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HERMITIAN KERNELS, HILBERT Q-MODULES AND ANDO DILATION

JAN PASEKA

ABSTRACT. A theory of hermitian kernels in the context of Hilbert Q-modules is presented. Applications are given, including an extension of the generalized Naimark dilation theorem and the KSGNS construction for Hilbert Q-modules and dilations of isometric representations.

In his influential paper [3], P. Halmos proved that every contraction on a Hilbert space has a unitary dilation. This is later generalized by Sz.-Nagy to his celebrated power dilation theorem: for every contraction T on a Hilbert space $\mathcal{H}$, there is a unitary operator U on a Hilbert space $\mathcal{H}'$ and an isometry V from $\mathcal{H}$ to $\mathcal{H}'$ such that every power T^n of T dilates to the unitary U^n i.e. the Sz.-Nagy dilation applies not just to T, but to the discrete semigroup generated by T. W. Stinespring [16] has proved that any completely positive map P of a space of bounded linear operators $\mathcal{B}(\mathcal{H})$ on a Hilbert space H into itself can be represented in the form $P(x) = V^* \circ \pi(x) \circ V$ for all $x \in \mathcal{B}(\mathcal{H})$; here $V : \mathcal{H} \to \mathcal{H}'$ is a bounded linear operator and π is a representation of $\mathcal{B}(\mathcal{H})$ on some Hilbert space $\mathcal{H}'$. The method used to produce a representation from a completely positive map is called the Gelfand-Naimark-Segal construction, or GNS construction, and provides one of the basic tools of C*-algebra theory (see [4]). The concept of a quantale was introduced in [7] as "a quantization" of the term locale.

In this paper we present analogues of the above theorems in the fully algebraic context of Hilbert Q-modules.

The organization of the paper is as follows. The first part of the paper recalls the notion of a Hilbert Q-module and of an adjointable map. Further, we introduce the notion of a hermitian kernel playing in the context of Hilbert Q-modules similar role as a positive kernel does for Hilbert C*-modules. We prove a lattice-theoretical version of the Kolmogorov decomposition theorem for Hilbert Q-modules. Using this result, we show first that a hermitian kernel

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has a minimal Naimark dilation if and only if it is a Toeplitz kernel and secondly we obtain a concrete representation of Hilbert Q-modules.

In the second part of the paper we show that for each isometric adjointable representation on a Hilbert Q-module M of an Ore semigroup we can find a unitary representation on a Hilbert Q-module M' of the corresponding group and containing M as a Hilbert Q-submodule. Applying this result we obtain an analogue of the celebrated theorem of Ando [1]. Our version of the theorem is also in a sense more general. Moreover, we obtain the KSGNS construction for Hilbert Q-modules and involutive quantales and the GNS construction for $\bigvee$-semilattices and arbitrary quantales. Subsequently, in the last part of the paper we make some concluding remarks.

The paper is closely related to the papers [9], [11] and [13] where the interested reader can find unexplained terms, notation and motivation concerning the subject. For facts concerning quantales in general we refer to [15]. We refer the reader to [4] and [6] for missing backround of C*-algebras and Hilbert C*-modules. The monograph [18] will be our reference for the terminologies and results of the Sz.-Nagy-Foias contraction theory.

1. Hermitian kernels and Hilbert Q-modules

As we shall be studying hermitian kernels in the context of Hilbert Q-module theory, we recall now some basic terminology, notation and results of that theory. The theory of Hilbert Q-modules (we refer the reader to [13] for details and examples) is a generalization of the theory of complete semilattices with a duality and it studies modules over an involutive quantale Q endowed with Q-valued inner products. Throughout this section, Q will be an involutive quantale (almost always unital).

Let Q be an involutive quantale, M a right (left) Q-module with a right (left) module action $\diamond$ ($\bullet$). We say that M is a *right* (*left*) pre-Hilbert Q-module (*right* (*left*) *Hilbert Q-module, strict Hilbert Q-module)* if M is equipped with a map $\langle -,-\rangle : M\times M\to Q$, called the *pre-inner product* (*inner product*), such that for all $a\in Q$, $m,n\in M$ and $m_i\in M$, where $i\in I$, the conditions (1)-(4) ((1)-(5), (1)-(6)) are satisfied

$$\langle m,n\rangle\cdot a=\langle m,n\diamond a\rangle \quad (a\cdot\langle m,n\rangle=\langle a\bullet m,n\rangle); \tag{1}$$

$$\bigvee_{i\in I}\langle m_i,n\rangle=\langle\bigvee_{i\in I} m_i,n\rangle; \tag{2}$$

$$\bigvee_{i\in I}\langle m,m_i\rangle=\langle m,\bigvee_{i\in I} m_i\rangle; \tag{3}$$

$$\langle m,n\rangle^*=\langle n,m\rangle; \tag{4}$$

$$\langle -,m\rangle=\langle -,n\rangle \ (\langle m,-\rangle=\langle n,-\rangle) \text{ implies } m=n; \tag{5}$$

$$\langle m, m\rangle = 0 \text{ implies } m = 0. \tag{6}$$

Let $f : M \to N$ a map between right (left) Hilbert Q-modules. We say that a map $f^* : N \to M$ is a **-adjoint to f* and f is *adjointable* if

$$\langle f(m), n\rangle = \langle m, f^*(n)\rangle$$

for all $m \in M$, $n \in N$. Automatically, f is then a Q-module homomorphism i.e. $f \in \mathcal{H}_Q(M, N)$.

The $\bigvee$-semilattice of all adjointable maps from M to N is denoted by $\mathcal{A}_Q(M, N)$. Note that $\mathcal{A}_Q(M)$, the $\bigvee$-semilattice of all adjointable maps from M to itself, is an involutive quantale. Moreover, if $Q = \mathbf{2}$, where $\mathbf{2}$ is the 2-element Boolean algebra then we put $\mathcal{Q}(M) = \mathcal{A}_{\mathbf{2}}(M)$. Moreover, $\bigvee$-semilattices with a duality (complete ortholattices) are exactly the Hilbert (strict) $\mathbf{2}$-modules.

Let us point out that, similarly as for Hilbert C*-modules, the existence of a *-adjoint is not automatic, unlike for $\bigvee$-semilattices with a duality (complete ortholattices).

Let $f : M \to N$ be a map between pre-Hilbert Q-modules M, N. We say that f is *inner-product preserving* if, for all $m_1, m_2 \in M$, $\langle m_1, m_2\rangle = \langle f(m_1), f(m_2)\rangle$. Any inner-product preserving map between Hilbert Q-modules is injective. We say that f is *isometric* if f is an inner-product preserving module homomorphism. Note that if f is a isometric map between Hilbert Q-modules then f may not be adjointable.

Let $f : M \to N$ be an adjointable map. Then f is said to be *unitary* if $f^* \circ f = \mathrm{id}_M$ and $f \circ f^* = \mathrm{id}_N$. In this case f is an isometric surjective Q-module homomorphism. Conversely, if f has these properties, it is unitary. If a unitary map from M to N exists, then M and N are said to be *unitarily equivalent.* A subset $P \subseteq M$ is said to be a *Hilbert Q-submodule of M* if P is a Q-submodule of M and a Hilbert Q-module w.r.t. the induced inner-product on P.

Definition 1.1. *Let T be an involutive semigroup and let Q, Q' be involutive quantales, M, M', M_v Hilbert Q-modules, S an arbitrary non-empty set. A map $k : S \times S \to Q'$ is said to be a* hermitian kernel *if $k(s,t)^* = k(t,s)$, for all $s, t \in S$ i.e. k is a hermitian matrix over Q'.*

Let $f : T \to Q'$ be a map. We shall say that f is hermitian *if the map $k_f : T \times T \to Q'$ defined by $k_f(a,b) = f(a^* \cdot b)$, for all $a, b \in T$ is a hermitian kernel.*

We shall say that $f : Q \to Q'$ is completely hermitian *if it is hermitian $\bigvee$-preserving map.*

Let $\varphi \in \mathcal{A}_Q(M)$, $\psi \in \mathcal{A}_Q(M')$. *The operator* ψ *is said to* dilate *the operator* φ *if there is an isometry* $w : M \to M'$ *such that* $\varphi = w^* \circ \psi \circ w$. *We say that* ψ *is a dilation of* φ.

A map $v : S \to \mathcal{A}_Q(M, M_v)$ *is said to be a* Kolmogorov decomposition for a kernel k, $k : S \times S \to \mathcal{A}_Q(M)$ *if* $k(s,t) = v(s)^* v(t)$, *for all* $s, t \in S$. *Moreover, if* M_v *is a strict Hilbert* Q*-module, we shall say that* v *is a* strict Kolmogorov decomposition for a kernel k. *If any* $p \in M_v$ *is of the form* $p = \bigvee\{v(s_i)(m_i) : s_i \in S, m_i \in M, i \in I\}$, *then* v *will be said to be* minimal.

Note that, for a unital involutive quantale Q, a map $f : Q \to Q'$ is completely hermitian if and only if it is a *- and $\bigvee$-preserving map.

Every hermitian kernel with values in $\mathcal{A}_Q(M)$ has an essentially unique minimal Kolmogorov decomposition. This is the content of the following result. The proof follows similar ideas as the proof for Hilbert C^*-modules in [9].

