathcal{F}}}} m_2(a)$, $m_1(b) {\sim_{C_{\mathcal{F}}}} m_2(b)$. Then there exist $\mathcal{F}$-connectors

$$a = \big(m(a), (R^{\uparrow}_{\gamma}(a), R^{\downarrow}_{\gamma}(a), \gamma \in \Gamma), m_1(a), m_2(a)\big), b = \big(m(b), (R^{\uparrow}_{\gamma}(b), R^{\downarrow}_{\gamma}(b), \gamma \in \Gamma), m_1(b), m_2(b)\big)$$

satisfying (3).

Put $m=m(a)\cap m(b)$, $m_1=m_1(a)\cap m_1(b)$, $m_2=m_2(a)\cap m_2(b)$, $R^{\downarrow}_{\gamma}=\{(x\wedge y,\{h\wedge y: h\in m'\}):\big(\ xR^{\downarrow}_{\gamma}(a)\ m'\ \&\ y\in m(b)\ \big)$ or $\big(\ xR^{\downarrow}_{\gamma}(b)\ m'\ \&\ y\in m(a)\ \big)\}$, $R^{\uparrow}_{\gamma}=\{(\{h\wedge y: h\in m'\},x\wedge y):\big(\ m'R^{\uparrow}_{\gamma}(a)\ x\ \&\ y\in m(b)\ \big)$ or $\big(\ m'R^{\uparrow}_{\gamma}(b)\ x\ \&\ y\in m(a)\ \big)\}$. Then $m_1,m_2\in\mathcal{F}_l(B)$ and we have to check that $c=\big(m,(R^{\uparrow}_{\gamma},R^{\downarrow}_{\gamma},\gamma\in\Gamma),m_1,m_2\big)$ is an $\mathcal{F}$-connector satisfying (3).

Checking (1), let $(m_1\subseteq k)$ & $[(nR^{\uparrow}_{\gamma}x)\ \&\ (n\subseteq k)\implies x\in k]$ & $[(xR^{\downarrow}_{\gamma}n)\ \&\ (x\in k)\implies n\subseteq k]$. We put $k(a)=\{z\in m(a):\{z\wedge h: h\in m_1(b)\}\subseteq k\}$. Then $(m_1(a)\subseteq k(a))$ & $[\ (nR^{\uparrow}_{\gamma}(a)\ x)\ \&\ (n\subseteq k(a))\implies x\in k(a)]$ & $[(xR^{\downarrow}_{\gamma}(a)\ n)\ \&\ (x\in k(a))\implies n\subseteq k(a)]$, which is easy to compute. Then $k(a)=m(a)$ i.e. $m(a)\cap m_1(b)\subseteq k$. Again, we put $k(b)=\{z\in m(b):\{z\wedge h: h\in m_1(a)\}\subseteq k\}$. Then $(m_1(b)\subseteq k(b))$ & $[\ \big(nR^{\uparrow}_{\gamma}(b)\ x\big)\ \&\ (n\subseteq k(b))\implies x\in k(b)]$ & $[\big(xR^{\downarrow}_{\gamma}(b)\ n\big)\ \&\ (x\in k(b))\implies n\subseteq k(b)]$. Then $k(b)=m(b)$ i.e. $m=m(a)\cap m(b)\subseteq k$.

In the same manner we may check the condition (2). That (3) holds is evident. Alltogether, we proved that $\sim_{C_{\mathcal{F}}}$actually preserves finite meets.

Let $m_1(a_i)\sim_{C_{\mathcal{F}}}m_2(a_i)$, $i\in I$, $\{m_1(a_i)\}$, $\{m_2(a_i)\}\in\mathcal{F}(\mathcal{F}_l(B))$. Then there exist $\mathcal{F}$-connectors

$$a_i=\big(m(a_i),(R^{\uparrow}_{\gamma}(a_i),R^{\downarrow}_{\gamma}(a_i),\gamma\in\Gamma),m_1(a_i),m_2(a_i)\big)\,,i\in I$$

satisfying (3). We put $a=\big(\bigcup m(a_i),(\bigcup R^{\uparrow}_{\gamma}(a_i),\bigcup R^{\downarrow}_{\gamma}(a_i),\gamma\in\Gamma),\bigcup m_1(a_i),\bigcup m_2(a_i)\big)$. Using (A3), we obtain that $\bigcup m_1(a_i),\bigcup m_2(a_i)\in\mathcal{F}_l(B)$.

