NATURAL OPERATORS
ON THE BUNDLE
OF CARTAN
CONNECTIONS

Ph.D. dissertation

Martin Panák

Brno 2001
Thesis advisor: prof. RNDr. Jan Slovák, DrSc.
Department of Algebra and Geometry
Faculty of Science, Masaryk University Brno
Janáčkovo nám. 2a
662 95 Brno
Czech Republic
Acknowledgement. I would like to thank my thesis advisor prof. Jan Slovák for numerous productive discussions on my thesis. I also thank him for his moral support, which has helped me a lot.
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There are several purposes of this PhD thesis. At first I want my PhD degree. At second it can serve as a brief introductory text to the study of Cartan geometries. Finally it gives a scent of (gauge) natural operators theory in Cartan geometries, where a method of searching these operators is introduced.

We assume that the reader has knowledge corresponding to a standard university course of differential geometry. A list of symbols is appended. If not explicitly defined the terminology used is taken from [KMS].

In the Episodes I and IV we provide a necessary introduction to the problem. We present there some basic facts on Cartan geometries and natural operators. In the Episode V we describe the method of finding natural operators. In each of these chapters there is a section devoted to the reductive geometries. The Episodes II and III are still mere subjects of speculations.

Already the ancient civilizations studied properties of the Euclidean space. Later some men of vision came upon with concepts of other geometries: spherical (navigators), hyperbolic, projective, ... Felix Klein came up with a unifying concept of known geometries. Each geometry has a group, say $G$, of motions which leaves all the studied properties intact. Now if we take a point $x$ of the underlying manifold, $M$ say, and consider the stabilizer of $x$ (the subgroup $S_x$ of the group $G$ of motions of the geometry which maps $x$ to itself) then $M \cong G/S_x$ as manifolds. Thus we can study geometries as pairs $(G, H)$ of Lie groups (groups together with a structure of a manifold; the multiplication is smooth), $H \subset G$ a closed subgroup (so-called Klein geometries). Unfortunately this is not enough to describe all complexities of the possible "tangles" of the space. There is a form on any Lie group $G$ in which nearly all the properties of $G$ are encoded. It is the Maurer-Cartan form (if we forget about the group structure and consider a Lie group just as a manifold, its group structure can be reconstructed just from the form given on the underlying manifold). If we leave out some of the properties of the Maurer-Cartan form we get the notion of the Cartan connection, the idea worked out by Elie Cartan (though the name "Cartan connection" was introduced later). The geometries which are described in this manner are more intrigued then the Klein geometries which are the simplest instance of these.

Natural operators are one of the key interests in differential geometry. Each geometry has a variety of objects associated to it (vector fields, tensor fields, ... ).
These objects can be transformed to other objects. Those assignments which are invariant under the action of the group $G$ are of a special importance. And though the meaning of the words "natural operator" varies in differential geometry from one working group to another, this is what natural operators at the bottom are.

For example, the Euclidean geometry deals with the area of polygons in the plane. The area of a given polygon (with vertices given in a coordinate system) can be evaluated as the determinant of a proper matrix which includes the coordinates of the vertices of the polygon. The determinant is then independent of the Euclidean change of coordinates, that is the area of a polygon does not change by rotating and translating the polygon. Thus "the area of a polygon" is a well defined notion in the Euclidean geometry.

Similarly we study in other geometries invariants which do not change under actions of various groups of transformations on various manifolds. Then the characterization of natural operators on a space gives a list of possible objects which can have in the given geometry a reasonable meaning. The main goal of this thesis is the study of natural operators on Cartan connections.

In [P] we have shown that Cartan connections on principal fibered bundles with a given structure group, say $H$, with values in $\mathfrak{g}$ ($H \subset G$ Lie groups, $\mathfrak{h}, \mathfrak{g}$ their Lie algebras) are (all) sections of a gauge natural bundle which we call the bundle of Cartan connections and we will write $C$ for it. In fact it is a bundle of elements of Cartan connections. It is a functor on the category $\mathcal{PB}_m(H)$ of principal bundles with a structure group $H$ and principal bundle morphisms with local diffeomorphisms as base maps. For each principal bundle $P$ the bundle $CP$ can be viewed as a subbundle of the bundle of principal connections on the associated bundle $P \times_H G$. We use the terms gauge natural bundle and gauge natural operator in the sense of [KMS].

We will study $r$-th order gauge natural operators on the bundle of Cartan connections with gauge natural bundles of the order $(1, 0)$ as target spaces. The bundles of the order $(1, 0)$ are bundles on which the induced action of morphisms from $\mathcal{PB}_m(H)$ depends only on 1-jets of underlying maps and only on values of morphisms in fibers. The notion of a natural sheaf will be used to describe the results. The natural sheaf is in a way the simplest structure we can introduce on the kernel or on the image of a natural operator. If we say that natural operators factorize through the curvature we have to specify what we mean by that: in general we mean that natural operators on Cartan connections factorize through the natural sheaf of their curvatures. Thus we reduce the problem of finding natural operators on Cartan connections to the problem of finding natural operators on the natural sheaf of curvatures of Cartan connections. This is in general still a complicated task. However in some specific cases, like torsion free geometries, we are able to say more about the structure of the natural sheaf of the curvatures. It is a subsheaf in the sheaf which is formed by all sections of an affine bundle.

One of the key results en route to the final theorem is that the natural sheaf of $r$-th order invariant jets of curvature functions of Cartan connections is of order $(1, 0)$, i.e. it
has a tensor character. The invariant jet is an object built with the help of the invariant derivation, which could be understood as a generalization of the covariant derivation for Cartan geometries. The tensor character of the invariant jets (among other nice properties of these) shows that they are really worth to notice.

Thus our results generalize the theorems from [KMS], sections 28 and 52. The first one says that $r$-th order natural operators on the bundle of symmetric linear connections factorize through the curvature operator and its covariant derivations up to order $r - 1$, while the second one shows that first order gauge natural operators on the bundle of principal connections factorize through the curvature operator. The same result (as in the second theorem) on the first order gauge natural operators but on the bundle of Cartan connections is obtained in [P].
There are at least two ways of defining a Cartan geometry. We will use the so-called bundle definition as in [S] (see also for the second definition).

**Definition.** A Cartan geometry \((P, \omega)\) on \(M\) modeled on \((g, h)\) with a group \(H\) is:

(i) a smooth manifold \(M\);
(ii) a principal right \(H\) bundle \(P\) over \(M\);
(iii) a \(g\)-valued form \(\omega\) on \(P\) satisfying following conditions:
   (\(\alpha\)) for each point \(u \in P\) \(\omega_u : T_u(P) \to g\) is an isomorphism;
   (\(\beta\)) \((r_h)^* \omega = \text{Ad}(h^{-1}) \omega\) for all \(h \in H\);
   (\(\gamma\)) \(\omega(\zeta_X) = X\) for all \(X \in h\), where \(\zeta_X\) is the fundamental field of \(X\).

The form \(\omega\) is then called the Cartan connection on \(P\) of type \((g, H)\).

**Example 1.** The canonical Maurer-Cartan form on a Lie group \(G\) is the Cartan connection on the principal fiber bundle \(G \to G/H\), \(H\) closed subgroup of \(G\).

**Example 2.** Let \(\gamma\) be a linear connection on the linear frame bundle \(P^1 M\). Then the affine connection \(\omega = \gamma \oplus \theta\), where \(\theta\) is the soldering form, is one of the best known examples of a Cartan connection.

The equivariance property of a Cartan connection \(\omega\) is translated into its inverse \(\omega^{-1}\) as follows:

**Lemma 1.1.**

\[ \omega^{-1}(uh)X = Tr_h \omega^{-1}(u)(\text{Ad}_h X), \]

for any \(X \in g\).

**Proof.** From the definition of the Cartan connection we have for the maps from \(T_u P\) to \(g\): \(\omega \circ Tr_h = \text{Ad}_{h^{-1}} \circ \omega\). For the inverse maps we get then \(Tr_{h^{-1}} \circ \omega^{-1}(uh) = \omega^{-1}(u) \circ \text{Ad}_h\). \(\square\)

**Principal and Cartan connections.**

There is a close relationship between Cartan and principal connections. At first we present a characterization of connection forms of principal connections.
Lemma 1.2. Let \( \gamma \in \Omega^1(P, g) \) be a \( g \)-valued 1-form on the \( H \)-principle bundle \( P \). Then \( \gamma \) is a connection form of a principal connection on \( P \) if the following two properties are satisfied:

(i) \((r_h)^*\omega = \text{Ad}(h^{-1})\omega \) for all \( h \in H \);
(ii) \( \gamma \) reproduces fundamental vector fields, i.e. \( \omega(\zeta_X) = X \) for all \( X \in \mathfrak{h} \).

Proof. See [KMS]. □

Remark. Conditions (i) and (ii) from the above lemma coincide with conditions (\( \beta \)) and (\( \gamma \)) from the definition of a Cartan connection.

On the other hand Cartan connections can be regarded as a special case of principal connections.