Theorem 1.2. *Let* Q *be an involutive quantale,* M *a Hilbert* Q*-module,* S *an arbitrary non-empty set and* k *a hermitian kernel from* $S \times S$ *to* $\mathcal{A}_Q(M)$. *Then there exists a minimal Kolmogorov decomposition for* k. *Moreover, if* $v : S \to \mathcal{A}_Q(M, M_v)$ *and* $w : S \to \mathcal{A}_Q(M, M_w)$ *are any two such minimal Kolmogorov decompositions, then there exists a unique unitary* $U : M_v \to M_w$ *such that* $U \circ v(s) = w(s)$ *for all* $a \in S$.

Proof. First, let us prove that M^S is a pre-Hilbert Q-module. Evidently, it is a Q-module. We shall endow M^S with a pre-inner product $\langle -, - \rangle_S : M^S \times M^S \to Q$ by setting

$$\langle f, g \rangle_S = \bigvee_{s,t \in S} \langle k(s,t)(f(t)), g(s) \rangle . \tag{7}$$

Now, let us prove the property (1). We have, for all $a \in Q$ and all $f, g : S \to M$,

$$\begin{aligned} a \cdot \langle f, g \rangle_S &= a \cdot \textstyle\bigvee_{s,t \in S} \langle k(s,t)(f(t)), g(s) \rangle = \bigvee_{s,t \in S} a \cdot \langle k(s,t)(f(t)), g(s) \rangle \\ &= \textstyle\bigvee_{s,t \in S} \langle k(s,t)(a \bullet f(t)), g(s) \rangle = \bigvee_{s,t \in S} \langle k(s,t)((a \bullet f)(t)), g(s) \rangle \\ &= \langle a \bullet f, g \rangle_S . \end{aligned}$$

Let us check the property (4) from the definition of pre-inner product. For all $f, g : S \to M$, we have

$$\begin{aligned} \langle f, g \rangle_S &= \textstyle\bigvee_{s,t \in S} \langle k(s,t)(f(t)), g(s) \rangle = \bigvee_{s,t \in S} \langle f(t), k(s,t)^*(g(s)) \rangle \\ &= \textstyle\bigvee_{s,t \in S} \langle f(t), k(t,s)(g(s)) \rangle = \bigvee_{s,t \in S} \langle k(t,s)(g(s)), f(t) \rangle^* \\ &= \left(\textstyle\bigvee_{s,t \in S} \langle k(t,s)(g(s)), f(t) \rangle\right)^* = \langle g, f \rangle_S^* . \end{aligned}$$

We shall prove the property (3) (note that the property (2) follows from the properties (3) and (4)). We have, for all $f, g_i : S \to M$, $i \in I$,

$$\begin{array}{rcl} \bigvee_{i\in I} \langle f, g_i\rangle_S &=& \bigvee_{i\in I}\bigvee_{s,t\in S} \langle k(s,t)(f(t)), g_i(s)\rangle \\ &=& \bigvee_{s,t\in S} \left\langle k(s,t)(f(t)), \bigvee_{i\in I} g_i(s)\right\rangle \\ &=& \bigvee_{s,t\in S} \left\langle k(s,t)(f(t)), (\bigvee_{i\in I} g_i)(s)\right\rangle \\ &=& \left\langle f, \bigvee_{i\in I} g_i\right\rangle_S. \end{array}$$

Hence, M^S is a pre-Hilbert Q-module. For any map $f : S \to M$, we shall define a map $K(f) : S \to M$ by setting $K(f)(s) = \bigvee_{t\in S} k(s,t)(f(t))$ and we shall denote by M_v the set of all these maps $K(f)$. Evidently, M_v is a Q-submodule of the $Q-$module M^S. Namely, if $K(f_i) \in M_v$, $i \in I$, then $(\bigvee_{i\in I} K(f_i))(s) = \bigvee_{i\in I, t\in S} k(s,t)(f_i(t)) = \bigvee_{t\in S} k(s,t)((\bigvee_{i\in I} f_i)(t)) = K(\bigvee_{i\in I} f_i)(s)$, for all $s \in S$. Similarly, for all $a \in Q$, $K(f) \in M_v$, $s \in S$, we have

$$\begin{array}{rcl} (a \bullet K(f))(s) &=& \bigvee_{t\in S} a \bullet k(s,t)(f(t)) = \bigvee_{t\in S} k(s,t)(a \bullet f(t)) \\ &=& \bigvee_{t\in S} k(s,t)((a \bullet f)(t)) = K(a \bullet f)(s). \end{array}$$

Moreover, we may endow M_v with an inner-product by setting

$$\langle K(f), K(g)\rangle_v = \langle f, g\rangle_S. \tag{8}$$

To prove the correctness of the definition (8) we shall check that, for all $f_1, f_2, g_1, g_2 : S \to M$ such that $K(f_1) = K(f_2)$ and $K(g_1) = K(g_2)$, we have

$$\begin{array}{rcl} \langle K(f_1), K(g_1)\rangle_v &=& \bigvee_{s,t\in S} \langle k(s,t)(f_1(t)), g_1(s)\rangle \\ &=& \bigvee_{s\in S} \left\langle \bigvee_{t\in T} k(s,t)(f_1(t)), g_1(s)\right\rangle \\ &=& \bigvee_{s\in S} \langle K(f_1)(s), g_1(s)\rangle = \bigvee_{s\in S} \langle K(f_2)(s), g_1(s)\rangle = \\ &=& \langle K(f_2), K(g_1)\rangle_v = \langle K(g_1), K(f_2)\rangle_v^* \\ &=& \langle K(g_2), K(f_2)\rangle_v^* = \langle K(f_2), K(g_2)\rangle_v. \end{array}$$

Moreover, we have that M_v is a pre-Hilbert Q-module. Let us check the last property (5). First of all, we shall define, for all $m \in M$ and all $t \in S$, a map $e_m^t : S \to M$ as follows:

$$e_m^t(s) = \begin{cases} m & \text{if } t = s, \\ 0 & \text{otherwise} \end{cases}$$

for all $s \in S$. Let us compute $K(e_m^t)$. We have $K(e_m^t)(s) = \bigvee_{t'\in S} k(s,t')e_m^t(t') = k(s,t)(m)$. Then $K(e_m^t) = k(-,t)(m)$.

Let $g_1, g_2 : S \to M$ be arbitrary, fixed mappings and let us assume that, for all maps $f : S \to M$, we have $\langle K(f), K(g_1)\rangle_v = \langle K(f), K(g_2)\rangle_v$. Then, for all $t \in S$ and all $m \in M$, we have

$$\begin{array}{rcl} \left\langle K(e_m^t), K(g_1)\right\rangle_v &=& \bigvee_{s\in S} \left\langle K(e_m^t)(s), g_1(s)\right\rangle = \bigvee_{s\in S} \langle k(s,t)(m), g_1(s)\rangle \\ &=& \bigvee_{s\in S} \langle m, k(s,t)^*(g_1(s))\rangle = \bigvee_{s\in S} \langle m, k(t,s)(g_1(s))\rangle \\ &=& \left\langle m, \bigvee_{s\in S} k(t,s)(g_1(s))\right\rangle = \langle m, K(g_1)(t)\rangle. \end{array}$$

Similarly, for all $t \in S$ and all $m \in M$, we have

$$\left\langle K(e^t_m), K(g_2)\right\rangle_v = \langle m, K(g_2)(t)\rangle .$$

Then, we have that $\langle m, K(g_1)(t)\rangle = \langle m, K(g_2)(t)\rangle$ for all $m \in M$ i.e. $K(g_1)(t) = K(g_2)(t)$ i.e. $K(g_1) = K(g_2)$. Thus, M_v is a Hilbert Q-module.

Now, let us find a minimal Kolmogorov decomposition v for k. For all $s \in S$, we shall define maps $v(s) : M \to M_v$ and $u(s) : M_v \to M$ as follows:

$$v(s)(m) = K(e^s_m) = k(-, s)(m) \quad \text{and} \quad u(s)(K(f)) = K(f)(s),$$

for each $m \in M$ and all maps $f : S \to M$. Then we have, for each $m \in M$ and all maps $f : S \to M$,

$$\langle m, u(s)(K(f))\rangle = \langle m, K(f)(s)\rangle = \langle K(e^s_m), K(f)\rangle_v = \langle v(s)(m), K(f)\rangle_v$$

i.e. $v(s)$ and $u(s)$ are adjointable. Moreover, if $s, t \in S$ and $m, n \in M$, then

$$\begin{array}{rcl}\langle (v(s)^* \circ v(t))(m), n\rangle & = & \langle v(t)(m), v(s)(n)\rangle_v = \left\langle K(e^t_m), K(e^s_n)\right\rangle_v \\ & = & \bigvee_{s' \in S} \langle k(s', t)(m), K(e^s_n)(s')\rangle = \langle k(s, t)(m), n\rangle\end{array}$$

i.e. $k(s, t)(m) = (v(s)^* \circ v(t))(m)$ i.e. $k(s, t) = v(s)^* \circ v(t)$. To prove the minimality of v note that any map $f : S \to M$ can be written as $f = \bigvee\{e^t_m : t \in S, m = f(t)\}$. Since $K(e^t(m)) = v(t)(m)$ we have that $K(f) = K(\bigvee\{e^t_m : t \in S, m = f(t)\}) = \bigvee\{K(e^t_m) : t \in S, m = f(t)\} = \bigvee\{v(t)(m) : t \in S, m = f(t)\}$.