Now, we will check (1). Let $(\bigcup m_1(a_i)\subseteq k)$ & $[\big(n\bigcup R^{\uparrow}_{\gamma}(a_i)x\big)\ \&\ (n\subseteq k)\implies x\in k]$ & $[\big(x\ \bigcup R^{\downarrow}_{\gamma}(a_i)n\big)\ \&\ (x\in k)\implies n\subseteq k]$. We put $k(a_i)=k\cap m(a_i)$ for $i\in I$. Then $(m_1(a_i)\subseteq k(a_i))$ & $[\big(nR^{\uparrow}_{\gamma}(a_i)\ x\big)\ \&\ (n\subseteq k(a_i))\implies x\in k(a_i)]$ & $[\big(xR^{\downarrow}_{\gamma}(a_i)\ n)\ \&\ (x\in k(a_i)\big)\implies n\subseteq k(a_i)]$. Clearly $k(a_i)=m(a_i)$, so we have $k=m$.

By the symmetry argument we may check the condition (2). That (3) holds is trivial. This gives us that $\sim_{C_{\mathcal{F}}}$ preserves $\mathcal{F}$-joins. □

We shall denote by $[m]$ the class of $m\in\mathcal{F}_l(B)$ in the factor partial frame $\mathcal{F}_l(B)/_{\sim_{C_{\mathcal{F}}}}$. For any $\gamma\in\Gamma$, we define a mapping $\tau_{\gamma}{:}L_{\gamma}\longrightarrow B$ to be the injection from L_{γ} to B in $\mathcal{M}eet$.

2.3 Proposition. *Given a system $f_{\gamma}: L_{\gamma}\longrightarrow C$ of morphisms in $\mathcal{S}_{\mathcal{F}}$, C being a partial frame, there exists a unique morphism $f:\mathcal{F}_l(B)/_{\sim_{C_{\mathcal{F}}}}\longrightarrow C$ of partial frames such that the following* *condition holds:*

$$f([\downarrow(x)] = \bigwedge\{f_\gamma(x_\gamma) : \gamma \in \Gamma_0\} \text{ for any } x \in B. \tag{4}$$

Proof. As in [6], we have only to check that f, which is uniquely determined by the prescription $f([u]) = \bigvee\{f([\downarrow(x)]) : x \in u\}$, is correctly defined (the join exists since u is $\mathcal{F}$-generated). In fact we will go first to $\mathcal{F}_l(B)$ and then factorize. Put $g(u) = \bigvee\{f([\downarrow(x)]) : x \in u\}$ for all $u \in \mathcal{F}_l(B)$. It is evident that g is an $\mathcal{F}$-morphism. Now, let $m_1 \sim_{C_\mathcal{F}} m_2$. Then there exists an $\mathcal{F}$-connector $a = \big(m, (R^{\updownarrow}_\gamma, R^{\downarrow}_\gamma, \gamma \in \Gamma), m_1, m_2\big)$ such that (3) holds. Put $k = \{x \in m : f([\downarrow(x)] \leq g(m_1)\}$. It is easy to check that k satisfies the condition (1) i.e. $k = m$. Then $g(m_2) \leq g(m_1)$. By a symmetry argument we have that $g(m_1) \leq g(m_2)$ i.e. $f([m_1]) = g(m_1) = g(m_2) = f([m_2])$. $\square$

2.4 Theorem. *The factor partial frame $\mathcal{F}_l(B)/_{\sim_{C_\mathcal{F}}}$ has $\mathcal{F}$-joins and finite meets given by the prescriptions*

$$\bigvee\{[m_i : i \in I\} = [\bigvee\{n_i : i \in I\}]$$
$$[m] \wedge [n] = [\{x \wedge y : x \in m, y \in n\}] \tag{5}$$

for all $m, n \in \mathcal{F}_l(B), \{m_i\} \in \mathcal{F}(\mathcal{F}_l(B))$.

Defining mappings $i_\gamma : L_\gamma \longrightarrow, \mathcal{F}_l(B)/_{\sim_{C_\mathcal{F}}}$ $\gamma \in \Gamma$, by $i_\gamma(x) = [\downarrow(\tau_\gamma(x))], x \in L_\gamma$, we have partial frame morphisms such that $\mathcal{F}_l(B)/_{\sim_{C_\mathcal{F}}}$ is a sum $\bigoplus_{\gamma\in\Gamma} L_\gamma$ of L_γ, $\gamma \in \Gamma$, with injections i_γ.