\( ^{\text{PB}}m(H) \subset \text{PB}_m(G) \). Associating a group \( G \) via the left multiplication to a principal \( H \)-bundle we get a \( G \)-principal bundle \( P' = P \times_H G \) with the right principal action \( \{u, g\} \cdot h = \{u, g \cdot h\} \), where \( \{u, g\} \) is the equivalence class corresponding to the pair \((u, g)\) in \( P \times_H G \). Thus the category \( \text{PB}_m(H) \), the category of principal bundles with structure group \( H \), can be seen as a "subcategory" of \( \text{PB}_m(G) \), the category of principal bundles with a structure group \( G \). Namely each principal \( H \)-bundle \( P \) can be identified with a subbundle of \( P \times_H G \) \( \{u, \{e\}\} \), and a \( \text{PB}_m(H) \)-morphism \( \Phi : P \to R \) yields a morphism \( \Phi' : P' \to R' \), \( \Phi'(\{u, g\}) = \{\Phi(u), g\} \). Further we can identify \( (\mathbb{R}^m \times H) \times_H G \) with \( \mathbb{R}^m \times G \) \( \{(x, h), g\} \mapsto (x, hg) \). Notice that the composition of the embeddings described above, \( \mathbb{R}^m \times H \to (\mathbb{R}^m \times H) \times_H G \to \mathbb{R}^m \times G \), is just the canonical embedding \( \mathbb{R}^m \times H \to \mathbb{R}^m \times G \).

Lemma 1.3. There is a bijective correspondence between Cartan connections on a principal \( H \)-bundle \( P \) and principal connections on \( P' = P \times_H G \) whose horizontal spaces do not meet \( TP \subset TP' \).

Proof. A Cartan connection on \( P \), that is a form \( \omega \in \Omega(P; g) \), can be extended as a pseudo-tensorial form of type \((\text{Ad}, g)\) (see [KN], II.5) over the whole \( P' \) \( (P \subset P' \), see above):

\[
\omega_{\{u, g\}} = \text{Ad}(g^{-1})\pi^*_P\omega(u) + \pi^*_G\omega_G(g),
\]

where \( \pi_P : P \times G \to P \) and \( \pi_G : P \times G \to G \) are canonical projections, and \( \omega_G \) is the left Maurer-Cartan form on \( G \), \( \{u, g\} \in P' \) the equivalence class corresponding to \((u, g)\). This is a correct definition due to the invariance properties of \( \omega \) and \( \omega_G \):

\[
\omega_{\{ua, a^{-1}g\}}(Tr_aX_u, Tl_{a^{-1}}Y_g) = \text{Ad}(g^{-1})\omega(ua)(Tr_aX_u) + \omega_G(a^{-1}g)(Tl_{a^{-1}}Y_g) = \text{Ad}(g^{-1})\text{Ad}(a)\omega(a^{-1}g)(X_u) + Tl_{g^{-1}a}(Tl_{a^{-1}}Y_g) = \text{Ad}(g^{-1})\omega(G(g))(Y_g) = (\text{Ad}(g^{-1})\pi^*_P\omega + \pi^*_G\omega_G(g))(X_u, Y_g).
\]
Since both the original Cartan connection $\omega$ on $P$ and the Maurer-Cartan form on $G$ satisfy the invariance property (\(\beta\)) from the definition of the Cartan connection as well as they reproduce the fundamental fields (condition (\(\gamma\))), the extended form $\omega$ on $P'$ satisfies both conditions too. But these two conditions characterize principal connections. Moreover the horizontal space of $\omega$ evidently does not meet the $TP \subset TP'$ bundle because $\omega|_{TP} : TP \to g$ is a bijection. Thus this construction gives rise to principal connections on $P'$, whose horizontal bundle does not meet the tangent bundle of $P$ viewed as the subbundle of $P'$ (see [S, Appendix A] for details). We will call this condition the condition $\dagger$.

Conversely the pull-back of a connection form of a principal connection which satisfies the condition $\dagger$, under the canonical inclusion $i : P \to P \times_H G$ is a Cartan connection on $P$ with values in $g$. The properties (\(\alpha\)) and (\(\beta\)) in the definition of a Cartan connection are satisfied by the original principal connection and does not change by the pull-back. The (\(\alpha\)) condition is satisfied due to the condition $\dagger$. □

Curvature.

In the study of Cartan geometries the curvature form of the given Cartan connection plays a key role.

Definition. We define the curvature of a Cartan connection by the structure equation:

$$K = \frac{1}{2}[[\omega, \omega] + d\omega].$$

Lemma 1.4. The curvature of a Cartan connection is a horizontal 2-form and satisfies the invariance property $r^*_a K = \text{Ad}(a^{-1})K$.

Proof. Let us show first the curvature is a form $K \in \Omega^2(P, g)$. We use the $f$-test. Let $\eta$, $\xi$ be vector fields on $P$, $f$ a function on $P$.

$$\begin{align*}
K(f \cdot \xi, \eta) &= f \cdot (\xi \cdot \omega(\eta)) - \eta \cdot (f \cdot \omega(\xi)) \\
&= -\omega([f \cdot \xi, \eta]) + f \cdot [\omega(\xi), \omega(\eta)]
\end{align*}$$

$$\begin{align*}
&= f \cdot (\xi \cdot \omega(\eta)) - (\eta \cdot f) \cdot \omega(\xi) - f \cdot (\eta \cdot \omega(\xi)) \\
&= -\omega(f \cdot [\xi, \eta] - (\eta \cdot f) \cdot \xi) + f \cdot [\omega(\xi), \omega(\eta)]
\end{align*}$$

$$\begin{align*}
&= f \cdot (\xi \cdot \omega(\eta)) - f \cdot (\eta \cdot \omega(\xi)) - f \cdot (\omega[\xi, \eta]) + f \cdot [\omega(\xi), \omega(\eta)]
\end{align*}$$

$$\begin{align*}
&= f \cdot K(\xi, \eta).
\end{align*}$$

Since $K$ is apparently antisymmetric it is a form. Let us show it is a horizontal one. Let $\omega^{-1}(Z)$, $Z \in h$ be a horizontal vector field on $P$ and $\eta$ a vector field on
Then the structure equation yields:

\[
K(\omega^{-1}(Z), \eta) = d\omega(\omega^{-1}(Z), \eta) + [Z, \omega(\eta)]
\]

\[
= i_{\omega^{-1}(Z)} \circ d\omega(\eta) + \text{ad}_Z \circ \omega(\eta)
\]

\[
= - \text{ad}_Z \circ \omega(\eta) + \text{ad}_Z \circ \omega(\eta) = 0.
\]

And the curvature is really a horizontal form.

The invariance property follows from that of \(\omega\). □

**Remark.** Thus the curvature of a Cartan connection on the principal bundle \(\pi : P \to M\) is a section of the bundle \((P \times \text{Ad} g) \otimes \Lambda^2(T^*M)\).

**Remark.** The geometries with zero curvature are locally isomorphic to the flat models \(G \to G/H\) with the canonical Maurer-Cartan form.

**Lemma 1.5.** For the curvature form \(K\) of the Cartan connection \(\omega\) of type \((g, H)\) on the principal \(H\)-bundle \(P\) and the curvature form \(\tilde{K}\) of a corresponding principal connection \(\tilde{\omega}\) on the bundle \(P \times_H G\) (see Lemma 1.3.) we have

\[
K = i^*(K),
\]

where \(i\) is a canonical inclusion \(P \to P \times_H G\), \(i(u) = \{u, e\}\).

**Proof.**

\[
K = \frac{1}{2} [\omega, \omega] + d\omega = \frac{1}{2} [i^*\tilde{\omega}, i^*\tilde{\omega}] + d(i^*\tilde{\omega})
\]

\[
= i^*(\frac{1}{2} [\tilde{\omega}, \tilde{\omega}] + d\tilde{\omega}) = i^*(\tilde{K}). \quad \square
\]

For the further purposes we define the torsion of a Cartan connection.

**Definition.** We define the torsion of a Cartan connection to be a \(g/\mathfrak{h}\)-part of its curvature form.

**The Invariant Differentiation.**

The following lemma is a necessary technical tool for dealing with sections of associated bundles.

**Lemma 1.6.** Let \(\lambda : H \to \text{GL}(V)\) be a linear representation and let \(P \to M\) be a principal bundle. Sections of the associated vector bundle \(P \times_\lambda V\) correspond bijectively to \(H\)-equivariant smooth functions \(s : P \to V\), \(s(u \cdot g) = \lambda(g^{-1}) s\). We denote the set of such invariant sections by \(C^\infty(P, V)^\lambda\).

**Proof.** See [Sl2]. □
**Definition.** The invariant differential corresponding to a Cartan connection $\omega$ is a mapping $\nabla^\omega : C^\infty(P, V) \to C^\infty(P, g^* \otimes V)$ defined by $\nabla^\omega_X s = \mathcal{L}_{\omega^{-1}(X)} s$.

**Remark.** The invariant derivative, i.e. for each $X \in g$ the mapping $\nabla^\omega_X : C^\infty(P, V) \to C^\infty(P, V)$, is called the universal covariant derivative in [S]. Both names have their foundations. The invariant differentiation gives rise to (practically) any differential operator in any Cartan geometry. In special cases (for example in Riemannian geometry) it can be decomposed into two separate operators and one of it coincides with the classical covariant derivative.