Suppose now that $v : S \to \mathcal{A}_Q(M, M_v)$ and $w : S \to \mathcal{A}_Q(M, M_w)$ are any two minimal Kolmogorov decompositions for k. We shall define a map $U : M_v \to M_w$ as follows: $U(\bigvee_{i\in I} v(s_i)(m_i)) = \bigvee_{i\in I} w(s_i)(m_i)$ for all $s \in S$ and all $m \in M$. First, let us check the correctness of U. Assume that $\bigvee_{i\in I} v(s_i)(m_i) = \bigvee_{j\in J} v(t_j)(n_j)$, $s_i, t_j \in S$, $m_i, n_j \in M$, $i \in I$, $j \in J$. Then we have

$$\begin{array}{l}\left\langle \bigvee_{i\in I} w(s_i)(m_i), w(s)(m)\right\rangle_w = \left\langle w(s)^*(\bigvee_{i\in I} w(s_i)(m_i)), m\right\rangle = \\ \left\langle \bigvee_{i\in I} w(s)^*(w(s_i)(m_i)), m\right\rangle = \left\langle \bigvee_{i\in I} k(s, s_i)(m_i), m\right\rangle = \\ \left\langle \bigvee_{i\in I} v(s)^*(v(s_i)(m_i)), m\right\rangle = \left\langle \bigvee_{i\in I} v(s_i)(m_i), v(s)(m)\right\rangle_v = \\ \left\langle \bigvee_{j\in J} v(t_j)(n_j), v(s)(m)\right\rangle_v = \left\langle \bigvee_{j\in J} v(s)^*(v(t_j)(n_j)), m\right\rangle = \\ \left\langle \bigvee_{j\in J} k(s, t_j)(n_j), m\right\rangle = \left\langle \bigvee_{j\in J} w(s)^*(w(t_j)(n_j)), m\right\rangle = \\ \left\langle \bigvee_{j\in J} w(t_j)(n_j), w(s)(m)\right\rangle_w ,\end{array}$$

for all $s \in S$ and all $m \in M$. This implies $\bigvee_{i\in I} w(s_i)(m_i) = \bigvee_{j\in J} w(t_j)(n_j)$. Similarly, we have a map $U' : M_w \to M_v$ defined as follows: $U'(\bigvee_{i\in I} w(s_i)(m_i)) = \bigvee_{i\in I} v(s_i)(m_i)$ for all $s \in S$ and all $m \in M$. Clearly, U and U' are inverse to each other. Now, we have

$$\left\langle U(\bigvee_{i\in I} v(s_i)(m_i)), \bigvee_{j\in J} w(t_j)(n_j)\right\rangle_w = \left\langle \bigvee_{i\in I} w(s_i)(m_i), \bigvee_{j\in J} w(t_j)(n_j)\right\rangle_w$$
$$= \bigvee_{i\in I, j\in J} \langle w(s_i)(m_i), w(t_j)(n_j)\rangle_w = \bigvee_{i\in I, j\in J} \langle w(t_j)^*(w(s_i)(m_i)), n_j\rangle =$$
$$\bigvee_{i\in I, j\in J} \langle k(t_j, s_i)(m_i), n_j\rangle = \bigvee_{i\in I, j\in J} \langle m_i, k(s_i, t_j)(n_j)\rangle =$$
$$\bigvee_{i\in I, j\in J} \langle m_i, v(s_i)^*(v(t_j)(n_j))\rangle = \bigvee_{i\in I, j\in J} \langle v(s_i)(m_i), v(t_j)(n_j)\rangle_v =$$
$$\left\langle \bigvee_{i\in I} v(s_i)(m_i), \bigvee_{j\in J} v(t_j)(n_j)\right\rangle_v = \left\langle \bigvee_{i\in I} v(s_i)(m_i), U'(\bigvee_{j\in J} w(t_j)(n_j)\right\rangle_v,$$

for all $s_i, t_j \in S$, $m_i, n_j \in M$, $i \in I$, $j \in J$. Hence, U is an adjointable map with $U^* = U' = U^{-1}$. Thus, U is a unitary map. Also, $U(v(s)) = w(s)$, for all $s \in S$. ▌

Theorem 1.3. *Let Q be an involutive quantale, M a strict Hilbert Q-module, S an arbitrary non-empty set and k a hermitian kernel from $S \times S$ to $\mathcal{A}_Q(M)$. Then there exists a minimal strict Kolmogorov decomposition for k. Moreover, if $v : S \to \mathcal{A}_Q(M, M_v)$ and $w : S \to \mathcal{A}_Q(M, M_w)$ are any two such minimal strict Kolmogorov decompositions, then there exists a unique unitary $U : M_v \to M_w$ such that $U \circ v(s) = w(s)$ for all $s \in S$.*

Proof. It is enough to check that the inner-product $\langle -, -\rangle_v$ defined in the proof of theorem 1.2 satisfies the condition 6. Let $f : S \to M$ be any map. Then $\langle K(f), K(f)\rangle = 0$ implies

$$\langle K(f), K(f)\rangle_v = \bigvee_{s,t\in S} \langle k(s,t)(f(t)), f(s)\rangle = \bigvee_{s,t\in S} \langle v(s)^*(v(t)(f(t))), f(s)\rangle =$$
$$\bigvee_{s,t\in S} \langle v(t)(f(t)), v(s)(f(s))\rangle = \left\langle \bigvee_{t\in S} v(t)(f(t)), \bigvee_{s\in S} v(s)(f(s))\right\rangle$$
$$= \left\langle \bigvee_{t\in S} v(t)(f(t)), \bigvee_{t\in S} v(t)(f(t))\right\rangle = 0.$$

Since M is strict we have that $\bigvee_{t\in S} v(t)(f(t)) = 0$ i.e. $v(s)^*(\bigvee_{t\in S} v(t)(f(t)))$ $= 0$ for all $s \in S$ i.e. $K(f)(s) = \bigvee_{t\in S} k(s,t)(f(t)) = \bigvee_{t\in S} v^*(s)(v(t)(f(t))) = 0$ for all $s \in S$ i.e. $K(f) = 0$. ▌

The following definition was motivated by the paper [14].

Definition 1.4. *Let S be a unital semigroup with unit e, Q an involutive quantale, M a Hilbert Q-module, $k : S \times S \to \mathcal{A}_Q(M, M)$ a hermitian kernel such that*

$$k(s,t) = k(cs, ct)$$

for all $c, s, t \in S$. Then k is said to be a Toeplitz kernel*. We say that a Toeplitz kernel is unital if $k(e,e) = \mathrm{id}_M$.*

We say that a hermitian kernel $k : S \times S \to \mathcal{A}_Q(M, M)$ admits a (unital) Naimark dilation *if there is a Hilbert Q-module M_v, an adjointable map (an isometry) $w \in \mathcal{A}_Q(M, M_v)$ and a semigroup $\{U_s\}_{s\in S}$ of isometries on M_v that dilates k, i.e.*

- $U_s \circ U_t = U_{st}$,
- $U_e = \mathrm{id}_{M_v}$,
- $k(s,t) = (U_s \circ w)^* \circ U_t \circ w$

and $U_s \in \mathcal{A}_Q(M_v, U_s(M_v))$ *for all* $s, t \in S$.

The Naimark dilation is called strict *if* M_v *is a strict Hilbert* Q*-module,* minimal *if* $M_v = \{\bigvee_{s \in T, m \in N} U_s(w(m)) : T \subseteq S, N \subseteq M\}$, adjointable *if* $U_s \in \mathcal{A}_Q(M_v, M_v)$ *for any* $s \in S$ *and* unitary *if* U_s *is unitary for each* $s \in S$.

Theorem 1.5. *Let* S *be a unital semigroup with unit* e, Q *an involutive quantale,* M *a Hilbert* Q*-module,* $k : S \times S \to \mathcal{A}_Q(M, M)$ *a hermitian kernel. Then* k *is a (unital) Toeplitz kernel if and only if it admits a minimal (unital) Naimark dilation. In this case the minimal Naimark dilation is unique up to an isomorphism. Moreover, if* $zS = S$ *for any* $z \in S$ *then the Naimark dilation is adjointable and unitary.*

Proof. Let v be a minimal Kolmogorov decomposition for k, $z \in S$ arbitrary, fixed. Let $x, y \in S$. Then

$$v(zx)^* \circ v(zy) = k(zx, zy) = k(x, y) = v(x)^* \circ v(y).$$

Let us put $v_z(x) = v(zx)$. Then $v_z : S \to \mathcal{A}_Q(M, M_v)$ is another Kolmogorov decomposition for k.