Proof. By the preceding we know that $\mathcal{F}_l(B)/_{\sim_{C_\mathcal{F}}}$ is a partial frame, with $\mathcal{F}$-joins and finite meets given by (5). Again by (5), i_γ are morphisms of partial frames. Given a system $f_\gamma : L_\gamma \longrightarrow C$ of partial frame morphisms , C being a partial frame, there exists by the preceding proposition a unique morphism $f : \mathcal{F}_l(B)/_{\sim_{C_\mathcal{F}}} \longrightarrow C$ of partial frames such that (4) holds i.e. $f \cdot i_\gamma = f_\gamma$. $\square$

2.6 Lemma. *Let $x, y \in B$. Then*

$$[\downarrow(x)] \leq [\downarrow(y)] \iff x \leq y. \tag{6}$$

Proof. At first, we check that for $x, y \in B$ we have:

$$[\downarrow(x)] = [\downarrow(y)] \iff x = y. \tag{7}$$

Let $[\downarrow(x)] = [\downarrow(y)]$. Then we have $\downarrow(x) \sim_{C_{\mathcal{F}}} \downarrow(y)$ i.e. there is an $\mathcal{F}$-connector $a = \big(m, (R^{\uparrow}_{\gamma}, R^{\downarrow}_{\gamma}, \gamma \in \Gamma), \downarrow(x), \downarrow(y)\big)$ such that (3) holds. Clearly, $\downarrow(x)$ satisfies (1) and $\downarrow(y)$ satisfies (2) i.e. $\downarrow(x) = m = \downarrow(y)$.

(6) then follows from (7) as in [6]. $\square$

Recall that this says that i_γ preserves arbitrary meets which exist in L_γ.

For any $X \subseteq B$ we define $s(X) := \{[\downarrow(x)] : x \in X\}$ i.e. we have defined a map $s : 2^B \to 2^L$.

3. Covers in partial frames

In the following we shall always assume that, for any partial frame K, $\mathcal{F}(K)$ is a lower set in the usual ordering i.e. any subset of an $\mathcal{F}$-set is an $\mathcal{F}$-set and moreover, that any finite subset of K is an $\mathcal{F}$-set.

Consider a system S_γ of covers of L_γ, $\gamma \in \Gamma$, such that $S_\gamma = \{1\}$ except for, at most, finitely many $\gamma \in \Gamma$. Then the system $\prod_{\gamma\in\Gamma} S_\gamma = \{\bigwedge i_\gamma(x_\gamma) : x_\gamma \in S_\gamma, \gamma \in \Gamma\}$ is a cover of L by (5).

Recall the following important lemma which is a generalization of a lemma for frames due to K from [7].

3.1 Lemma. *Let $T = \{[\downarrow(t)] : t \in T'\}$, $T' \subseteq\subseteq B$ finite, be a cover of L. Then there is a system S_γ of covers of L_γ, $\gamma \in \Gamma$, such that $S_\gamma = \{1\}$ except for, at most, finitely many $\gamma \in \Gamma$ and $\{\bigwedge i_\gamma(x_\gamma) : x_\gamma \in S_\gamma, \gamma \in \Gamma\}$ refines T.*

Proof. We shall slightly modify the proof of K. We can easily find a finite system $\{K_t : t \in T'\}$ of finite subsets $K_t \subseteq\subseteq \Gamma$ and elements $a(\alpha, t), \alpha \in K_t, t \in T'$ such that

$$T = \{\bigwedge_{\alpha\in K_t} i_\alpha(a(\alpha, t)) : t \in T'\}.$$

We put $H := \bigcup_{t\in T'} K_t$. We shall denote by $C_\beta, \beta \in H$ the finite meet semi-lattice generated in L_β by the finite subset $\{a(\beta, t) : \beta \in K_t, t \in T'\}$.

We define

$$k := \{z \in B : (\forall\beta \in H)(\exists B_\beta \subseteq\subseteq C_\beta)(\bigvee i_\beta(B_\beta) \geq [\downarrow(z)]) \text{ and } \bigwedge_{\beta\in H} i_\beta(B_\beta) < T\}.$$

Since T is a cover of L there is an $\mathcal{F}$-connector $c = \big(m, (R^{\uparrow}_{\gamma}, R^{\downarrow}_{\gamma}, \gamma \in \Gamma), m_1, m_2\big)$, where $m = m_2 = B, m_1 = \bigcup_{t\in T'} \downarrow(t) \in \mathcal{F}_l(B)$.

Evidently, $k \subseteq m$, k is a lower set. It is easy to see that to prove $1 \in k = m = B$ (this is in fact our proposition) we have only to check the following implication

$$(nR^{\updownarrow}_{\gamma}x) \ \& \ (n \subseteq k) \Longrightarrow x \in k.$$

It is enough to do it for a standard relation $S^{\updownarrow}_{\gamma}$.