The following lemma shows that the invariant differentiation ”behaves well”, namely that invariant derivatives are sections of a nice associated bundle.

**Lemma 1.7.** $\nabla^\omega$ is a mapping from $C^\infty(P, V)^\lambda$ to $C^\infty(P, V)^{\lambda \otimes \text{Ad}^*}$. That is

$$\nabla^\omega_X s(ph) = \lambda(h^{-1})\nabla^\omega_{\text{Ad}(h)X}s(u).$$

**Proof.** See [S]. □

The invariant derivation can be interpreted as a derivation in the direction of $\omega$-constant vector fields. The derivation in the direction of $\omega^{-1}(X)$ for $X \in h$ is not very interesting, it is the so-called algebraic part of the derivation. This is in fact a straight consequence of the following lemma which we will use also later.

**Lemma 1.8.** Let $P \times_\lambda V$ be an associated vector bundle and $H$ a connected Lie group. Then the following are equivalent:

1. a mapping $s : P \to V$ is $H$-equivariant (i.e. is a section of $P \times_\lambda V$)
2. $\zeta_X s = -\lambda'(X)s$,

where $\lambda'$ is the Lie algebra representation $\lambda' : h \to \text{gl}(V)$ and the derivative at the identity of the representation $\lambda : H \to \text{GL}(V)$.

**Proof.** Let $\zeta_X(u)$ be a fundamental field on $P$. Then $\zeta_X(u)$ is of the form $\frac{\partial}{\partial t}|_0 u \cdot \exp(tX)$. Then for the $H$-equivariant $s : P \to V$ we get:

$$\zeta_X \cdot s(u) = \frac{\partial}{\partial t}|_0 s(u \cdot \exp(tX)) =$$

$$= \frac{\partial}{\partial t}|_0 \lambda((\exp(tX))^{-1})s(u) =$$

$$= \frac{\partial}{\partial t}|_0 \lambda(\exp(-tX))s(u) = -\lambda'(X)s(u).$$

The inverse implication follows from the same identities. □
Corollary 1.9. For $X \in \mathfrak{h}$ and $s \in C_\infty(P,V)^H$ we have $\nabla_X s = -\lambda'(X)s$, where $\lambda'$ is again the Lie algebra representation. □

Well, we see that the invariant differentiation in the direction of fibers of the principal bundle $P$ is not particularly interesting. Then what are the interesting directions? One would guess it should be something like directions of the tangent vectors of the base manifold. Unfortunately there are no such canonical directions in the general. In the reductive case ($\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_-$ as $H$-modules), $\omega^{-1}(\mathfrak{g}_-)$ gives the desired preferred directions. The invariant differentiation in these directions coincides with the usual covariant derivative.

Let us state an easy consequence of the definition of the Cartan connection which we will need later.

Lemma 1.10. Let $\omega$ be a Cartan connection of the type $(\mathfrak{g},H)$ on the $H$-principal bundle $P$ with the projection $\pi : P \to M$ on the base. Then there is an isomorphism $P \times \text{Ad} (\mathfrak{g}/\mathfrak{h}) \simeq TM$, $\{u,[X]\} \simeq T\pi(\omega^{-1}(u)(X))$, where $[X]$ is the equivalent class corresponding to $X \in \mathfrak{g}$ in $\mathfrak{g}/\mathfrak{h}$.

Proof. Firstly the definition of the isomorphism is correct since for $Z \in \mathfrak{h}$ we have

$$T\pi(\omega^{-1}(u)(X + Z)) = T\pi(\omega^{-1}(u)(X) + \omega^{-1}(u)(Z)) = T\pi(\omega^{-1}(u)(X)),$$

since $\omega^{-1}(u)(Z)$ is a vertical vector.

Further we know from the definition of the Cartan connection that $\omega^{-1}(u) : \mathfrak{g} \to T_uP$ is an isomorphism. This means $T\pi(\omega^{-1}(u)(\mathfrak{g})) = T\pi(T_uP) = T_xM$, where $x = \pi(u)$, and $T\pi(\omega^{-1}(u)(\mathfrak{h})) = 0_x$. Thus $T\pi(\omega^{-1}(u)(\mathfrak{g}/\mathfrak{h})) = T_xM$. We have from the definition $\text{dim}(\mathfrak{g}/\mathfrak{h}) = \text{dim} M$, and inevitably $T\pi|_{\omega^{-1}(u)(\mathfrak{g}_-)}$ has to be an isomorphism. □

Reductive Geometries.

Definition. A Cartan geometry modeled on $(\mathfrak{g},\mathfrak{h})$ with group $H$ is reductive if there is an $H$-module decomposition $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_-$ ($H$ acts on $\mathfrak{g}$ and its subalgebras via the Ad representation).

Lemma 1.11. Let $(P,\omega)$ be a reductive Cartan geometry modeled on $(\mathfrak{g},\mathfrak{h})$. Let $\omega$ be a Cartan connection of type $(\mathfrak{g},H)$ on $P$. Then the $\omega_h$-part of $\omega$ is a principal connection on $P$.

Proof. It is sufficient to check the conditions i) and ii) from Lemma 1.2. As for the condition i) $\omega_h$ reproduces fundamental vector fields from the definition of the Cartan connection. Let us check $\omega_h$ is also $H$-equivariant. We know $(r_h)^*\omega = \text{Ad}(h^{-1})\omega$, that is $(r_h)^*(\omega_h + \omega_{\mathfrak{g}_-}) = \text{Ad}(h^{-1})(\omega_h + \omega_{\mathfrak{g}_-})$. Since $\mathfrak{g}_- \subset \mathfrak{g}$ is an $H$-module, comparing $\mathfrak{h}$-parts of the last identity we get the $H$-equivariance for $\omega_h$. □
Example. Let $\gamma$ be a linear connection on the linear frame bundle $P^1M$ and $\theta$ the soldering form. Then we already know that an affine connection $\omega = \gamma \oplus \theta$ is a Cartan connection on $P^1M$. Further the horizontal lift $\omega^{-1}(u)(X)$ of a $X \in \mathfrak{g}_-$ coincides with a horizontal lift to $u$ of $T\pi(\omega^{-1}(u)(X)) \in T_xM$, $x = \pi(u)$ with respect to the linear connection $\gamma$. Thus the invariant derivation with respect to $\omega$ coincides with the classical covariant differentiation with respect to the linear connection $\gamma$.

For the rest of this section we will devote our attention to even more specific case of reductive geometries, namely we will suppose that (in the previous notation) $\mathfrak{g}_- \subset \mathfrak{g}$ is an abelian ideal. This restriction still covers quite a wide range of geometries, including $G$-structures.

Lemma 1.12. Let $\omega = \omega_h \oplus \omega_{\mathfrak{g}_-}$ be a Cartan connection on an $H$-bundle $P$ in a reductive Cartan geometry modeled on $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{g}_-$, $\mathfrak{g}_-$ an abelian ideal. Then the torsion $\kappa_{\mathfrak{g}_-}$ of $\omega$ is the exterior covariant derivative of $\omega_{\mathfrak{g}_-}$ with respect to the principal connection $\omega_h$ on $P$.

Proof. Follows immediately from the structure equation and the fact that $w_{\mathfrak{g}_-}$ is horizontal. $\Box$

General case.

In the general case (i.e. not in the reductive one) things tend to be more intrigued. Still we can choose $\mathfrak{g}_- \subset \mathfrak{g}$ a complementary vector subspace to $\mathfrak{h}$, then $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_-$ as vector spaces. However we will not deal with such a general situation. We will suppose, that $\mathfrak{g}_-$ is a subalgebra of $\mathfrak{g}$.

Definition. Let $s : P \to V$ be an $H$-invariant function, $\omega$ a Cartan connection on $P$. We define the mapping

$$j^1_\omega s := (s, \nabla^\omega s) : P \to V \oplus \mathfrak{g}_-^* \otimes V,$$

and call it the invariant first jet prolongation of $s$ with respect to $\omega$. However we are going to "misuse" the shorter term invariant 1-jet for it (the invariant jet should rather mean the value of the prolongation in one point only).