Let us define a map $V_z : M^S \to M^S$ by the prescription

$$V_z(f)(x) = \bigvee\{f(s) : zs = x\}$$

for all $x \in S$. Note that $V_z(M^S) = \{f \in M^S : f(s) = 0 \text{ for any } s \notin zS\}$. We shall show that V_z is a module homomorphism. Let $F \subseteq M^S$. Then

$$V_z(\bigvee F)(x) = \bigvee\{\bigvee_{f \in F} f(s) : zs = x\} = \bigvee_{f \in F} \bigvee\{f(s) : zs = x\} = \bigvee_{f \in F} V_z(f)(x).$$

Similarly, let $a \in Q$, $f \in M^S$. Then

$$(a \bullet V_z(f))(x) = a \bullet \bigvee\{f(s) : zs = x\} = \bigvee\{a \bullet f(s) : zs = x\} = V_z(a \bullet f)(x).$$

We shall prove that V_z is an isometry. Evidently,

$$\begin{array}{rcl}
\langle V_z(f), V_z(g)\rangle_S & = & \bigvee_{x,y \in S} \langle k(x,y)(V_z(f)(y)), V_z(g)(x)\rangle \\
& = & \bigvee_{x,y \in S} \langle k(x,y)(V_z(f)(y)), V_z(g)(x)\rangle \\
& = & \bigvee_{x,y \in S} \langle k(x,y)(\bigvee\{f(t) : zt = y\}), \bigvee\{g(s) : zs = x\}\rangle \\
& = & \bigvee_{x,y \in S} \langle \bigvee\{k(x,y)(f(t)) : zt = y\}, \bigvee\{g(s) : zs = x\}\rangle \\
& = & \bigvee_{x,y \in S} \langle \bigvee\{k(zs,zt)(f(t)) : zt = y\}, \bigvee\{g(s) : zs = x\}\rangle \\
& = & \bigvee_{s,t \in S} \langle k(s,t)(f(t)), g(s)\rangle = \langle f, g\rangle_S .
\end{array}$$

Evidently, $M_{v_z} = \{K(V_z(f)) : f \in M^S\} \subseteq M_v$ is a pre-Hilbert Q-submodule of M_v. The validity of the last property (5) follows by the same arguments as in theorem 1.2 applied on functions $e_m^{zt} \in V_z(M^S)$, $t \in S$. So we have that M_{v_z} is a Hilbert Q-module.

Let us define a map $V_z' : M^S \to M^S$ by

$$V_z'(g)(v) = g(zv)$$

for all $v \in S$. Then, for all $f \in M^S$ and $g \in V_z(M^S)$,

$$\begin{array}{rcl} \langle V_z(f), g\rangle_S & = & \bigvee_{s,t\in S} \langle k(s,t)(V_z(f)(t)), g(s)\rangle \\ & = & \bigvee_{s,t\in zS} \langle k(s,t)(\bigvee\{f(v) : zv = t\}), g(s)\rangle \\ & = & \bigvee_{s\in zS, v\in S} \langle k(s,zv)(f(v)), g(s)\rangle \\ & = & \bigvee_{u\in S, v\in S} \langle k(zu,zv)(f(v)), g(zu)\rangle \\ & = & \bigvee_{u\in S, v\in S} \langle k(u,v)(f(v)), V'_z(g)(u)\rangle = \langle f, V'_z(g)\rangle_S . \end{array}$$

This gives us, for all $f \in M^S$ and all $g_1, g_2 \in V_z(M^S)$ such that $K(g_1) = K(g_2)$,

$$\begin{array}{rcl} \langle K(f), K(V'_z(g_1))\rangle_S & = & \langle f, V'_z(g)_1\rangle_S = \langle V_z(f), g_1\rangle_S \\ & = & \langle V_z(f), g_2\rangle_S = \langle f, V'_z(g_2)\rangle_S = \langle K(f), K(V'_z(g_2))\rangle_S \end{array}$$

i.e. $K(V'_z(g_1)) = K(V'_z(g_2))$. Recall that $M_v \subseteq V'_z(V_z(M^S))$. Namely, since $K(f)(s) = \bigvee_{t\in S} k(zs,zt)(f(t))$ we have

$$V'_z(V_z(K(f)))(s) = V_z(K(f))(zs) = \bigvee\{K(f)(s') : zs' = zs\} = K(f)(s).$$

Let us define a map $U_z : M_v \to M_v$ by $U_z(K(f)) = K(V_z(f))$. Let us check that U_z is correctly defined. Assume that $K(f_1) = K(f_2)$, $g \in V_z(M^S)$. We have

$$\begin{array}{rcl} \langle K(V_z(f_1)), K(g)\rangle_S & = & \langle V_z(f_1), g\rangle_S = \langle f_1, V'_z(g)\rangle_S \\ & = & \langle f_2, V'_z(g)\rangle_S = \langle K(V_z(f_2)), K(g)\rangle_S \end{array}$$

i.e. $K(V_z(f_1)) = K(V_z(f_2))$. U_z is a module homomorphism such that $U_z(v(s)m) = K(V_z(e^s_m)) = K(e^{zs}_m) = v(zs)(m)$. Similarly, for all $f \in M^S$ and all $g \in V_z(M^S)$,

$$\langle U_z(K(f)), K(g)\rangle_S \;\; = \;\; \langle V_z(f), g\rangle_S = \langle f, V'_z(g)\rangle_S = \langle K(f), K(V'_z(g))\rangle_S$$

i.e. the restriction $U_z : M_v \to M_{v_z}$ is adjointable with an adjoint U_z^*. We shall prove that U_z is an isometry such that $U_z^* \circ U_z = \mathrm{id}_{M_v}$. Clearly,

$$\begin{array}{rl} \langle (U_z^* \circ U_z)(K(f)), K(g)\rangle_S = & \!\!\langle U_z(K(f)), U_z(K(g))\rangle_S = \langle K(V_z(f)), K(V_z(g))\rangle_S \\ = & \!\!\langle V_z(f), V_z(g)\rangle_S = \langle f, g\rangle_S = \langle K(f), K(g)\rangle_S . \end{array}$$

Moreover, we have

$$(U_s \circ U_t)(K(f)) = U_s(K(V_t(f))) = K(V_s(V_t(f))) = K(V_{st}(f)) = U_{st}(K(f))$$

i.e. $U_s \circ U_t = U_{st}$. Similarly,

$$U_e(K(f)) = K(V_e(f)) = K(f) = \mathrm{id}_{M_v}(K(f))$$

i.e. $U_e = \mathrm{id}_{M_v}$. Thus, the map, $U : S \to \mathcal{H}_Q(M_v)$ is an isometric representation of S. Let us put $w = v(e)$. Then

$$(U_s \circ w)^* \circ U_t \circ w = (U_s \circ v(e))^*) \circ (U_t \circ v(e)) = v(se)^* \circ v(te) = v(s)^* \circ v(t) = k(s,t).$$

Moreover, if k is a unital Toeplitz kernel then $\mathrm{id}_M = k(e,e) = v(e)^* v(e) = w^* w$ i.e. w is an isometry.

Now, we shall prove that our Naimark dilation is minimal. We have

$$\begin{aligned} M_v &= \{\bigvee_{s\in T, m\in N} v(s)(m) : T \subseteq S, N \subseteq M\} \\ &= \{\bigvee_{s\in T, m\in N} U(s)(v(e)(m)) : T \subseteq S, N \subseteq M\} \\ &= \{\bigvee_{s\in T, m\in N} U(s)(w(m)) : T \subseteq S, N \subseteq M\}. \end{aligned}$$

To prove the uniqueness, let there is a Hilbert Q-module $M_{v'}$, an isometry $w' \in \mathcal{A}_Q(M, M_{v'})$ and a semigroup $\{U'_s\}_{s\in S}$ of isometries on $M_{v'}$ that dilates k. We shall define an operator $W : M_v \to M_{v'}$ by $W(\bigvee_{i\in I} v(s_i)(m_i)) = \bigvee_{i\in I} v'(s_i)(m_i))$ i.e. $W(\bigvee_{i\in I} U_{s_i}(v(e)(m_i)) = \bigvee_{i\in I} U'_{s_i}(v'(e)(m_i))$.

Since, for any $m, m' \in M$ and $s, t \in S$,

$$\big\langle U_t(v(e)(m)), U_s(v(e)(m'))\big\rangle = \big\langle k(s,t)(m), m'\big\rangle = \big\langle U'_t(v'(e)(m)), U'_s(v'(e)(m'))\big\rangle$$

one can easily see that the operator W is correctly defined, isometric, and in view of minimality it is onto.

Assume that $zS = S$ for any $z \in S$. Then $M_v = M_{v_z}$ and $U_z : M_v \to M_{v_z}$ is adjointable. This gives us that $U_z \in \mathcal{A}(M_v)$. We have $U_z^* \circ U_z = \mathrm{id}_{M_v}$. Let us show that $U_z \circ U_z^* = \mathrm{id}_{M_v}$. Let $g \in M_v = M_{v_z}$. Then $g = U_z(f)$ for some $f \in M_v$. We have $(U_z \circ U_z^*)(g) = U_z(U_z^*(U_z(f))) = U_z(f) = g = \mathrm{id}_{M_v}(g)$.

Let us prove the converse of this theorem. Assume that $\{U_s\}_{s\in S}$ is an isometric representation of S on a Hilbert Q-module M_v, $w : M \to M_v$ an adjointable map. Since $k : S \times S \to \mathcal{A}_Q(M)$ is defined by $k(s,t) = (U_s \circ w)^* \circ U_t \circ w$, we have $k(cs, ct) = (U_{cs} \circ w)^* \circ U_{ct} \circ w = (U_c \circ U_s \circ w)^* \circ U_c \circ U_t \circ w = (U_s \circ w)^* \circ U_c^* \circ U_c \circ U_t \circ w = k(s,t)$. Moreover, if w is an isometry we have that $k(e,e) = w^* \circ U_e^* \circ U_e \circ w = w^* \circ w = \mathrm{id}_M$. So we infer that k is a (unital) Toeplitz kernel. ▌

Corollary 1.6. *Let S be a unital semigroup with unit e, Q an involutive quantale, M a strict Hilbert Q-module, $k : S \times S \to \mathcal{A}_Q(M, M)$ a hermitian kernel. Then k is a (unital) Toeplitz kernel if and only if it admits a minimal strict (unital) Naimark dilation. In this case the minimal Naimark dilation is unique up to an isomorphism.*

As an easy corollary, we obtain the following analogue of the well known Naimark dilation theorem for groups (see [10]).