Let $(x \neq 0)\&\big(x_\gamma = \bigvee m'_\gamma\big)$ & $(x = x_\gamma \wedge \bigwedge X_p)$ & $(m' = m'_\gamma \wedge \bigwedge X_p)$ and $m' \subseteq k, m'_\gamma \in \mathcal{F}(L_\gamma)$ for some $\gamma \in \Gamma$. We have to prove that $x \in k$. If $\gamma \notin H$ then $C_\gamma = \{1\}$ and the rest is evident. Let $\gamma \in H$. For any $y \in m'_\gamma$ we put

$$K_y = \{B^y \subseteq\subseteq C_\gamma : \bigvee i_\gamma(B^y) \geq i_\gamma(y) \wedge [\downarrow(\bigwedge X_p)]\}$$

and

$$K = \{K_y : y \in m'_\gamma\}.$$

Evidently, the set $K \subseteq 2^{C_\gamma}$ is finite. We shall define for any $E \in K$ an element $w_E = \bigvee\{y \in m'_\gamma : K_y = E\}$.

Then there are sets $B^E_\beta \subseteq\subseteq C_\beta$, $\beta \in H$ such that

$$\bigvee i_\beta(B^E_\beta) \geq i_\gamma(w_E) \wedge [\downarrow(\bigwedge X_p)] \quad \text{and} \quad \bigwedge_{\beta \in H} i_\beta(B^E_\beta) < T.$$

We put

$$B_\beta = \begin{cases} \bigcup_{E \in K} B^E_\gamma & \text{if } \beta = \gamma \\ \\ \bigwedge_{E \in K} B^E_\beta & \text{otherwise.} \end{cases}$$

We have then for any $\beta \in H$

$$\bigvee i_\beta(B_\beta) \geq \bigvee_{E \in K} i_\gamma(w_E) \wedge [\downarrow(\bigwedge X_p)] = \bigvee_{y \in m'_\gamma} i_\gamma(y) \wedge [\downarrow(\bigwedge X_p)]$$

and

$$\bigwedge_{\beta \in H} i_\beta(B_\beta) = \bigcup_{E \in K} \left(i_\gamma(B^E_\gamma) \wedge \bigwedge_{\beta \in H - \{\gamma\}} i_\beta(B_\beta) \right) < \bigcup_{E \in K} \bigwedge_{\beta \in H} i_\gamma(B^E_\beta) < T. \quad \square$$

In the following we shall generalize our results for frames obtained in [11]. We shall show that under some restricted assumptions certain covering properties of partial frames are preserved in their sums. Recall that Axiom of Choice is prerequisite of our considerations (except in the case of compact partial frames).

Definition 3.2. Let L be a partial frame, $C, D \subseteq L$. We shall say that D *finitely regularly (strongly) refines* C if, for each $d \in D$, there exists $F \subseteq\subseteq C$ $(c \in C)$ such that $d \triangleleft \bigvee F$ i.e. there is $e \in L$ such that $e \vee \bigvee F = 1$ and $e \wedge d = 0$ $(d \triangleleft c)$. We shall write $D \triangleleft_f C$ $(D \triangleleft C)$. Clearly, $\triangleleft \subseteq \triangleleft_f$. Putting $\sigma(C) = \bigcup\{X \subseteq L : X \triangleleft_f C\}$ we have that $\sigma(C) \triangleleft_f C$.

Recall that the relations $\triangleleft_f$, $\triangleleft$ are preserved by morphisms of partial frames and that the following holds:

(i) $E < D \triangleleft_f C < F$ implies $E \triangleleft_f F$,

(ii) $E < D \triangleleft C < F$ implies $E \triangleleft F$,

(iii) $D_i \triangleleft_f C_i,\ i \in I,\ I \neq \emptyset$ implies $\bigcup D_i \triangleleft_f \bigcup C_i$,

(iv) $D_i \triangleleft C_i,\ i \in I,\ I \neq \emptyset$ implies $\bigcup D_i \triangleleft \bigcup C_i$,

(v) $B \triangleleft_f A,\ D \triangleleft_f C$ implies $B \wedge D \triangleleft_f A \wedge C$,

(vi) $B \triangleleft A,\ D \triangleleft C$ implies $B \wedge D \triangleleft A \wedge C$.

We shall say that a partial frame L is *finitely (cover) regular* if each cover of L has a finitely regular (strong) refinement which is a cover as well. A partial frame L is said to be regular if any its element a is a join of some $\mathcal{F}$-subset of the set $\sigma(a) = \{x \in L : x \triangleleft a\}$. Regular partial frames are cover regular and cover regular partial frames are finitely regular.