Now we will study the properties of invariant jets. As we have already said we would like to see invariant derivatives $(\nabla^\omega s)$, $s \in \mathcal{C}\infty(P,V)^\lambda$ to be sections of "nice" associated bundles as in the reductive case. Unfortunately this is impossible for the individual derivatives. However we can view the invariant jet $(s, \nabla^\lambda_X)$ as a section of the bundle $P \times \mathcal{J}^1(\lambda)(V \oplus (\mathfrak{g}_-^* \otimes V))$, where $\mathcal{J}^1(\lambda)$ is the so-called jet prolongation of the action $\lambda$ (see the next definition). The form of the $\mathcal{J}^1(\lambda)$ action comes from the requirement, that the invariant derivation along vector fields from $\mathfrak{h}$ should give the action $\lambda$ up to the minus sign (see Lemma 1.8.).
Definition. Let $\lambda : H \to \text{GL}(V)$ be a representation of $H$ on the vector space $V$. Then we define a mapping $\mathcal{J}^1(\lambda) : \mathfrak{h} \to \text{gl}(\mathcal{J}^1(V))$, where $\mathcal{J}^1(V) := V \oplus (\mathfrak{g}^*_+ \otimes V) :$

$$\mathcal{J}^1(\lambda)(Z)(v, \varphi) = (\lambda(Z)(v), \lambda(Z) \circ \varphi - \varphi \circ \text{ad}_-(Z) + \lambda(\text{ad}_h(Z)(\cdot))(v),$$

where $\text{ad}_-(Z) : \mathfrak{g}_- \to \mathfrak{g}_-$, $X \to [Z, X]_-$ and $\lambda(\text{ad}_h(Z)(\cdot))(v) : \mathfrak{g}_- \to V,$

$$\lambda(\text{ad}_h(Z)(\cdot))(v)(X) = \lambda([Z, X]_h)(v).$$

Lemma 1.13. The mapping $\mathcal{J}^1(\lambda)$ is a representation of $\mathfrak{h}$ on $\mathcal{J}^1(V)$. We call it the jet prolongation of the representation $\lambda$.

Proof. See [CSS]. □

Remark. We should rather use the term "jet-like" prolongation, but we stick to the terminology introduced in [CSS].

We can view the above construction also from a different angle. The action $\lambda$ coincides with a canonical action of $\mathfrak{h}$ on the standard fiber of the first jet prolongation of the associated bundle $G \times_\lambda V$. Thus we get an identification $J^1(G \times_\lambda V) \simeq P \times_{\mathcal{J}^1(\lambda)} \mathcal{J}^1(V)$. For the computing of the canonical action of $\lambda$ on $J^1(V)$ see [Sl].

The $\mathcal{J}^1$ can be extended to a functor on the category of $\mathfrak{h}$-representations: for an $\mathfrak{h}$-module homomorphism $f$ we define $\mathcal{J}^1(f)(v, \varphi) := (f(v), f \circ \varphi)$.

**Higher order derivatives and prolongations.**

The functor $\mathcal{J}^1$ on the category of $\mathfrak{h}$-representations enables us to define functors $\mathcal{J}^r$ on the same category. On vector spaces we get higher jet prolongations $\mathcal{J}^r(V) = \bigoplus_{i=0}^r (\otimes^i \mathfrak{g}^*_+ \otimes V)$ with the representation $\lambda$ of $\mathfrak{h}$ induced from the original $\lambda$ on $V$: firstly we define $\mathcal{J}^2(V)$. $\mathcal{J}^2(V)$ is the subspace of $\mathcal{J}^1(\mathcal{J}^1(V)) = (V \oplus (\mathfrak{g}^*_+ \otimes V)) \oplus \mathfrak{g}^*_+ \otimes (V \oplus (\mathfrak{g}^*_+ \otimes V)) = (V \otimes \mathfrak{g}^*_+ \otimes V) \oplus (\mathfrak{g}^*_+ \otimes V) \oplus (\mathfrak{g}^*_+ \otimes \mathfrak{g}^*_+ \otimes V)$, where the two middle components are equal. One can easily check that then $\lambda$ really induces an action on $\mathcal{J}^2(V)$. It looks like this: for $(v, \varphi, \psi) \in \mathcal{J}^2(V) = V \oplus (\mathfrak{g}^*_+ \otimes V) \oplus (\mathfrak{g}^*_- \otimes \mathfrak{g}^*_+ \otimes V)$, $Z \in \mathfrak{h}$ we have

$$\mathcal{J}^2\lambda(Z)(v, \varphi, \psi) = (\lambda(Z)v, \lambda(Z) \circ \varphi - \varphi \circ \text{ad}_-(Z) + \lambda(\text{ad}_h(Z)(\cdot))(v),$$

$$\lambda(Z) \circ \psi - \psi \circ (\text{id}_{\mathfrak{g}_-}, \text{ad}_-(Z)) - \psi \circ (\text{ad}_-(Z), \text{id}_{\mathfrak{g}_-})$$

$$+ 2\lambda(\text{ad}_h(Z)(\cdot)) \circ \varphi$$

$$- \varphi \circ \text{ad}_-(\text{ad}_h(Z)(\cdot)) + \lambda(\text{ad}_h(Z)(\cdot))(\cdot)v).$$

Inductively we can construct the $r$-th jet prolongation $\mathcal{J}^r(V)$ and the action $\mathcal{J}^r(\lambda)$ on it.
**Definition.** Let $s : P \to V$ be an $H$-invariant function, $\omega$ a Cartan connection on $P$. We define the mapping

$$j^k_\omega s := (s, \nabla^w s, \ldots, (\nabla^w)^k s) : P \to \mathcal{J}^k(V),$$

and call it the invariant $k$-jet of $s$ with respect to $\omega$.

Similarly to the first order case we would like the invariant jets $j^r_\omega s$ to be sections of an associated bundle. Again the individual $r$-th order derivatives are not in general sections of an associated bundle, the ”proper” objects are the invariant jets.

**Lemma 1.14.** The invariant $r$-jet of a section $s \in C^\infty(P \to V)^\lambda$ is an equivariant mapping $j^r_\omega s : P \to \mathcal{J}^r(V)$ with respect to the action $\mathcal{J}^r(\lambda)$ constructed above.

**Proof.** The action $\mathcal{J}^1(\lambda)$ was constructed in such a way so as the invariant 1-jet was the section of $P \times \mathcal{J}^1(\lambda) \mathcal{J}^1(V)$. Then by induction we get the statement also for the invariant $r$-jets. □

Thus we have an identification of the semiholonomic $r$-th jet prolongation $\mathcal{J}^r(P \times_\lambda V)$ of the associated vector bundle $P \times_\lambda V$ with the associated vector bundle $P \times_{\mathcal{J}^r(\lambda)} \mathcal{J}^r(V)$.

**Bianchi and Ricci identities.**

**Definition.** Let $\omega$ be a Cartan connection, $K$ its curvature. Then the formula

$$\kappa(X,Y) = K(\omega^{-1}(X),\omega^{-1}(Y)),$$

well defines a $H$-equivariant function $\kappa : P \to g^*_+ \otimes g^*_- \otimes g$, which is called the curvature function. The curvature function is $H$-equivariant with respect to the action $\lambda : H \to \text{GL}(g^*_+ \otimes g^*_- \otimes g)$:

$$\lambda(a)\kappa(p) = \text{Ad}(a)\kappa(p)(\text{Ad}_-(a^{-1})(\_), \text{Ad}_-(a^{-1})(\_)),$$

where $\text{Ad}_- : h \to \text{gl}(g_-)$ is the $g_-$-part of the Ad representation: $\text{Ad}_-(a)(X) = [\text{Ad}(a)X]_{g_-}$. And the curvature function is a section in the associated bundle $P \times_\lambda g^*_+ \otimes g^*_- \otimes g$.

The curvature function is an important tool for handling Cartan geometries. It is not an arbitrary section of $P \times_\lambda g^*_+ \otimes g^*_- \otimes g$ but it is bounded by Bianchi identities. The Ricci identities describe the interchangeability of the invariant derivatives in different directions.
Lemma 1.15.

\[(\nabla)^k s(u)(X, Y, \ldots, Z) = (\mathcal{L}_{\omega^{-1}(Z)} \circ \cdots \circ \mathcal{L}_{\omega^{-1}(Y)} \circ \mathcal{L}_{\omega^{-1}(X)}) s(u).\]

In particular we obtain

\[ (\nabla^\omega X \nabla^\omega Y - \nabla^\omega Y \nabla^\omega X)s = (\nabla^\omega [X,Y] - \nabla^\omega \kappa_{\cdot \cdot \cdot}(X,Y)) + \lambda(\kappa h(X, Y)) s, \]

the Ricci identity.

Proof. See [CSS]. The prove there is just for special groups and algebras but it works in the general case as well. □

Lemma 1.16. The Bianchi identity.

\[ \sum_{\text{cyclic}} ([\kappa(X,Y),Z] - \kappa(\kappa_{\cdot \cdot \cdot}(X,Y),Z) + \nabla^\omega_Z \kappa(X,Y) + \kappa([X,Y],Z)) = 0, \]

for \(X, Y, Z \in \mathfrak{g}_{-}\).

Proof. See [CSS] (again just for special geometries but can be easily modified for the general case). □
There are some more subtle notions discussed in naturality, not only the invariance to the various groups of transformations but also smoothness and globality. Also the language of various geometers is not always the same (the mathematical as well as the native one). Thus there are several notions of naturality in differential geometry. But the gist of all of these definitions is the same. We will use the terminology from the book [KMS].

**Definition.** A bundle functor on $\mathcal{Mf}_m$ or a natural bundle over $m$-dimensional manifolds is a covariant functor $F : \mathcal{Mf}_m \to \mathcal{FM}$ with following properties:

(i) (Prolongation) $B \circ F = \text{Id}_{\mathcal{Mf}_m}$, where $B : \mathcal{FM} \to \mathcal{Mf}_m$ is the base functor and the induced projections form a natural transformation $p : F \to \text{Id}_{\mathcal{Mf}_m}$.