Theorem 1.7. *Let Q be an involutive quantale, M a (strict) Hilbert Q-module, G a group, $\rho : G \to \mathcal{A}_Q(M)$ a map such that $k(x,y) = \rho(x^{-1}y)$ is a hermitian kernel from $G \times G$ to $\mathcal{A}_Q(M)$. Then there exists a minimal (strict) Kolmogorov decomposition $v : G \to \mathcal{A}_Q(M, M_v)$ for k. Moreover, there exists an adjointable map $w : M \to M_v$ and a unitary representantion u of G on M_v such that $\rho(x) = w^* \circ u(x) \circ w$, for all $x \in G$ and $M_v = \{\bigvee_{i\in I}(u(x_i) \circ w)(m_i) : x_i \in G, m_i \in M, i \in I\}$. Moreover, if $u : G \to \mathcal{A}_Q(M_v, M_v)$, $u' : G \to \mathcal{A}_Q(M_{v'}, M_{v'})$ are any two such minimal unitary representations, then there exists a unique unitary $W : M_v \to M_{v'}$ such that $W \circ v(g) = v(g)'$ and $u(g) = W^* \circ u'(g) \circ W$ for all $g \in G$.*

Proof. Let v be a minimal Kolmogorov decomposition for k. Let $x, y, z \in G$. Then

$$v(zx)^* \circ v(zy) = \rho(x^{-1}z^{-1}zy) = \rho(x^{-1}y) = v(x)^* \circ v(y)$$

i.e. k is a Toeplitz kernel. The rest is evident. ▌

The unitary representation $u : G \to \mathcal{A}_Q(M_v)$ is called a *dilation* of ρ. Moreover, if $\rho(e) = \mathrm{id}_M$ then the map w is an isometry.

The following proposition is the Hilbert Q-module version of the far reaching Sz.-Nagy Dilation Theorem.

Proposition 1.8. *Let Q be an involutive quantale, M a (strict) Hilbert Q-module, $\varphi \in \mathcal{A}_Q(M)$. Then there is a (strict) unitary dilation $\psi \in \mathcal{A}_Q(M_v)$ of φ such that every power φ^n dilates to the unitary ψ^n.*

Proof. Let us define a map $\rho_{\mathbf{Z}} : \mathbf{Z} \to \mathcal{A}_Q(M)$ by

$$\rho_{\mathbf{Z}}(m) = \begin{cases} \varphi^m; & \text{if } m > 0 \\ (\varphi^*)^m; & \text{if } -m > 0 \\ \mathrm{id}_M; & \text{if } m = 0 > 0. \end{cases}$$

and a map $k_{\mathbf{Z}} : \mathbf{Z} \times \mathbf{Z} \to \mathcal{A}_Q(M)$ by

$$k_{\mathbf{Z}}(m, n) = \rho_{\mathbf{Z}}(n - m).$$

Evidently, $k_{\mathbf{Z}}$ is a Toeplitz kernel satisfying the assumptions of the theorem 1.7 i.e. there is a unitary representation u of $\mathbf{Z}$ on M_v such that $\rho_{\mathbf{Z}}(m) = w^* \circ u(m) \circ w$, $u(m) = u(1)^m$, $\rho_{\mathbf{Z}}(m) = \varphi^m$ for $m > 0$, $\psi = u(1)$ i.e. $\varphi^m = w^* \circ \psi^m \circ w$. ▌

Definition 1.9. *Let S and T be $\bigvee$-lattices with a duality and let $Q \subseteq \mathcal{Q}(S)$ be an involutive subquantale of $\mathcal{Q}(S)$. Let M be a $\bigvee$-sublattice of $\mathcal{A}_{\mathbf{2}}(S, T)$ and suppose that the following conditions are satisfied:*

1. *If $f \in M$, $a \in Q$, then $f \circ a \in M$;*
2. *If $f, g \in M$ then $f^* \circ g \in Q$;*
3. *M is a right Hilbert Q-module.*

We call M a concrete *Q-module.*

The following result states that all Hilbert Q-modules can be represented as concrete Hilbert ones.

Theorem 1.10. *Let Q be an involutive quantale and let M be a Hilbert Q-module. Then there is a faithful representation $\pi : Q \to \mathcal{Q}(S)$, S being a $\bigvee$-lattice with a duality, and an isometric adjointable isomorphism $\mu : M \to F$, F being a concrete Hilbert $\pi(Q)$-submodule of $\mathcal{A}_{\mathbf{2}}(S, T)$, T being a $\bigvee$-lattice with a duality such that*

$$\langle \mu(m), \mu(n) \rangle = \pi(\langle m, n \rangle) \quad \textit{and} \quad \mu(m \diamond a) = \mu(m) \circ \pi(a) \tag{9}$$

for all $m, n \in M$, $a \in Q$.

Proof. Let $\pi : Q \to \mathcal{Q}(S)$ be any faithful representation of Q (see [12]). Then we have a kernel $k : M \times M \to \mathcal{Q}(S)$ defined by $k(m,n) = \pi(\langle m,n \rangle)$. Namely, we have $k(m,n)^* = \pi(\langle m,n \rangle)^* = \pi(\langle m,n \rangle^*) = \pi(\langle n,m \rangle) = k(n,m)$ for all $m, n \in M$. Then, by the theorem 1.2, k admits a classical minimal Kolmogorov decomposition $v : M \to \mathcal{A}_{\mathbf{2}}(S,T)$, where T is a $\bigvee$-lattice with a duality such that $k(m,n) = v(m)^* \circ v(n)$ and $t \in T$ iff $t = \bigvee_{i \in I} v(m_i)(s_i)$ for suitable $m_i \in M$ and $s_i \in S$, $i \in I$. Let us put $F = v(M)$. Then F is a $\bigvee$-sublattice of $\mathcal{A}_{\mathbf{2}}(S,T)$ for which $F \circ \pi(Q) \subseteq F$ and $F^* \circ F \subseteq \pi(Q)$. Namely, we have, for all $a \in Q, m, n \in M$, $s \in S$, $(\pi(a)^* \circ v(m)^* \circ v(n))(s) = (\pi(a)^* \circ \pi(\langle m,n \rangle))(s) = (\pi(a^* \cdot \langle m,n \rangle)(s) = (\pi(\langle a^* \bullet m, n \rangle)(s) = k(a^* \bullet m, n)(s) = (v(a^* \bullet m)^* \circ v(n))(s)$ i.e. $\pi(a)^* \circ v(m)^* = v(a^* \bullet m)^*$ i.e. $v(m) \circ \pi(a) = v(a^* \bullet m) = v(m \diamond a)$, since T is a Hilbert $\mathbf{2}$-module $\bigvee$-generated by the elements of the form $v(n)(s)$, $n \in M$, $s \in S$. Let us put $\mu : M \to F$ by $\mu(m) = v(m)$ for all $m \in M$. We shall prove that μ is an injective map. Let $v(m_1) = v(m_2)$. Then, for all $n \in M$, $a \in S$, $\pi(\langle m_1, n \rangle)(s) = (v(m_1)^* \circ v(n))(s) = (v(m_2)^* \circ v(n))(s) = \pi(\langle m_2, n \rangle)(s)$ i.e. $\pi(\langle m_1, n \rangle) = \pi(\langle m_2, n \rangle)$ for all $n \in M$ i.e. $\langle m_1, n \rangle = \langle m_2, n \rangle$ for all $n \in M$ i.e. $m_1 = m_2$.

Similarly, for all $m, n \in M$, we have $v(m)^* \circ v(n) = k(m,n) = \pi(\langle m,n \rangle) \in \pi(Q)$. So we may define a scalar product on F by $\langle v(m), v(n) \rangle = k(m,n) = \pi(\langle m,n \rangle)$. It is an easy task to check that the $\pi(Q)$-module F is a Hilbert $\pi(Q)$-module. ▌

2. Dilations of isometries and the KSGNS construction

Definition 2.1. *An* Ore semigroup S (*see* [5]) *is a cancellative semigroup such that $S \cdot s \cap S \cdot t \neq \emptyset$ for every pair $s, t \in S$.*

Let us recall that abelian cancellative semigroups and normal generating semigroups (see [8]) are Ore semigroups. Moreover, any Ore semigroup S can be preordered by a relation $x \preceq_r y$ iff $y \in Sx$ or $x = y$. This preorder is directed i.e. for all $x, y \in S$ there is an element $z \in S$ such that $x \preceq_r z$ and $y \preceq_r z$.

Recall the following well known theorem (see [2], [5]).

Theorem 2.2. *A semigroup S can be embedded in a group G with $S^{-1}S = G$ if and only if it is an Ore semigroup. In this case, the group is determined up to canonical isomorphism and every semigroup homomorphism $\varphi : S \to H$ from S into a group H extends to a group homomorphism $\varphi : G \to H$.*

The following theorem is an easy generalization of the Theorem 2.3. in [8]. Let us recall that a map $\varphi : S \to S'$ is an *isometric representation* of a semigroup S into an involutive monoid S' with an involution * and the unit e if $\varphi(s)^* \varphi(s) = e$ for all $s \in S$.