Moreover, it is easy to prove that any paracompact (compact) partial frame is finitely regular.

In the sequel, let $\mathcal{B}, \mathcal{E}$ be functions with the domain $\mathcal{S}_{\mathcal{F}}$ and the codomain $\mathcal{S}et$ satisfying the following conditions:

(E0) $L \in \mathcal{S}_{\mathcal{F}}$ implies $1 \in \mathcal{B}(L),\ \mathcal{B}(L) \subseteq L$ is closed under finite meets,

(E1) $L \in \mathcal{S}_{\mathcal{F}},\ t \in \mathcal{B}(L)$ implies $\{t\} \in \mathcal{E}(L),\ \mathcal{E}(L) \subseteq 2^{\mathcal{B}(L)} \cap \mathcal{F}(L)$,

(E2) $L \in \mathcal{S}_{\mathcal{F}},\ t \in \mathcal{B}(L),\ E \in \mathcal{E}(L)$ implies $\{t\} \wedge E \in \mathcal{E}(L)$,

(E3) $L \in \mathcal{S}_{\mathcal{F}},\ E \in \mathcal{E}(L)$ and for each $f \in \mathcal{E}(L)$ there is $E_f \in \mathcal{E}(L)$ such that $f = \bigvee E_f$ implies $\bigcup\{E_f : f \in E\} \in \mathcal{E}(L)$,

(E4) $L, K \in \mathcal{S}_{\mathcal{F}}, E \in \mathcal{E}(L), H \in \mathcal{E}(K)$ implies $i_L(E) \wedge i_K(H) \in \mathcal{E}(L \oplus K)$.

If $E \in \mathcal{E}(L)$, we shall speak about a set of a type $\mathcal{E}$. It is sometimes convenient to suppose the following strengthening of the condition **(E1)**:

(E1') $L \in \mathcal{S}_{\mathcal{F}},\ C \subseteq\subseteq \mathcal{B}(L)$ implies $C \in \mathcal{E}(L),\ \mathcal{E}(L) \subseteq 2^{\mathcal{B}(L)} \cap \mathcal{F}(L)$.

Definition 3.3.. We shall say that a partial frame L is an *$\mathcal{E}$-partial frame* (*$\mathcal{ED}$-partial frame*) if $C \in \mathcal{C}(L)$ implies there is $D \in \mathcal{C}(L) \cap \mathcal{E}(L)$ ($D \in \mathcal{D}(L) \cap \mathcal{E}(L)$) such that $D < C$ i.e. every cover of L has a (dense) refinement of the type $\mathcal{E}$.

A morphism $f : K \to L$ of partial frames is said to be $\mathcal{E}$-good if $f(\mathcal{E}(K)) \subseteq \mathcal{E}(L)$.

Remark 3.4. In the sequel, let L_γ, $\gamma \in \Gamma$, be a family of finitely regular $\mathcal{E}$-partial frames, $K \subseteq B$, $\{\downarrow[(u)] : u \in K\} \in \mathcal{F}(L)$, $\bigvee s(K) = 1$, $\overline{K} = \{x \in B : x \in M' \subseteq\subseteq B, \bigvee s(M') = 1$ implies there is a subset Q'_x of $B, s(Q'_x) < s(K \cup (M' - \{x\}))$, $s(Q'_x) \lhd_f s(K \cup (M' - \{x\}))$, $s(Q'_x) \in \mathcal{E}(L) \cap \mathcal{C}(L)\}$.

Lemma 3.5. *Let $S_\gamma, \gamma \in \Gamma$ be a system of covers of $L_\gamma, \gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. Then there are $T_\gamma, \gamma \in \Gamma$, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathcal{C}(L_\gamma) \cap \mathcal{E}(L_\gamma), \gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$.*

Proof. Let $\Gamma_0 \subseteq \Gamma$ be finite such that $\gamma \notin \Gamma_0$ implies $S_\gamma = \{1\}$. For each $\gamma \in \Gamma_0$ there is T_γ, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathcal{C}(L_\gamma) \cap \mathcal{E}(L_\gamma)$, for each $\gamma \notin \Gamma_0$ we put $T_\gamma = \{1\}$ i.e. $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathcal{C}(L_\gamma) \cap \mathcal{E}(L_\gamma)$ by **(E0), (E1).** □