(ii) (Locality) If $i : U \to M$ is an inclusion of an open submanifold, then $FU = p^{-1}_M(U)$ and $Fi$ is the inclusion of $p^{-1}_M(U)$ into $\mathcal{FM}$.

(iii) (Regularity) If $f : P \times M \to N$ is a smooth map such that for all $p \in P$ the maps $f_p : f(p, \cdot) : M \to N$ are local diffeomorphisms, then $\tilde{F}f : P \times \mathcal{FM} \to FN$, defined by $\tilde{F}f(p, \cdot) = Ff_p$, $p \in P$, is smooth, i.e. smoothly parametrized systems of local diffeomorphisms are transformed into smoothly parametrized systems of fibered local isomorphisms.

$F_0\mathbb{R}^m := p^{-1}_\mathbb{R}^m(0)$ is called the standard fiber of the bundle functor $F$.

A natural bundle $F : \mathcal{Mf}_m \to \mathcal{FM}$ is said to be of finite order $r$, $0 \leq r < \infty$, if for all local diffeomorphisms $f, g : M \to N$ and every point $x \in M$, the equality $j^r_x f = j^r_x g$ implies $Ff|F_x M = Fg|F_x M$ and $r$ is the smallest natural number with this property.

**Example.** The tangent functor $T$ is a first order natural operator.

**Definition.** Associated maps. Let $F$ be a natural bundle, $M, N$ be $m$ dimensional manifolds. We define the associated maps of the bundle functor $F$ as mappings $F_{M,N} : \text{inv } J^r(M, N) \times_M \mathcal{FM} \to FN$, $(j^r_x f, y) \mapsto Ff(y)$.

**Proposition.** The associated maps are smooth.

**Proof.** See [KMS]. □
Definition. Let $Y$ and $Z$ be fibered manifolds over a manifold $M$. A local operator $A : C^\infty Y \to C^\infty Z$ is such a map that for every section $s : M \to Y$ and every point $x \in M$ the value $As(x)$ depends on the germ of $s$ at $x$ only. An operator $A$ is said to be of order $k$, if $k$ is the smallest number such that $j_x^k s = j_x^k q$ implies $As(x) = Aq(x)$. Finally an operator $A : C^\infty Y \to C^\infty Z$ is called a regular operator if every smoothly parametrized family of sections of $Y$ is transformed into a smoothly parametrized family of sections of $Z$.

Definition. Let $A : C^\infty Y \to C^\infty Z$ be a $k$-th order operator. The map $A(j_x^k s) = As(x)$ is called the associated map to the $k$-th order operator $A$.

Lemma 4.1. The associated map to a finite order operator is smooth if and only if $A$ is regular.

Proof. See [KMS]. \hfill \Box

Definition. The induced action $l$ of the jet group $G^r_m = \text{inv } J^r_m(\mathbb{R}^m, \mathbb{R}^m)_0$ on the standard fiber of the $r$-th order natural bundle $F$ is the restriction of the associated map $F_{\mathbb{R}^m,\mathbb{R}^m}|G^r_m \times S$.

Remark. The map $F_{\mathbb{R}^m,M}|_{P^r M \times S} : P^r M \times S \to FM$ well defines a diffeomorphism between the associated bundle $P^r M \times_1 S$ and $FM$. Thus the values of any natural bundle lie in the category of bundles with structure group $G^r_m$ and standard fiber $S$. $Ff : FM \to FN$ is then identified with $(P^r f, \text{id}_S) : P^r M \times_{G^r_m} S \to P^r N \times_{G^r_m} S$.

Definition. A natural operator $A : F \to G$ between two natural bundles $F$ and $G$ is a system of regular operators $A_M : C^\infty(FM) \to C^\infty(GM)$, $M \in O(Mf_m)$, satisfying

(i) for every section $s \in C^\infty(FM \to M)$ and every diffeomorphism $f : M \to N$ it holds

$$A_N(Ff \circ s \circ f^{-1}) = Gf \circ A_M s \circ f^{-1},$$

(ii) $A_U(s_U) = (A_M s)|_U$ for every $s \in C^\infty(FM)$ and every open submanifold $U \subset M$.

Though our main goal is the study of gauge natural operators we present here for the sake of complexity also a fundamental result on natural operators.

Theorem 4.2. Let $F$ and $F'$ be natural bundles on $Mf_m$ of finite orders $r$ or $r'$, with standard fibers $S$ or $S'$. There is a canonical bijective correspondence between the set of all $k$-th order natural operators $A : F \to F'$ and the set of all smooth $G^r_m$-equivariant maps between the left $G^r_m$-spaces $T^k_m S$ and $S'$, where $q = \max\{r + k, r\}$. The group $G^r_m$ acts on $T^k_m S$ (the standard fiber of a natural bundle $J^k \circ F'$) and on $S'$ via trivial extension of the induced action of $G^{r+k}_m$ on $T^k_m S$ and $G^r_m$ on $S'$.

Proof. See [KMS] (together with the explicit description of the induced actions). \hfill \Box
**Example.** The Lie bracket of vector fields is a first order natural operator between the bundles $T \oplus T$ and $T$.

In the study of natural operators often an orbit reduction theorem plays a key role. We present following version from [KMS]:

**Theorem 4.3. The orbit reduction theorem.** Let $p : G \to H$ be a Lie group homomorphism with kernel $K$, $S$ be a $G$-space, $T$ be an $H$-space, and let $\pi : S \to T$ be a $p$-equivariant surjective submersion, i.e. $\pi(gx) = p(g)\pi(x)$ for all $x \in S$, $g \in G$. Given $p$, we can consider every $H$-space $N$ to be $G$-space:

$$ gy = p(g)y, \quad g \in G, \quad y \in N.$$  

If each $\pi^{-1}(t)$, $t \in T$, is a $K$-orbit in $S$, then there is a bijection between the $G$-equivariant maps $f : S \to N$ and the $H$-equivariant maps $\varphi : T \to N$ given by $f = \varphi \circ \pi$.

**Proof.** See [KMS]. □

As we have already mentioned there are different kinds of naturality. For example we can change the condition (i) in the definition of a natural operator in such a manner that we require the invariance of the operator just to some distinguished set of morphisms $f : M \to N$, not to all of them. These sets of morphisms naturally arises when manifolds are equipped with an extra structure.

We have seen in the previous chapter, in our formulation of Cartan geometries, principal bundles and their morphisms play an important role. Thus, if we work with the category $\mathcal{PB}_m(H)$ of principal fiber bundles and their local isomorphisms instead of $\mathcal{M}_m$, we restrict the invariance to principal bundle morphisms and their base maps. And we get the notion of gauge natural bundles and gauge natural operators:

**Definition.** A gauge natural bundle over $m$-dimensional manifolds is a functor $F : \mathcal{PB}_m(H) \to \mathcal{F}M$ such that

(i) every $\mathcal{PB}_m(H)$ object $\pi : P \to BP$ is transformed into a fibered manifold $q_P : FP \to BP$ over $BP$, where $BP$ is a base functor,

(ii) every $\mathcal{PB}_m(H)$ morphism $\Phi : P \to Q$ is transformed into a fibered morphism $F\Phi : FP \to FQ$ over $B\Phi$,

(iii) for every open subset $U \subseteq BP$, the inclusion $i : \pi^{-1}(U) \to P$ is transformed into the inclusion $Fi : q_P^{-1}(U) \to FP$.

**Definition.** A gauge natural operator $A : F \to G$ between two gauge natural bundles $F$ and $G$ over $m$-dimensional manifolds is a system of regular operators $A_P : C^\infty(FP) \to C^\infty(GP)$ for all $\mathcal{PB}_m(H)$-objects $\pi : P \to BP$ satisfying

(i) for every section $s \in C^\infty(FP)$ and every $\mathcal{PB}_m(H)$-isomorphism $\Phi : P \to Q$ it holds

$$ A_Q(F\Phi \circ s \circ B\Phi^{-1}) = G\Phi \circ A_P s \circ B\Phi^{-1}, $$

(ii) $A_\pi^{-1}U(s_U) = (A_P s)|_U$ for every $s \in C^\infty(FP)$ and every open subset $U \subseteq P$. 

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**Definition.** Let \( \Phi, \Psi : P \to \overline{P} \) be \( \mathcal{PB}_m(H) \)-morphisms. Then \( j^*_x \Phi = j^*_x \Psi \) implies \( j^*_r \Phi = j^*_r \Psi \) for all \( p \in P \), \( \pi(p) = \pi(u) = x \). We will write \( j^*_r \Phi = j^*_r \Psi \), \( x \in BP \).

A gauge natural bundle \( F \) is said to be of order \( r \) if \( j^*_x \Phi = j^*_x \Psi \) implies \( F \Phi|F_x P = F \Psi|F_x P \) and \( r \) is the smallest integer with this property.

A regular gauge natural bundle transforms every smoothly parametrized family of \( \mathcal{PB}_m(G) \)-morphisms into a smoothly parametrized family of fibered morphism.