Theorem 2.3. *Let S be an Ore semigroup, $G = S^{-1}S$ a group and let $\varphi : S \to S'$ be an isometric representation of S into an involutive monoid S'.*

Then there exists a unique extension $\varphi' : G \to S'$ of φ having the following properties:

1. $\varphi'(xs) = \varphi'(x)\varphi(s)$ *for all* $x \in G$ *and* $s \in S$*;*
2. $\varphi'(x)^* = \varphi'(x^{-1})$ *for all* $x \in G$*.*

Proof. Since S is an Ore semigroup, every element x of G can be written in the form $x = s^{-1}t$, for some elements s and t of S. So, the uniqueness of φ' is trivial. To get existence, we set $\varphi'(x) = \varphi(s)^*\varphi(t)$, if $x = s^{-1}t$, where $s, t \in S$. We shall show that φ' is well defined. Let us suppose that we can also write $x = u^{-1}v$, where $u, v \in S$. This gives us that $t^{-1} = v^{-1}us^{-1}$. Then there are elements $m, n \in S$ such that $mt = nv$ i.e. $m = nvt^{-1} = nvv^{-1}us^{-1} = nus^{-1}$ i.e. $ms = nu$. Applying φ, we have $\varphi(m)\varphi(s) = \varphi(n)\varphi(u)$ i.e. $\varphi(n)^*\varphi(m)\varphi(s) = \varphi(u)$. We then have $\varphi(s)^*\varphi(t) = \varphi(s)^*\varphi(m)^*\varphi(m)\varphi(t) = \varphi(s)^*\varphi(m)^*\varphi(n)\varphi(v) = (\varphi(n)^*\varphi(m)\varphi(s))^*\varphi(v) = \varphi(u)^*\varphi(v)$. Evidently, the conditions (1) and (2) are satisfied. ▌

Proposition 2.4. *Let Q be an involutive quantale, M a Hilbert Q-module, S an Ore semigroup, $G = S^{-1}S$ a group and let $\rho : S \to \mathcal{A}_Q(M)$ be an isometric representation. Then there exists a unique extension $\rho' : G \to \mathcal{A}_Q(M)$ having the following properties:*

1. $\rho'(xs) = \rho'(x) \circ \rho(s)$ *for all* $x \in G$ *and* $s \in S$*;*
2. $\rho'(x)^* = \rho'(x^{-1})$ *for all* $x \in G$*.*

Moreover, ρ' is hermitian.

Proof. Since $A_Q(M)$ is an involutive monoid the first part of the proposition follows from the theorem 2.3. It remains to show that ρ' is hermitian. Let us put $k'(x, y) = \rho'(x^{-1}y)$ for all $x, y \in G$. Then we have $k'(x, y)^* = \rho'(x^{-1}y)^* = \rho'(y^{-1}x) = k'(y, x)$ i.e. k' is a kernel. ▌

Theorem 2.5. *Let Q be an involutive quantale, M a (strict) Hilbert Q-module, S an Ore semigroup, $G = S^{-1}S$ a group and let $\rho : S \to \mathcal{A}_Q(M)$ be an isometric representation. Then there exists a (strict) Hilbert Q-module M', an isometry $w : M \to M'$ and a unitary representation $u : G \to \mathcal{A}_Q(M')$ such that $\rho(s) = w^* \circ u(s) \circ w$, for all $s \in S$ and $M' = \{\bigvee_{i \in I}(u(x_i) \circ w)(m_i) : x_i \in G, m_i \in M, i \in I\}$.*

Proof. By the proposition 2.4 we have a hermitian map $\rho' : G \to \mathcal{A}_Q(M)$. Applying to it the theorem 1.7 we have the required isometry and the unitary representation. ▌

Suppose that $\mathcal{H}$ is a Hilbert space, while T_1, T_2 are two commutative contractions on $\mathcal{H}$. A celebrated theorem of Ando [1], [17] states the existence of a larger Hilbert space $\mathcal{H}'$ and of a pair of commutative unitaries U_1, U_2 on $\mathcal{H}'$ and an isometry $W : \mathcal{H} \to \mathcal{H}'$ such that for all integers $n_1, n_2 \geq 0$ we have

$$T_1^{n_1} \circ T_2^{n_2} = W^* \circ U_1^{n_1} \circ U_2^{n_2} \circ W.$$

We shall now prove an analogue of this theorem for Hilbert Q-modules.

Definition 2.6. *Let S be a non-empty subset, Q an involutive quantale, M a Hilbert Q-module, $\{T_s\}_{s\in S} \subseteq \mathcal{A}_Q(M)$ and $T_s \circ T_t = T_t \circ T_s$ for all $s,t \in S$. We say that a system $\{V_s\}_{s\in S} \subseteq \mathcal{A}_Q(M')$ in a Hilbert Q-module M' is said to be an* Ando dilation *of the system $\{T_s\}_{s\in S}$ if*

- *there is an isometry $W \in \mathcal{A}_Q(M, M')$,*
- *$V_s \circ V_t = V_t \circ V_s$ for all $s, t \in S$,*
- *$T_{s_1}^{n_1} \circ \cdots \circ T_{s_l}^{n_l} = W^* \circ V_{s_1}^{n_1} \circ \cdots \circ U_{s_l}^{n_l} \circ W$ for all $n_1, \ldots, n_l \in \mathbf{N}$, $s_1, \ldots, s_l \in S$, $l \in \mathbf{N}$.*

The Ando dilation is called strict *if M' is a strict Hilbert Q-module and* minimal *if*

$$M' = \{\bigvee_i V_{s_1}^{n_1} \circ \cdots \circ V_{s_m}^{n_m} \circ W(x_i) : s_1, \ldots, s_l \in S, n_1, \ldots, n_l \in \mathbf{N}, l \in \mathbf{N}, x_i \in M\}.$$

The Ando dilation $\{V_s\}_{s\in S}$ is said to be isometric (unitary) *if any of its elements is an isometry (unitary).*

Theorem 2.7. *Let Q be an involutive quantale, M a (strict) Hilbert Q-module and $\{T_s\}_{s\in S} \subseteq \mathcal{A}_Q(M)$ a set of commuting isometries. Then there exists a (strict) Hilbert Q-module M', an isometry $W : M \to M'$ and unitary commuting operators $\{T_s\}_{s\in S} \subseteq \mathcal{A}_Q(M')$ such that $T_{s_1}^{n_1} \circ \cdots \circ T_{s_l}^{n_l} = W^* \circ U_{s_1}^{n_1} \circ \cdots \circ U_{s_l}^{n_l} \circ W$ for all $n_1, \ldots, n_l \in \mathbf{N}$, $s_1, \ldots, s_l \in \mathbf{S}$, $l \in \mathbf{N}$ and M' is minimal. Moreover, any two minimal unitary Ando dilations of $\{T_s\}_{s\in S}$ are unitarily equivalent.*

Proof. Let us take the free commutative semigroup $F_c(S)$ over the set S. Then we may assume that $F_c(S)$ is a subsemigroup of $\mathbf{N}^S$ generated by the elements $e_s = (e_{st})_{t\in S}$, $e_{st} = \begin{cases} 1 & \text{if } s = t \\ 0 & \text{otherwise.} \end{cases}$ Then $F_c(S)$ is an Ore semigroup since any commutative cancellative semigroup is Ore. Evidently, the free commutative group $G = G_c(S)$ over set S satisfies $G = (-F_c(S)) + F_c(S) \subseteq \mathbf{Z}^S$. Let us define a map $\rho : F_c(S) \to \mathcal{A}_Q(M)$ by $\rho(0) = \mathrm{id}_M$ and $\rho(e_s) = T_s$. Clearly, ρ is an isometric representation of $F_c(S)$. By the theorem 2.5 there exists a Hilbert Q-module M', an isometry $W : M \to M'$ and a unitary representantion $u : G \to \mathcal{A}_Q(M')$ such that $\rho(s) = W^* \circ u(s) \circ W$, for all $s \in S$. Let us put $U_s = u(e_s)$. Then $\{U_s\}_{s\in S}$ are commuting unitaries and $u(n_1 \cdot e_{s_1} + \cdots + n_l \cdot e_{s_l}) = U_{s_1}^{n_1} \circ \cdots \circ U_{s_m}^{n_m}$. We have $M' = \bigvee u(G)(W(M)) = \{\bigvee_i V_{s_1}^{n_1} \circ \cdots \circ V_{s_m}^{n_m} \circ W(x_i) : s_1, \ldots, s_l \in S, n_1, \ldots, n_l \in \mathbf{N}, l \in \mathbf{N}, x_i \in M\}$. The unitary equivalence of two minimal unitary Ando dilations follows from theorem 1.2. ▌

Corollary 2.8. *Let M be a complete ortholattice and $\{T_s\}_{s\in S} \subseteq \mathcal{Q}(M)$ a set of commuting $\bigvee$-preserving injective maps on M. Then there exists a complete ortholattice M', an injective $\bigvee$-preserving map $W : M \to M'$ and commuting*

$\bigvee$-preserving automorphisms $\{T_s\}_{s\in S} \subseteq \mathcal{Q}(M')$ such that $T_{s_1}^{n_1} \circ \cdots \circ T_{s_m}^{n_m} = W^* \circ U_{s_1}^{n_1} \circ \cdots \circ U_{s_m}^{n_m} \circ W$ for all $(n_1, \ldots, n_m) \in \mathbf{N}^m$, $(s_1, \ldots, s_m) \in \mathbf{S}^m$, $m \in \mathbf{N}$ and M' is minimal. Moreover, any two minimal unitary Ando dilations of $\{T_s\}_{s\in S}$ are $\bigvee$-isomorphic.

The next result gives us a lattice-theoretical version of the Stinespring dilation theorem for involutive quantales.