Lemma 3.6. *Let $t \in B, s(t) \in \mathcal{B}(L), A \subseteq\subseteq B, W \subseteq\subseteq \overline{K}, (K \cup A) \cap W = \emptyset, s(t) \lhd_f s(A \cup W)$. Then there is $T \subseteq B$ such that $s(T) < s(K \cup A), s(T) \lhd_f s(K \cup A), s(T) \in \mathcal{E}(L), \bigvee s(T) = s(t)$.*

Proof. The proof will be done by induction according to card W. If card $W = 0$ we are ready – we put $T = \{t\}$. Let card $W = n > 0$ and the lemma holds for all $k < n$. Then there is an element $w \in W$ such that $\bigvee s(D(t) \cup A \cup (W - \{w\}) \cup \{w\}) = 1$, where $D(t) \subseteq\subseteq B$ such that $\bigvee s(D(t)) \wedge s(t) = 0$ and $\bigvee s(D(t)) \vee \bigvee s(A \cup W) = 1$. Since $w \in \overline{K}$,$D(t) \cup A \cup (W - \{w\})$ is finite there is $Q_w \subseteq B, s(Q_w) < s(K \cup (D(t) \cup A \cup (W - \{w\}))$, $s(Q_w) \lhd_f s(K \cup D(t) \cup A \cup (W - \{w\})$, $s(Q_w) \in \mathcal{E}(L) \cap \mathcal{C}(L)\}$.

We put $S = \{t\} \wedge Q_w$. Then $s(S) < s(K \cup A \cup (W - \{w\}))$. Let $z \in S$. Then $z = t \wedge y$ for some $y \in Q_w$. We have $1 = s(y)^* \vee s(K' \cup D(t) \cup A \cup (W - \{w\})), K' \subseteq\subseteq K$. This implies $1 = s(z)^* \vee s(K' \cup A \cup (W - \{w\}))$ i.e. $s(z) \lhd_f s(K \cup A \cup (W - \{w\})), \bigvee s(S) = s(t), s(S) \in \mathcal{E}(L)$ by **(E2)**. By induction, for each $z \in S$ there is $T_z \subseteq B$ such that $s(T_z) < s(K \cup A), s(T_z) \lhd_f s(K \cup A), \bigvee s(T_z) = s(z), s(T_z) \in \mathcal{E}(L)$.

We put $T = \bigcup\{T_z : z \in S\}$. Then by **(E3)** $s(T) \in \mathcal{E}(L), \bigvee s(T) = \bigvee\{T_z : z \in S\} = \bigvee s(S) = s(t), s(T) \lhd_f s(K \cup A)$. $\square$

Proposition 3.7. *It holds* $\overline{K} = B$.

Proof. Evidently, $K \subseteq \overline{K}$. Namely, if $k \in K,\ k \in M' \subseteq\subseteq B, \bigvee s(M') = 1$ then by 3.1 there are finite covers $S_\gamma, \gamma \in \Gamma$ of $L_\gamma, \gamma \in \Gamma,\ S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma, \mathbb{X} S_\gamma < s(M')$. Then by 3.5 there are $T_\gamma, \gamma \in \Gamma,\ T_\gamma < S_\gamma,\ T_\gamma \lhd_f S_\gamma,\ T_\gamma \in \mathcal{C}(L_\gamma) \cap \mathcal{E}(L_\gamma), \gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. We put $Q_k = \mathbb{X} T_\gamma$. Then $Q_k = s(Q_k'), Q_k' \subseteq B$, and by **(E4)** and that $M' \subseteq\subseteq K \cup (M' - \{k\})$ we have that $Q_k < s(K \cup (M' - \{k\})),\ Q_k \lhd_f s(K \cup (M' - \{k\})),\ Q_k \in \mathcal{E}(L) \cap \mathcal{C}(L)$.

Since K is a cover of L there is an $\mathcal{F}$-connector $c = \big(m, (R_\gamma^\uparrow, R_\gamma^\downarrow, \gamma \in \Gamma), m_1, m_2\big)$, where $m = m_2 = B, m_1 = \bigcup_{u \in K} \downarrow(u) \in \mathcal{F}_l(B)$.

Evidently, $\overline{K} \subseteq m, \overline{K}$ is a lower set. Namely, let $x \in \overline{K},\ z \leq x,\ z \in B,\ z \in M' \subseteq\subseteq B, \bigvee s(M') = 1$. Then $\bigvee \left(s\left((M' - \{z\}) \cup \{x\}\right)\right) = 1$ and it implies there is a subset Q_x' of $B, s(Q_x') < s(K \cup (M' - \{z\})),\ s(Q_x') \lhd_f s(K \cup (M' - \{z\})),\ s(Q_x') \in \mathcal{E}(L) \cap \mathcal{C}(L)\}$. We may put $Q_z' = Q_x'$. Then $z \in \overline{K}$.

