Věta 2.7. *Derivace Beziérova křivkového segmentu v bodě t se spočítá takto:*

$$\frac{d^p}{dt^p}\boldsymbol{b}(t) = \frac{n!}{(n-p)!}\sum_{i=0}^{n-p}\Delta^p\boldsymbol{b}_iB_i^{n-p}(t)$$
$$= \frac{n!}{(n-p)!}\Delta^p\boldsymbol{b}_{n-p}^{n-p}(t),$$

kde

$$\Delta^0\boldsymbol{b}_i = \boldsymbol{b}_i$$
$$\Delta^p\boldsymbol{b}_i = \Delta^{p-1}\boldsymbol{b}_{i+1} - \Delta^{p-1}\boldsymbol{b}_i$$
$$\Delta^0\boldsymbol{b}_i^j(t) = \boldsymbol{b}_i^j(t)$$
$$\Delta^p\boldsymbol{b}_i^j(t) = \Delta^{p-1}\boldsymbol{b}_{i+1}^j(t) - \Delta^{p-1}\boldsymbol{b}_i^j(t) \qquad i = j, \ldots, n-p$$

Důkaz. 1. $\frac{d^p}{dt^p}\boldsymbol{b}(t) = \frac{n!}{(n-p)!}\sum_{i=0}^{n-p}\Delta^p\boldsymbol{b}_iB_i^{n-p}(t)$
Matematickou indukcí:
a) $p = 1$.
Využijeme vztahu $\frac{d}{dt}B_i^n(x) = n(B_{i-1}^{n-1}(x) - B_i^{n-1}(x))$.

$$\frac{d}{dt}\boldsymbol{b}(t) = [\sum_{i=0}^{n}\boldsymbol{b}_iB_i^n(t)]'$$
$$= -\boldsymbol{b}_0n(1-t)^{n-1} + \sum_{i=1}^{n-1}\boldsymbol{b}_in[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)] + \boldsymbol{b}_nnt^{n-1}$$
$$= n[B_0^{n-1}(t)(\boldsymbol{b}_1 - \boldsymbol{b}_0) + B_1^{n-1}(t)(\boldsymbol{b}_2 - \boldsymbol{b}_1) + \cdots + B_{n-2}^{n-1}(t)(\boldsymbol{b}_{n-1} - \boldsymbol{b}_{n-2}) + B_{n-1}^{n-1}(t)(\boldsymbol{b}_n - \boldsymbol{b}_{n-1})]$$

Pro $p = 1$ tedy tvrzení platí.
b) Předpokládejme, že tvrzení platí pro p. Dokážeme, že platí pro $p + 1$.

$$\frac{d^{p+1}}{dt^{p+1}}\boldsymbol{b}(t) = \frac{d}{dt}(\frac{n!}{(n-p)!}\sum_{i=0}^{n-p}\Delta^p\boldsymbol{b}_iB_i^{n-p}(t))$$
$$= \frac{n!}{(n-p)!}\frac{(n-p)!}{(n-p-1)!}\sum_{i=0}^{n-p-1}\Delta^1(\Delta^p\boldsymbol{b}_i)B_i^{n-p-1}(t)$$
$$= \frac{n!}{(n-p-1)!}\sum_{i=0}^{n-p-1}\Delta^{p+1}\boldsymbol{b}_iB_i^{n-p-1}(t)$$

2. $\frac{d^p}{dt^p}\boldsymbol{b}(t) = \frac{n!}{(n-p)!}\Delta^p\boldsymbol{b}_{n-p}^{n-p}(t)$
Matematickou indukcí.
a) $p = 1$.
Vycházíme z důkazu první části věty pro $p = 1$.

$$\begin{aligned}
\frac{d}{dt}\boldsymbol{b}(t) &= [\sum_{i=0}^{n}\boldsymbol{b}_i B_i^n(t)]' \\
&= -\boldsymbol{b}_0 n(1-t)^{n-1} + \sum_{i=1}^{n-1}\boldsymbol{b}_i n[B_{i-1}^{n-1}(t) - B_i^{n-1}(t)] + \boldsymbol{b}_n n t^{n-1} \\
&= n[B_0^{n-1}(t)(\boldsymbol{b}_1 - \boldsymbol{b}_0) + B_1^{n-1}(t)(\boldsymbol{b}_2 - \boldsymbol{b}_1) + \cdots + B_{n-2}^{n-1}(t)(\boldsymbol{b}_{n-1} - \boldsymbol{b}_{n-2}) + B_{n-1}^{n-1}(t)(\boldsymbol{b}_n - \boldsymbol{b}_{n-1})] \\
&= n[(B_0^{n-1}(t)\boldsymbol{b}_1 + B_1^{n-1}(t)\boldsymbol{b}_2 + \cdots + B_{n-1}^{n-1}(t)\boldsymbol{b}_n) \\
&- (B_0^{n-1}(t)\boldsymbol{b}_0 + B_1^{n-1}(t)\boldsymbol{b}_1 + \cdots + B_{n-1}^{n-1}(t)\boldsymbol{b}_{n-1})] \\
&= n(\boldsymbol{b}_n^{n-1} - \boldsymbol{b}_{n-1}^{n-1})
\end{aligned}$$

Pro $p = 1$ tedy tvrzení platí.

b)Předpokládejme, že tvrzení platí pro p, dokážeme, že platí i pro $p+1$.

$$\begin{aligned}
\frac{d^{p+1}}{dt^{p+1}}\boldsymbol{b}(t) &= \frac{d}{dt}[\frac{n!}{(n-p)!}\Delta^p \boldsymbol{b}_{n-p}^{n-p}(t)] \\
&= \frac{n!}{(n-p)!}(n-p)\Delta^1(\Delta^p \boldsymbol{b}_{n-p}^{n-p}(t)) \\
&= \frac{n!}{(n-p-1)!}\Delta^p \boldsymbol{b}_{n-p+1}^{n-p}(t) - \Delta^p \boldsymbol{b}_{n-p}^{n-p}(t) \\
&= \frac{n!}{(n-p-1)!}\Delta^{p+1}\boldsymbol{b}_{n-p}^{n-p}(t)
\end{aligned}$$

$\square$