Theorem 2.9. *Let Q and Q' be involutive quantales, Q' unital, let M be a (strict) Hilbert Q-module and let $\rho : Q' \to \mathcal{A}_Q(M)$ be a completely hermitian map. Then there is a (strict) Hilbert Q-module M_ρ, a homomorphism of involutive quantales $\pi_\rho : Q' \to \mathcal{A}_Q(M_\rho)$ and an element $w_\rho \in \mathcal{A}_Q(M, M_\rho)$ such that $\rho(b) = w_\rho^* \circ \pi_\rho(b) \circ w_\rho$, for all $b \in Q'$. Moreover, $M_\rho = \{\bigvee_{i\in I}(\pi_\rho(b_i) \circ w_\rho)(m_i) : b_i \in Q', m_i \in M, i \in I\}$. In this case the minimal Stinespring dilation is unique up to an isomorphism.*

Proof. We know, by the theorem 1.2, that the kernel $k : Q' \times Q' \to \mathcal{A}_Q(M)$, $k(a, b) = \rho(a^* \cdot b)$, for all $a, b \in Q'$, has a minimal Kolmogorov decomposition $v : Q' \to \mathcal{A}_Q(M, M_v)$. Moreover, v is $\bigvee$-preserving. Namely, we have, for all $m \in M$ and all $c \in Q'$, that

$$\begin{aligned} (v(\textstyle\bigvee_{i\in I} b_i)^* \circ v(c))(m) &= \rho((\textstyle\bigvee_{i\in I} b_i)^* \cdot c)(m) = \rho(\textstyle\bigvee_{i\in I} b_i^* \cdot c)(m) \\ &= \textstyle\bigvee_{i\in I} \rho(b_i^* \cdot c)(m) = \textstyle\bigvee_{i\in I}(v(b_i)^* \circ v(c))(m) \\ &= ((\textstyle\bigvee_{i\in I} v(b_i)^*) \circ v(c))(m). \end{aligned}$$

Hence, since any element of M_v is a join of the elements $v(c)(m)$, $c \in Q'$, $m \in M$, we have that $v(\bigvee_{i\in I} b_i)^* = \bigvee_{i\in I} v(b_i)^*$ i.e. $v(\bigvee_{i\in I} b_i) = \bigvee_{i\in I} v(b_i)$, for all $b_i \in Q'$, $i \in I$.

Now, let us put $M_\rho = M_v$ and $\pi_\rho(b)(\bigvee_{i\in I} v(c_i)(m_i)) = \bigvee_{i\in I} v(b \cdot c_i)(m_i)$. First of all, let us prove the correctness of our definition. Assume that $\bigvee_{i\in I} v(c_i)(m_i) = \bigvee_{j\in J} v(d_j)(n_j)$, $c_i, d_j \in Q'$, $m_i, n_j \in M$, $i \in I$, $j \in J$. Then we have

$$\begin{aligned} &\left\langle \textstyle\bigvee_{i\in I} v(b \cdot c_i)(m_i), v(d)(m) \right\rangle_v = \left\langle v(d)^*(\textstyle\bigvee_{i\in I} v(b \cdot c_i)(m_i)), m \right\rangle = \\ &\left\langle \textstyle\bigvee_{i\in I} v(d)^*(v(b \cdot c_i)(m_i)), m \right\rangle = \left\langle \textstyle\bigvee_{i\in I} k(d, b \cdot c_i)(m_i), m \right\rangle = \\ &\left\langle \textstyle\bigvee_{i\in I} \rho(d^* \cdot (b \cdot c_i))(m_i), m \right\rangle = \left\langle \textstyle\bigvee_{i\in I} \rho((d^* \cdot b) \cdot c_i)(m_i), m \right\rangle = \\ &\left\langle \textstyle\bigvee_{i\in I} k(b^* \cdot d, c_i)(m_i), m \right\rangle = \left\langle \textstyle\bigvee_{i\in I} v(b^* \cdot d)^*(v(c_i)(m_i)), m \right\rangle = \\ &\left\langle \textstyle\bigvee_{i\in I} v(c_i)(m_i), v(b^* \cdot d)(m) \right\rangle_v = \left\langle \textstyle\bigvee_{j\in J} v(d_j)(n_j), v(b^* \cdot d)(m) \right\rangle_v = \\ &\left\langle \textstyle\bigvee_{j\in J} v(b \cdot d_j)(n_j), v(d)(m) \right\rangle_v, \end{aligned}$$

for all $s \in S$ and all $m \in M$. This implies that $\bigvee_{i\in I} v(b \cdot c_i)(m_i) = \bigvee_{j\in J} v(b \cdot d_j)(n_j)$.

Let us show that π_ρ is a homomorphism of involutive quantales. Evidently, it preserves arbitrary suprema. Let $a, b \in Q'$. Then, for all $c_i \in Q'$, $m_i \in M$, $i \in I$, we have $\pi_\rho(a \cdot b)(\bigvee_{i\in I} v(c_i)(m_i)) = \bigvee_{i\in I} v((a \cdot b) \cdot c_i)(m_i) = \bigvee_{i\in I} v(a \cdot (b \cdot c_i))(m_i) = \pi_\rho(a)(\bigvee_{i\in I} v(b \cdot c_i)(m_i)) = \pi_\rho(a)(\pi_\rho(b)(\bigvee_{i\in I} v(c_i)(m_i))) = (\pi_\rho(a) \circ$

$\pi_\rho(b))(\bigvee_{i\in I} v(c_i)(m_i))$ i.e. $\pi_\rho(a\cdot b) = \pi_\rho(a)\circ\pi_\rho(b)$. Now, let $a\in Q'$. We have to show that $\pi_\rho(a)^* = \pi_\rho(a^*)$. Assume that $c_i, d_j \in Q'$, $m_i, n_j \in M$, $i\in I$, $j\in J$. We have, for all $d\in Q'$ and all $m\in M$

$$\begin{aligned}
&\langle \pi_\rho(a)(v(c)(m)), v(d)(n)\rangle_v = \langle v(a\cdot c)(m), v(d)(n)\rangle_v = \\
&\langle v(d)^*(v(a\cdot c)(m)), n\rangle = \langle k(d, a\cdot c)(m), n\rangle = \\
&\langle \rho(d^*\cdot(a\cdot c))(m), n\rangle = \langle \rho((d^*\cdot a)\cdot c))(m), n\rangle = \\
&\langle k(a^*\cdot d, c)(m), n\rangle = \langle v(a^*\cdot d)^*(v(c)(m)), n\rangle = \\
&\langle v(c)(m), v(a^*\cdot d)(n)\rangle_v = \langle v(c)(m), \pi_\rho(a^*)(v(d)(n))\rangle_v,
\end{aligned}$$

i.e. $\pi_\rho(a)$ is adjointable to $\pi_\rho(a^*)$. We shall put $w_\rho = v(e)$. Let us show that $\rho(b) = w_\rho^*\circ\pi_\rho(b)\circ w_\rho$, for all $b\in Q'$. We have, for all $m, n\in M$,

$$\begin{aligned}
\left\langle (w_\rho^*\circ\pi_\rho(b)\circ w_\rho)(m), n\right\rangle &= \langle (v(e)^*(\pi_\rho(b)(v(e)(m))), n\rangle \\
&= \langle (v(e)^*\circ v(b))(m), n\rangle = \langle \rho(b)(m), n\rangle .
\end{aligned}$$

Note that we have $m'\in M_\rho$ iff $m' = \bigvee_{a\in A} v(a)(m_a) = \bigvee_{a\in A}(\pi_\rho(a)\circ v(e))(m_a) = \bigvee_{a\in A}(\pi_\rho(a)\circ w_\rho)(m_a)$ for suitably chosen $m_a\in M$, $a\in A\subseteq Q'$.

To prove the uniqueness, let there is a Hilbert Q-module N, an isometry $w\in\mathcal{A}_Q(M,N)$ and an involutive quantale homomorphism $\pi : Q'\to\mathcal{A}_Q(N)$ such that $\rho(b) = w^*\circ\pi(b)\circ w$, for all $b\in Q'$ and $N = \{\bigvee_{i\in I}(\pi(b_i)\circ w)(m_i) : b_i\in Q', m_i\in M, i\in I\}$. We shall define an operator $W : M_\rho\to N$ by

$$W(\bigvee_{i\in I}(\pi_\rho(b_i)\circ w_\rho)(m_i)) = \bigvee_{i\in I}(\pi(b_i)\circ w)(m_i)).$$

Since

$$\begin{aligned}
\langle \pi_\rho(b)\circ w_\rho)(m), \pi_\rho(b')\circ w_\rho)(m')\rangle &= \langle k(b,b')(m), m'\rangle \\
&= \langle \pi(b)\circ w)(m), \pi(b')\circ w)(m')\rangle
\end{aligned}$$

for any $m, m'\in M$ and $b, b'\in Q'$, we can easily obtain by the same considerations as in theorem 1.2 that the operator W is unitary. Moreover, $w = W\circ w_\rho$ and $\pi(a) = W\circ\pi_\rho(a)\circ W^{-1}$, for all $a\in Q'$. ▌

The unique (up to unitary equivalence) triple $(M_\rho, \pi_\rho, w_\rho)$ obtained from ρ as in theorem 2.9 will be called the KSGNS construction (see [6]) associated with ρ (for Kasparov, Stinespring, Gel'fand, Naimark, Segal).

The last construction we consider is the case of $\bigvee$-semilattices (without a duality). Let us recall the following lemma.