It is easy to see that to prove $1 \in \overline{K} = m = B$ we have only to check the following implication as in 3.1

$$(n R_{\gamma_0}^\uparrow x)\ \&\ (n \subseteq \overline{K}) \Longrightarrow x \in k,$$

$\gamma_0 \in \Gamma$ arbitrary, fixed.

It is enough to do it for a standard relation $S_{\gamma_0}^\uparrow$.

Let $(x \neq 0) \& (x_{\gamma_0} = \bigvee W_{\gamma_0})\ \ \&\ \ (x = x_{\gamma_0} \wedge \bigwedge X_p)\ \& (W = W_{\gamma_0} \wedge \bigwedge X_p)$ and $W \subseteq \overline{K}, W_{\gamma_0} \in \mathcal{F}(L_{\gamma_0})$. We have to prove that $x \in \overline{K}$.

Let $x \in M' \subseteq\subseteq B,\ \bigvee s(M') = 1$. Again by 3.1 we have that there are finite covers $S_\gamma, \gamma \in \Gamma$ of $L_\gamma, \gamma \in \Gamma,\ S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$ and $\mathbb{X} S_\gamma < s(M')$. We put $S_{\gamma_0}' = \{y \in S_{\gamma_0} : y \not\leq x_{\gamma_0}\} \cup W_{\gamma_0}, S_\gamma' = S_\gamma$ for $\gamma \neq \gamma_0$. By 3.5 there are $T_\gamma, \gamma \in \Gamma,\ T_\gamma < S_\gamma',\ T_\gamma \lhd_f S_\gamma'$, $T_\gamma \in \mathcal{C}(L_\gamma) \cap \mathcal{E}(L_\gamma), \gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. We put $T = \mathbb{X} T_\gamma, S' = \mathbb{X} S_\gamma'$. Then $T < S' < s((M' - \{x\}) \cup W), \bigvee T = 1$ and clearly $T \lhd_f s(S'), T \in \mathcal{E}(L)$ by **(E4)** .

Evidently, for each $s(t) \in T$ there are

$$A_t \subseteq\subseteq (M' - \{x\} \cup (W \cap K) \subseteq (M' - \{x\} \cup K,$$

$$W_t \subseteq\subseteq (W - K) \cap (B - (M' - \{x\})) \subseteq \overline{K}$$

such that $(A_t \cup K) \cap W_t = \emptyset$ and $s(t) \triangleleft_f s(A_t \cup W_t)$. By the preceding lemma, there are sets $T_t \subseteq B$ such that $s(T_t) < s(K \cup A_t), s(T_t) \triangleleft_f s(K \cup A_t), s(T_t) \in \mathcal{E}(L), \bigvee s(T_t) = s(t)$. We put $Q'_x = \bigcup\{T_t : t \in T\}$. Then $Q'_x \subseteq B, s(Q'_x) < s(K \cup (M' - \{x\})),\ s(Q'_x) \triangleleft_f s(K \cup (M' - \{x\})),\ s(Q'_x) \in \mathcal{E}(L) \cap \mathcal{C}(L)\}$ by **(E3)**. We have that $x \in \overline{K}$. □

Theorem 3.8 General cover theorem. *Let $L_\gamma, \gamma \in \Gamma$ be a family of finitely regular $\mathcal{E}$-partial frames. Then the sum $L = \bigoplus_{\gamma \in \Gamma} L_\gamma$ is a finitely regular $\mathcal{E}$-partial frame.*

Proof. Let $C \in \mathcal{C}(L)$. Then there is a basic refinement $s(K),\ K \subseteq B$ of C. By 3.7. $1 \in \overline{K}$ i.e. putting $M' = \{1\}$ there is a subset Q of $B, s(Q) < s(K), s(Q) \triangleleft_f s(K), s(Q) \in \mathcal{E}(L) \cap \mathcal{C}(L)\}$. □

Corollary 3.9. *Let $L_\gamma, \gamma \in \Gamma$ be a family of finitely regular partial framess. Then the sum $L = \bigoplus_{\gamma \in \Gamma}$ is a finitely regular partial frame.*

Proof. We put, for any partial frame K, $\mathcal{B}(K) = K, \mathcal{E}(K) = \mathcal{F}(K)$. Clearly, the conditions **(E0)-(E4)** are satisfied. □

Theorem 3.10. *Let $L_\gamma, \gamma \in \Gamma$ be a family of compact partial frames. Then the sum L is a compact partial frame.*