**Example.** The curvature \( K \) of a Cartan connection \( \omega \) is a second order gauge natural operator between the bundle of Cartan connections and \( (P \times \text{Ad} \mathfrak{g}) \otimes \Lambda^2 (T^* M) \).

**Definition.** Analogously to the case of natural bundles we define associated maps of the gauge natural bundle \( F, F_P, \overline{P} : J^r(P, \overline{P}) \times_{BP} FP \to F \overline{P} \), \( (j^*_r \Phi, u) = F \Phi(u) \).

The induced action of \( W^r_m G = J_j^r(\mathbb{R}^m \times G, \mathbb{R}^m \times G)_0 \) on the standard fiber \( S = F_0(\mathbb{R}^m \times G) \) is given by the restriction of \( F_{\mathbb{R}^m \times G, \mathbb{R}^m \times G} \) to the jets at \( 0 \in \mathbb{R}^m \).

**Principal prolongation of principal bundles.** Now similarly to the case of natural bundles, where every natural bundle is bundle associated to \( P^r \) via the induced action of the jet group \( G^r_m \) on the standard fiber, gauge natural bundles are bundles associated to \( W^r \) via the induced action of the principal prolongation \( W^r_m G \) of the Lie group \( G \) on the standard fiber.

**Definition.** For a principal bundle \( P \in \text{Ob}(\mathcal{PB}_m(G)) \) we define

\[
W^r P := \{ j^*_r \Phi(0, e); \Phi : \mathbb{R}^m \times G \to P \text{ a } \mathcal{PB}_m(G) \text{-morphism} \},
\]

the \( r \)-th principal prolongation of the principal bundle \( P \).

There is a canonical action of \( W^r_m G \) on \( W^r P \) given by the composition of jets and we have

**Proposition 4.4.** Every \( r \)-th order regular gauge natural bundle is a fiber bundle associated to \( W^r \).

**Proof.** See [KMS]. \( \square \)

**Proposition 4.5.** The \( k \)-th order gauge natural operators \( F \to E \) are in a canonical bijection with the natural transformations \( J^k F \to E \).

**Proof.** See [KMS]. \( \square \)

As a conclusion of the previous proposition we get:

**Proposition 4.6.** There is a canonical bijection between \( k \)-th order gauge natural operators \( F \to E \) and the set of \( W^r_m G \)-equivariant maps of their standard fibers, where \( r \) is the maximum of the orders of \( J^k F \) and \( E \). \( \square \)

**Remark.** Let \( F \) be a gauge natural bundle, \( P \) a \( H \)-principal bundle. Then \( FP \) is a fibered manifold and the morphisms \( \Phi \) of \( \mathcal{PB}_m(H) \) act on sections \( s \) of \( FP \) \( \Phi_{s,s} = F \Phi \circ s \circ f^{-1} \), where \( f \) is a base map of \( \Phi \). Thus an operator \( D \) is natural if \( D(\Phi_{s,s}) = \Phi_{s}(Ds) \). The action of \( \mathcal{PB}_m(H) \)-morphisms on the frame forms of sections of associated bundles is then the following one:
Lemma 4.7. Let \( s \in C^\infty(P \times \lambda V) \) be a section of the associated bundle \( P \times \lambda V \). Further let \( \Phi \) be \( \mathcal{P} \mathcal{B}_m(H) \)-morphism and \( [s] \) the frame form of the section \( s \), \( H \)-equivariant map \( P \to V \) corresponding to \( s \). Then \( \Phi \) acts on the frame form as follows:

\[
\Phi \cdot [s] = [\Phi \cdot s] = [\Phi \circ s \circ f^{-1}] = [s] \circ \Phi^{-1}.
\]

Proof. Let \( \sigma : M \to P \) be a local section of \( P \). Then \( s \in C^\infty(P \times \lambda V) \) is given as \( \{\sigma(u), [s](\sigma(u))\} \) and we have

\[
[\Phi \circ s \circ f^{-1}] = [\Phi \circ \{\sigma(x), [s](\sigma(x))\} \circ f^{-1}]
\]

\[
= [\Phi \circ \{\Phi^{-1}(s(x)), [s](\Phi^{-1}(s(x)))\}]
\]

\[
= [[\Phi \circ \Phi^{-1}(s(x)), [s](\Phi^{-1}(s(x)))]]
\]

\[
= [[(s(x)), [s](\Phi^{-1}(s(x)))] = [s] \circ \Phi^{-1}. \quad \square
\]

Lemma 4.8. Let \( \omega \) be a Cartan connection of type \( (g, H) \) on a principal bundle \( P \), \( s \in (P, V)^\lambda \) a section of an associated bundle \( P \times \lambda V \). Then the invariant jet \( (s, \nabla^\omega s) : CP \oplus C^\infty(P, V)^\lambda \to C^\infty(P, V \oplus V \otimes g^\lambda) \) is a gauge natural operator.

Proof. Let \( \Phi \in \text{Mor}(\mathcal{P} \mathcal{B}_m(H)) \) be a principal bundle morphism. The gauge naturality of \( \nabla^\omega s \) means \( \nabla^\Phi \omega \Phi_s s = \Phi_s (\nabla^\omega s) \), that is \( ((\Phi_s \omega)^{-1}(X)) \cdot \Phi_s s = \Phi_s (\omega^{-1}(X) \cdot s) \). Further we have

\[
\Phi_s \omega(\Phi(u))(\Phi_s (\omega^{-1}(X)(\Phi(u)))) = \Phi_s \omega(\Phi(u))(T \Phi \circ \omega^{-1}(X)(u)) = (\omega \circ T(\Phi^{-1}) \circ T \Phi \circ \omega^{-1}(X))(u) = X.
\]

Thus \( (\Phi_s \omega)^{-1}(X) = \Phi_s (\omega^{-1}(X)) \) and we get

\[
((\Phi_s \omega)^{-1}(X)) \cdot \Phi_s s = \Phi_s (\omega^{-1}(X)) \Phi_s s = T \Phi_s \circ T \Phi^{-1} \circ T \Phi \circ \omega^{-1}(X) \circ \Phi^{-1} = (\omega^{-1}(X) \cdot s) \circ \Phi^{-1} = \Phi_s (\omega^{-1}(X) \cdot s). \quad \square
\]

Remark. We have already used in the previous lemma the fact, that Cartan connections of the given type form a gauge natural bundle \( C \) over \( \mathcal{P} \mathcal{B}_m(H) \). This was in fact already implicitly proven. For the explicit formulation see Lemma 4.9.

Gauge natural bundle of Cartan connections.

We will construct a gauge natural bundle \( C \), which we will call the bundle of Cartan connection. Then Cartan connections on the \( H \)-principal bundle \( P \) with values
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in \( g \) uniquely correspond to sections of \( CP \). We will use the construction of the bundle of principal connections from [KMS].

**The bundle of principal connections.** Let \( P \to M \) be a principal bundle with the structure group \( H \) and \( r : P \times G \to P \) be the principal action. Then there is a canonical right action of \( G \) on \( J^1P \): \( (j^1_P)_g = j^1_P(r_g \circ s) \). Principal connections can be viewed as right invariant sections of the \( J^1P \to P \) (fibered over targets of jets). We define \( QP \) to be the set of orbits \( J^1P/G \). Then \( QP \) is fibered over \( M \). Moreover the action of \( \mathcal{PB}_m(H) \)-morphism \( \Phi \) on a jet \( j^1_P s \in J^1P \), satisfies

\[
J^1\Phi(j^1_P(r_g \circ s)) = j^1_{\bar{\Phi}(x)}(\Phi \circ r_g \circ s \circ f^{-1}) = j^1_{\bar{\Phi}(x)}(r_g \circ \Phi \circ s \circ f^{-1}).
\]

Hence the map \( J^1\Phi : J^1P \to J^1\bar{\Phi} \) factors to a map \( Q\Phi : QP \to Q\bar{\Phi} \). The smooth structure on \( Q(\mathbb{R}^m \times G) \) is defined with the help of the canonical representant \( j^1_P s \), \( s(x) = (x, e) \) in each orbit of \( J^1(\mathbb{R}^m \times G)/G \). We have \( J^1(\mathbb{R}^m \times G) \simeq \mathbb{R}^m \times J^1(\mathbb{R}^m, G) \), \( j^1_P s \mapsto (x, j^1_P x) \), where \( t_x \) is a translation by \( x \) on \( \mathbb{R}^m \). Consequently we get \( Q(\mathbb{R}^m \times G) \simeq \mathbb{R}^m \times J^1(\mathbb{R}^m, G) \), which defines the smooth structure. Then the morphism \( Q\Phi \), for \( \Phi = (f, \varphi) \in \text{Mor}(\mathcal{PB}_m(H)) \) is evidently smooth:

\[
Q\Phi(j^1_P s) = j^1_{\bar{\Phi}(x)}(r_{f^{-1}(x)} \circ \Phi \circ s \circ f^{-1}).
\]

The smooth structure is then extended over the whole category \( \mathcal{PB}_m(H) \) via local trivializations.