Lemma 2.10. *Let L be $\bigvee$-semilattice, $f : L\to L$ a $\bigvee$ −preserving map. Then the map $\overline{f} = f\times f^{\dashv} : L\times L^{op}\to L\times L^{op}$ is $\bigvee$-preserving and $L\times L^{op}$ is a $\bigvee$-semilattice with a duality $'$; here $f^{\dashv}$ is the right adjoint to f and $(a,b)' = (b,a)$. Moreover, there is an injective $\bigvee$-preserving map $\iota_f : L\to L\times L^{op}$ and a surjective $\bigvee$-preserving map $\nu_f : L\times L^{op}\to L$ such that $f = \nu_f\circ(f\times f^{\dashv})\circ\iota_f$.*

Proof. Let us define $\iota_f(l) = (l, 1)$ and $\nu_f(l,k) = l$ for all $l, k\in L$. Then both maps preserve arbitrary joins, ι_f is an injective map and ν_f is an surjective map. Since $f^{\dashv}$ preserves arbitrary meets in L it preserves arbitrary joins in L^{op}.

Then $f \times f^{\dashv}$ is $\bigvee$-preserving and $(\nu_f \circ (f \times f^{\dashv}) \circ \iota_f)(l) = (\nu_f \circ (f \times f^{\dashv}))(l, 1) = \nu_f(f(l), f^{\dashv}(1)) = f(l)$. ∎

Proposition 2.11. *L be $\bigvee$-semilattice, $f : L \to L$ a $\bigvee$ $-$preserving map. Then there is a $\bigvee$-semilattice K with a duality, an injective $\bigvee$-preserving map $\mathbf{i}_L : L \to K$, a surjective $\bigvee$-preserving map $\mathbf{n}_L : K \to L$ and $\bigvee$-automorphism $g : K \to K$ such that every power f^n dilates to the $\bigvee$-automorphism g^n i.e. $f^n = \mathbf{n}_L \circ g^n \circ \mathbf{i}_L$.*

Proof. By the theorem 1.8 there is a unitary dilation $g : K \to K$ of the map $f \times f^{op}$ such that $(f \times f^{op})^n = w^* \circ g^n \circ w$. Let us put $\mathbf{i}_L = w \circ \iota_f$, $\mathbf{n}_L = \nu_f \circ w^*$. Since w is an injective $\bigvee$-preserving map and w^* a surjective $\bigvee$-preserving map we have that $\mathbf{i}_L : L \to K$ is injective $\bigvee$-preserving map and $\mathbf{n}_L : L \to K$ is surjective $\bigvee$-preserving map. We have, for all $l \in L$, $(\mathbf{n}_L \circ g^n \circ \mathbf{i}_L)(l) = \nu_f((w^* \circ g^n \circ w)(\iota_f(l))) = \nu_f((f \times f^{op})^n)(l, 1)) = \nu_f(f^n(l), (f^{op})^n(1)) = f^n(l)$. ∎

Similarily, we have the following

Lemma 2.12. *Let Q be a quantale, Q^d a quantale with the same join operation $\bigvee$ and a new operation of multiplication $\cdot_c$ defined by $a \cdot_c b = b \cdot a$, L a $\bigvee$-semilattice with a duality, $f : Q \to \mathcal{Q}(L)$ a $\bigvee$-preserving map. Then we have an involutive $\bigvee$-preserving map $F : Q \times Q^d \to \mathcal{Q}(L \times L^{op})$, an injective morphism of quantales $i_Q : Q \to Q \times Q_d$ such that $\nu_{\mathrm{id}} \circ F(i_Q(a)) \circ \iota_{\mathrm{id}} = f(a)$ for all $a \in Q$.*

Proof. First, we put $i_Q(a) = (a, 0)$ for all $a \in Q$. Clearly, i_Q is an injective morphism of quantales. Note that evidently $Q \times Q^d$ is an involutive quantale with involution $(a, b)^* = (b, a)$.

Let us define now the map $F : Q \times Q^d \to \mathcal{Q}(L \times L^{op})$ as follows:

$$F(a, b) = f(a) \times f(b)^{\dashv} : L \times L^{op} \to L \times L^{op}$$

for all $a, b \in Q$. Then $f(a)$ resp. $f(b)^{\dashv}$ preserve all suprema in L resp. in L^{op} i.e. the pair $f(a) \times f(b)^{\dashv}$ belongs to $\mathcal{Q}(L \times L^{op})$. Let us show that F preserves arbitrary joins and the involution.

Now, assume $X \subseteq Q \times Q^d$. Then $F(\bigvee X) = F(\bigvee\{(a, b) : (a, b) \in X\}) = F(\bigvee\{a : (a, b) \in X\}, \bigvee\{b : (a, b) \in X\}) = f(\bigvee\{a : (a, b) \in X\}) \times f^{\dashv}(\bigvee\{b : (a, b) \in X\}) = \bigvee\{f(a) \times f(b)^{\dashv} : (a, b) \in X\} = \bigvee\{F(a, b) : (a, b) \in X\} = \bigvee F(X)$.

Now, we shall show that F preserves involution. First, let us denote the right adjoint to $F(a, b)$ by $G(a, b)$. Evidently, $G(a, b) = (f(a)^{\dashv}, f(b))$. Note that we have $F(a, b)^* = {}' \circ G(a, b) \circ {}'$ (see [11], [12]). Then, for all $l, k \in L$,

$$\begin{aligned} F(a,b)^*(l,k) &= (G(a,b)((l,k)'))' = (G(a,b)((k,l)))' = (f(a)^{\dashv}(k), f(b)(l))' \\ &= (f(b)(l), f(a)^{\dashv}(k)) = F(b,a)(l,k) = F((a,b)^*)(l,k). \end{aligned}$$

We have, for all $l \in L$,

$$\begin{aligned}(\nu_{\mathrm{id}} \circ F(i_Q(a)) \circ \iota_{\mathrm{id}})(l) &= (\nu_{\mathrm{id}} \circ F(a,0))(l,1) = (\nu_{\mathrm{id}}(f(a), f(0)^{\dashv}))(l,1) \\ &= \nu_{\mathrm{id}}(f(a)(l), f(0)^{\dashv}(1)) = f(a)(l).\end{aligned} \quad \blacksquare$$

Recall that (unital) quantales play for involutive quantales the same role as operator algebras (norm closed subalgebras of the space $B(\mathcal{H})$ of bounded linear operators on a Hilbert space $\mathcal{H}$) to C*-algebras. Similarly as for operator algebras we shall establish the Gelfand-Naimark-Segal construction for arbitrary quantales.

Theorem 2.13. *Let Q be a unital quantale, let L be a $\bigvee$-semilattice and let $\rho : Q \to \mathcal{Q}(L)$ be a $\bigvee$-preserving map. Then there is a $\bigvee$-semilattice L_ρ with a duality, a quantale homomorphism $\pi_\rho : Q \to \mathcal{Q}(L_\rho)$, an injective $\bigvee$-preserving map $w_\rho : L \to L_\rho$ and a surjective $\bigvee$-preserving map $v_\rho : L_\rho \to L$ such that $\rho(b) = v_\rho \circ \pi_\rho(b) \circ w_\rho$, for all $b \in Q$.*

Proof. Let us apply the theorem 2.9 to the induced map $F : Q \times Q^d \to \mathcal{Q}(L \times L^{op})$ from the preceding lemma. Then we have a $\bigvee$-semilattice L_F with a duality, a homomorphism of unital quantales $\pi_F : Q \times Q^d \to \mathcal{Q}(L_F)$, and an injective $\bigvee$-preserving map $w_F : L \times L^{op} \to L_F$ such that $F(b,c) = w_F^* \circ \pi_F(b,c) \circ w_F$, for all $(b,c) \in Q \times Q^d$.

Let us put $L_\rho = L_F$, $\pi_\rho = \pi_F \circ i_Q$, $w_\rho = w_F \circ \iota_{\mathrm{id}_L}$, $v_\rho = \nu_{\mathrm{id}_L} \circ w_F^*$.

We have, for all $b \in Q$,

$$\begin{aligned}(v_\rho \circ \pi_\rho(b) \circ w_\rho)(l) &= (v_\rho \circ \pi_F(i_Q(b)) \circ w_F)(l,1) \\ &= \nu_{\mathrm{id}_L}(w_F^* \circ \pi_F(i_Q(b)) \circ w_F)(l,1)) \\ &= \nu_{\mathrm{id}_L}(F(b,0)(l,1)) = \nu_{\mathrm{id}_L}((\rho(b) \times \rho(0)^{\dashv})(l,1)) = \rho(b)(l).\end{aligned} \quad \blacksquare$$

3. Conclusion and future work

We have introduced hermitian kernels for Hilbert Q-modules and established basic dilation theorems in the Hilbert Q-module context. We have no doubt that other applications of this technology will follow in the course of time. In particular, we used purely elementary algebraic methods that suggest that there should be a common denominator between the category of Hilbert spaces and bounded mappings (Hilbert C*-modules and adjointable mappings) and the category of $\bigvee$-semilattices with a duality and $\bigvee$-preserving mappings (Hilbert Q-modules and adjointable mappings).

There are several topics in our approach we would like to study in more detail. We hope that after some effort using the KSGNS construction we may obtain more information what the right concept of a point in a quantale should be. Another possibility is to study tensor products of involutive quantales using their representations via $\bigvee$-semilattices with a duality.

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Department of Mathematics, Masaryk University, Janáčkovo nám. 2a, 66295 Brno, Czech Republic

E-mail address: `paseka@math.muni.cz`