Proof. We put $\mathcal{B}(L) = L, \mathcal{F}(L) = \mathcal{F}in(L) = \{A : A \subseteq\subseteq L\}$. One can easily check that **(E0)-(E4** are satisfied. □

Theorem 3.11. *Let $L_\gamma, \gamma \in \Gamma$ be a family of paracompact partial frames. Then the sum L is paracompact.*

Proof. We put $\mathcal{B}(L) = L, \mathcal{F}(L) = \mathcal{LF}in(L) = \{A : A \in \mathcal{F}(L), A$ is locally finite$\}$. Since every paracompact locale is finitely regular the rest follows 3.10. □

Recall that a partial frame is *κ-compact*, κ being a a regular cardinal, if every cover has subcover of a cardinality $< \kappa$. Proceeding in the same vein as above we have

Theorem 3.12.. *Let $L_\gamma, \gamma \in \Gamma$ be a family of κ-compact finitely regular partial frames. Then the product L is κ-compact.*

Proof. Put $\mathcal{B}(L) = L, \mathcal{E}(L) = \{F : F \in \mathcal{F}(L), cardF \in \kappa\}$. $\square$

Theorem 3.13. *Let $L_\gamma, \gamma \in \Gamma$ be a family of supercompact partial frames. Then the sum L is supercompact.*

Proof. We put $\mathcal{B}(L) = \{1\}, \mathcal{E}(L) = \{\{1\}\}$. $\square$

Remark 3.14. From now, let $L_\gamma,\ \gamma \in \Gamma$, be a family of finitely regular $\mathcal{ED}$-partial frames, $K \subseteq B$, $\{\downarrow[(u)] : u \in K\} \in \mathcal{F}(L)$, $\bigvee s(K) = 1$, $\widehat{K} = \{x \in B : x \in M' \subseteq\subseteq B, \bigvee s(M') = 1$ implies there is a subset Q'_x of $B, s(Q'_x) < s(K \cup (M' - \{x\}))$, $s(Q'_x) \lhd_f s(K \cup (M' - \{x\}))$, $s(Q'_x) \in \mathcal{E}(L) \cap \mathcal{D}(L)\}$.

Lemma 3.15. *Let $S_\gamma, \gamma \in \Gamma$ be a system of covers of $L_\gamma, \gamma \in \Gamma$, $S_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$. Then there are $T_\gamma, \gamma \in \Gamma$, $T_\gamma < S_\gamma$, $T_\gamma \lhd_f S_\gamma$, $T_\gamma \in \mathcal{D}(L_\gamma) \cap \mathcal{E}(L_\gamma), \gamma \in \Gamma$ such that $T_\gamma = \{1\}$ for all but finitely many $\gamma \in \Gamma$.*

Proof. It is evident. $\square$

Lemma 3.16. *Let $t \in B, s(t) \in \mathcal{B}(L), A \subseteq\subseteq B, W \subseteq\subseteq \widehat{K}, (K \cup A) \cap W = \emptyset, s(t) \lhd_f s(A \cup W)$. Then there is $T \subseteq B$ such that $s(T) < s(K \cup A), s(T) \lhd_f s(K \cup A), s(T) \in \mathcal{E}(L), (\bigvee s(T))^* = s(t)^*$.*

Proof. The proof follows the same arguments as the proof of lemma 2.2 in [12]

Proposition 3.17. *It holds $\widehat{K} = B$.*

Proof. It is a straightforward modification of 3.7. $\square$

Now, we have

Theorem 3.18 General dense cover theorem. *Let $L_\gamma, \gamma \in \Gamma$ be a family of finitely regular $\mathcal{ED}$-partial frames. Then the sum $L = \bigoplus_{\gamma \in \Gamma} L_\gamma$ is a finitely regular $\mathcal{ED}$-partial frame.*

Theorem 3.19. *Let $L_\gamma, \gamma \in \Gamma$ be a family of finitely regular almost paracompact (almost κ-compact) partial frames. Then the sum $L = \bigoplus_{\gamma \in \Gamma}$ is a finitely regular almost paracompact (almost κ-compact) partial frame.*

Proof. Recall that a partial frame is almost paracompact (almost κ-compact) if every its cover has a locally finite dense refinement (a refinement of a cardinality $< \kappa$.) We put $\mathcal{B}(L) = L, \mathcal{F}(L) = \{A : A \subseteq\subseteq L\, A \text{ is locally finite}\}$ ($\mathcal{F}(L) = \{A : A \subseteq\subseteq L\, cardA \in \kappa\}$). One can easily check that **(E0)-(E4)** are satisfied. The rest is evident. $\square$

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