**Lemma 4.9.** Cartan connections form sections of a gauge natural bundle, the Gauge natural bundle of Cartan connections.

**Proof.** We will show that for any \( P \in \text{Ob}(\mathcal{PB}_m(H)) \), \( CP \) is a subbundle in \( Q(P \times_H G) = QP' \). Cartan connections on a principal bundle \( P \) are in one-to-one correspondence with principal connections on the bundle \( P' \) whose kernels do not meet \( TP \subset T(P \times_H G) \) (condition \( \dagger \)), see Lemma 1.3.

A representant \( j^1_P s \in J^1(P \times_H G) \) of an element of a Cartan connection is as well given by the horizontal space (of the element of a connection) \( H_s(x)P' \) in \( s(x) \), namely \( T_s(T_xM) \). Then the condition \( \dagger \) reads as \( H_s(x)P' \cap T_s(x)P = \{0\} \), for \( s(x) \in P \subset P' \). Then the condition \( \dagger \) defines an open subset of \( Q(P \times_H G) \), thus a submanifold (if \( H_s(x)P' \cap T_s(x)P = \{0\} \) then since \( T_s(T_xM) \) changes smoothly with \( x \) we have \( H_{s(y)}P' \cap T_{s(y)}P = \{0\} \) for \( y \in U, U \subset M \) some open neighbourhood of \( x \in M \).

Due to the dimension reasons \( H_s(x)P' \) and \( T_s(x)P \) span \( T_s(x)P' \). The element \( J^1\Phi(j^1_P s) \) for \( \mathcal{PB}_m(H) \)-morphism \( \Phi \) is then given by the horizontal space \( T\Phi(H_s(x)P') \). Since \( \Phi \) is a diffeomorphism, \( T\Phi : T_uP' \to T_{\Phi(u)}P' \) is an isomorphism of vector spaces for any \( u \in P' \). Hence \( T\Phi(H_s(x)P') \) and \( T\Phi(T_s(x)P) \) also span \( T_{\Phi(s(x))}P' \) and \( T\Phi(H_s(x)P') \cap T_{\Phi(s(x))}P' = \{0\} \). Thus condition \( \dagger \) is preserved by \( Q\Phi \) for \( \Phi \in \text{Mor}(\mathcal{PB}_m(H)) \). For \( P \in \text{Ob}(\mathcal{PB}_m(H)) \) we define \( CP \) to be a subbundle of \( Q(P \times_H G) \) of those elements \( (j^1_P s \in J^1(P \times_H G)) \) which satisfy the condition \( \dagger \). We have just shown that \( Q\Phi|_{CP} : CP \to CR \) for any principle bundle morphism \( \Phi : P \to R \). Cartan connections are then sections of the bundle \( CP \). \( \square \)
Lemma 4.10. The bundle of Cartan connections is a subbundle in the affine bundle of principal connections.

Proof. We define the affine structure on the whole bundle of principal connections on the G-principal bundle P. The standard fiber of the bundle of principal connections is $S = \mathbb{R}^{m*} \otimes \mathfrak{g}$. Let $s : \mathbb{R}^m \to G, s(0) = e$, then $j_0^1s \in S$. Further let $\Phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G, f(0) = 0$, be a $\mathcal{PB}_m(G)$-morphism, then $j_0^1(\Phi)(0, e) \in W^1_m(G)$ (this corresponds to the decomposition $W^1_m(G) = (G^1_m \times G) \times (\mathfrak{g} \otimes \mathbb{R}^{m*})$, with the multiplication $(A, a, Z)(B, b, Y) = (A \circ B, ab, \text{Ad}(b^{-1})Z \circ B^{-1} + Y))$. Then the action $k$ of $W^1_mG$ on $S$ can be written as

$$(A, a, Z)(Y) \simeq Q\Phi(j_0^1s) = j_0^1(\text{conj}(a) \circ \mu \circ (l_{a^{-1}} \circ \varphi, s) \circ f^{-1}) = \text{Ad}(a)(X + Y) \circ A^{-1}.$$

Thus $QP$ is an associated bundle $W^1P \times_k (\mathbb{R}^{m*} \otimes \mathfrak{g})$. We will show it has also an affine structure corresponding to the vector bundle $W^1P \times_{G \times G^1_m} (\mathbb{R}^{m*} \otimes \mathfrak{g})$. Let us first describe associated vector bundle $W^1P \times_{G \times G^1_m} \mathbb{R}^{m*} \otimes \mathfrak{g}$; there is an action of the product of Lie groups $G \times G^1_m$ on $\mathbb{R}^{m*} \otimes \mathfrak{g}$: $(a, A)(Y) = \text{Ad}(a)(Y) \circ A^{-1}$. $G \times G^1_m$ is a subset $W^1_mG$ and thus it acts also on $W^1P$. One immediately sees that the action of $(G \times G^1_m)$ on $\mathbb{R}^{m*} \otimes \mathfrak{g}$ respects the vector space structure on $\mathbb{R}^{m*} \otimes \mathfrak{g}$.

As for the affine structure on $W^1P \times_k \mathbb{R}^{m*} \otimes \mathfrak{g}$ we have for the difference of two elements in $\mathbb{R}^{m*} \otimes \mathfrak{g}$:

$$k(A, a, Z)(Y_1) - k(A, a, Z)(Y_2) = \text{Ad}(a)(Y_1 - Y_2) \circ A^{-1},$$

and it transforms according to the vector bundle action of $G \times G^1_m$.

See [KMS], 17.6. for more details.

Now the Cartan connections are a subbundle in this affine bundle, but they are not an affine subbundle. The reason is that the $\omega_{\mathfrak{g}/h}$-part of the connection has to satisfy the regularity condition $\dagger$. On the other hand, the connections with fixed $\omega_{\mathfrak{g}/h}$-part form an affine subbundle of the affine bundle of principal connections. □

Natural sheafs.

Sometimes it is useful not to work with natural bundles but with natural sheafs. This is a notion introduced by D.J. Eck in [Eck]. Some geometrical objects on a manifold are more likely to form a natural sheaf than a natural bundle. For example if we take the natural bundle of real functions on a manifold it is a natural sheaf but if we restrict ourselves to constant functions they cannot be considered as a natural bundle (they are sections in the bundle of all real functions but not all the sections) but they are still a natural sheaf. Similarly as in case of natural bundles we will define natural sheafs as functors and we give the definition just for the natural sheafs over the category $\mathcal{PB}_m(H)$ though they can be defined over different categories as well.
Definition. A natural sheaf \( \mathcal{F} \) over the category \( \mathcal{PB}_m(H) \) is a functor on \( \mathcal{PB}_m(H) \) such that for each principal bundle \( P \to M \), \( \mathcal{F} \) is a sheaf of modules over the sheaf of Lie algebras \( \mathcal{X}^r(P) \) of right invariant vector fields on \( P \), that is for any open subset \( U \) of \( M \), there is a vector space \( \mathcal{F}(U) \) on which the Lie algebra acts by continuous linear operators \( L_{\mathcal{X}} \). Moreover for any open subsets \( V \subset U \) the operators are equivariant with respect to the sheaf restriction \( \mathcal{F}(U) \to \mathcal{F}(V) \).

Example. Let \( E \) be a natural vector bundle on a manifold \( M \). Then the space \( C^\infty(EU) \) of local sections of \( EU \) form a natural sheaf over \( M \). Vector fields \( \mathcal{X}(U) \) on \( U \) act on \( EU \) via Lie derivatives of sections. The topology on sections is induced by the vector bundle atlas of \( E \). It is a natural sheaf over the category \( M_{f_m} \) of \( m \)-dimensional manifolds and their local diffeomorphisms. There we have the action of the sheaf of Lie algebras of all vector fields on \( M \).

Remark. We restrict ourselves to natural sheafs over \( \mathcal{PB}_m(H) \), where the sheaf of right invariant vector fields acts by Lie derivatives, i.e. to subsheafs of gauge natural bundles.

Remark. The definition of the natural sheaf enables us to speak about natural operators between natural sheafs even if they are not natural bundles: if the operators are even linear we substitute the naturality condition \( \Phi^* (D(s)) = D(\Phi^* s) \) with its infinitesimal version \( D(\mathcal{L}_X s) = \mathcal{L}_X (D(s)) \) (\( D \) is a natural operator, \( s \) element of a sheaf, \( \Phi \) a \( \mathcal{PB}_m(H) \)-morphism and \( X \) a right invariant vector field). However for nonlinear operators we are not able to compare \( D(\mathcal{L}_X s) \) and \( \mathcal{L}_X (D(s)) \). We have to deal with the vertical prolongation of a natural operator: let \( D : C^\infty(Y \to M) \to C^\infty(Y' \to M) \) be a natural operator, and \( q \in C^\infty(VY \to M) \) a section. Then \( q \) is of the form \( \frac{\partial}{\partial t} \big|_0 s_t \) for a family \( s_t \in C^\infty(Y) \) and we define the vertical prolongation \( VD : C^\infty(VY \to M) \to C^\infty(VY' \to M) \) as

\[
VD(q) = VD\left( \frac{\partial}{\partial t} \big|_0 s_t \right) = \frac{\partial}{\partial t} \big|_0 (Ds_t) \in C^\infty(VY' \to M).
\]

Then the infinitesimal version of naturality reads as \( VD(\mathcal{L}_X s) = \mathcal{L}_X (D(s)) \). See [CS] for details on infinitesimal naturality.

Lemma 4.11. Every gauge natural operator on \( \mathcal{PB}_m(H) \) consists of infinitesimally gauge natural operators \( D_M \) (i.e. "every gauge natural operator is infinitesimally gauge natural").

Proof. See [CS]. □

Remark. If the group \( G \) is connected and we restrict ourselves to the category of principal bundles with morphisms whose base maps preserve orientation of base manifolds, then the implication from the previous lemma holds also in the opposite direction: every infinitesimally gauge operator \( D_P : C^\infty(FP) \to C^\infty(EP) \) extends to a unique gauge natural operator \( D : F \to E \).
Corollary 4.12. The kernel and the image of a finite order gauge natural operator is a (gauge) natural sheaf. □

**Curvature Spaces.**

The construction of the jet prolongations of $\mathfrak{h}$-modules can be applied to the vector space $\mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}$ with the action $\lambda : H \to \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}$, $\lambda(a)\varphi = \text{Ad}(a)\varphi((\text{Ad}_{-}(a^{-1}))(\_))$, respectively to the derivation of this action $\lambda : \mathfrak{g} \to \mathfrak{g}^* \wedge \mathfrak{g}^* \otimes \mathfrak{g}$ (we denote it by the same letter $\lambda$). Thus the $r$-th invariant jet $(\nabla)^r\kappa$ of the curvature function is a section of the bundle $P \times J^r(\lambda) J^r(\Lambda^2(\mathfrak{g}^*_+) \otimes \mathfrak{g})$.

But not any section of this bundle is the curvature function of some Cartan connection. Nevertheless as a corollary of Lemma 4.8. and Corollary 4.12 we obtain

**Corollary 4.13.** The curvatures of the Cartan connections in the bundle of Cartan connections $C$ form a natural sheaf. The $r$-th order invariant jets of the curvature function of Cartan connections on $P$ form a natural sheaf. It is a subsheaf in the bundle $P \times J^r(\lambda) \oplus \sum_{i=0}^{r}(\otimes^i(\mathfrak{g}^*_+) \otimes (\Lambda^2\mathfrak{g}^*_+ \otimes \mathfrak{g}))$. □

**Notation.** We will write $\mathcal{K}_r$ for the natural sheaf of $r$-th order invariant jets of the curvature function.

In this paragraph we are going to describe more explicitly the sheaf $\mathcal{K}_r$. We know that the curvature function and its invariant derivatives satisfy the Bianchi and Ricci identities and its consequences obtained by the invariant derivation. In the reductive geometries we will be able to show, that $r$-th order invariant jets of the curvature function are sections of an affine subbundle in $P \times J^r(\lambda) \oplus \sum_{i=0}^{r}(\otimes^i(\mathfrak{g}^*_+) \otimes (\Lambda^2\mathfrak{g}^*_+ \otimes \mathfrak{g}))$.

**Remark.** So far we were able to avoid computations in local coordinates which are sometimes boring sometimes non-transparent but always something one does not like. Unfortunately the order of operators and their invariance is connected to the action of jet prolongation of Lie groups and the author is not able to handle the action but in coordinates.

**Notation.** We indicate the splitting $\mathfrak{g} = \mathfrak{h} + \mathfrak{g}_-$ by indices. We use $p, q, \ldots$ indices for values in the whole $\mathfrak{g}$ (or $G$), $v, w, \ldots$ indices for values in $\mathfrak{h}$ (or $H$), and $i, j, \ldots$ for values in $\mathfrak{g}_-$. Since $T_xM$ and $\mathfrak{g}_-$ are isomorphic we use the the $i, j, \ldots$ indices also for coordinates on $M$ (or $\mathbb{R}^m$). Thus, for example, we write $\omega^i_j dx^i = \omega^i_v dx^i + \omega^i_p dx^i$ for the $\mathfrak{h}$ and $\mathfrak{g}_-$ parts of the "horizontal" part of a Cartan connection.

**Lemma 4.14.** Let $\omega$ be a Cartan connection on $P$, $x^i$, $y^p$ be coordinates on $P$ given by a local trivialization $P = \mathbb{R}^m \times H$. Then $\omega = \omega^i_p dy^v - \omega^p_i dx^i = \omega_H - \omega^p_i dx^i$, where $\omega_H$ is the Maurer-Cartan form on $H$. □

**Proof.** The tangent space in each point $(x, h)$ of the trivial bundle $\mathbb{R}^m \times H$ naturally splits to $\mathbb{R}^m \oplus T_hH$. If we consider a Cartan connection at $(x, h)$ restricted to $T_hH$ then from the definition it just reproduces the fundamental fields generated by $\mathfrak{h}$, that is it
Coordinate form of curvature and its invariant derivatives. We will write \( \Gamma^p_i = \omega^p_i(0, e) \) for the Christoffel symbols of Cartan connections. The coordinates on the \( r \)-th jet prolongation of the standard fiber of the bundle of Cartan connections will be \( \Gamma^p_i, \Gamma^p_{ij} \ldots \Gamma^p_{i1 \ldots i_r} = \partial \Gamma^p_i / \partial x^{i_1} \ldots \partial x^{i_r} \).

The curvature form \( K \) of a Cartan connection is given by the structure equation \( K = d\omega + \frac{1}{2}[\omega, \omega]. \) Since the curvature is a horizontal 2-form its values on the whole tangent space of the principal fiber bundle are given by values on any subbundle complementary to the vertical subbundle. In particular any local trivialization \( P = R^m \times H \) defines the horizontal bundle \( TR^m \times 0 \subset TR^m \times TH = TP \) and the curvature is fully determined by values on this subbundle only. Moreover as we have already mentioned, the curvature is \( H \)-equivariant along the fiber and thus its value on \( TR^m \times 0 \) is given by its value on \( R^m \times 0. \) In the local trivialization \( P = R^m \times H \) with \( x^i \) coordinates on \( R^m \) and \( y^v \) on \( H \) we can write according to lemma 4.14 \( \omega = \omega^p_i dy^v - \omega^j_i dx^i. \) If we substitute this local expression to the structure equation we get the coordinate expression of the horizontal part of the curvature

\[
K^p_{ij} dx^i \wedge dx^j = \Gamma^p_{[ij]} dx^i \wedge dx^j + \epsilon^p_{qr} \Gamma^q_{ij} dx^i \wedge dx^j,
\]

where the \( \epsilon^p_{qr} \) are the structure constants of \( G: [e_q, e_r] = \epsilon^p_{qr} e_p, \) \( e_p \) is a base in \( g. \)

For each \( u \in P, \pi(u) = x \) we have an isomorphism \( (T\pi) \circ (\omega(u))^{-1} : g_- \to T_x M \) \( (\pi : P \to M \) projection to the base manifold).

Now if we write \( \rho = \omega^p_i dx^i = \omega^v_i dx^i + \omega^j_i dx^i \) according to splitting \( g = \mathfrak{h} + g_- \), the matrix \( \omega^j_i \) represents the inverse isomorphism \( T_x M \to g_- \) in given bases. We can choose the base in \( g_- \) in such a way that \( \omega^j_i \) is the identity matrix at some point in the fiber, let’s say at \( (0, e), \) that is \( \Gamma^i_j = \delta^i_j. \) Now we can interpret the \( K^p_{ij} \) as the coordinate expression of the curvature function \( \kappa \in C^\infty(P, g_-^\ast \otimes g_- \otimes g), \) \( \kappa(u)(X, Y) = K(\omega^{-1}(u)(X), \omega^{-1}(u)(Y)). \) Since in the local trivialization \( P = R^m \times H \) vectors \( ((T\pi)\omega^{-1}(u)(X), 0) \) and \( \omega^{-1}(u)(X) \) differ only by a vertical vector, the curvature form has the same value on both of them. If we take the base in \( g_- \) as above, we can write \( \kappa(0, e)(X, Y) = K(\omega^{-1}(0, e)(X), \omega^{-1}(0, e)(Y)) = K((X, 0)_{(0, e)}, (Y, 0)_{(0, e)}) = K^p_{ij} X^1 Y^j. \) And \( K^p_{ij} \) are the coordinates of \( \kappa(0, e)(X, Y). \)

The invariant differentiation of the curvature function in local coordinates. We express the invariant derivative of the \( r \)-th invariant jet of the curvature function in terms of local coordinates on \( J^1_r C, \) namely the Christoffel symbols and their partial derivatives. As before we can assume that \( \Gamma^i_j = \delta^i_j, \) let further \( (K^p_{ij}; K^p_{ij1}, \ldots, K^p_{ij1 \ldots i_r}) \) be the coordinate expression of the invariant jet of the curvature function in \( (0, e). \) That is \( K^p_{ij1 \ldots i_r} \simeq (\nabla)^r \kappa(0, e)(\frac{\partial}{\partial x^{i_1}}, \ldots, \frac{\partial}{\partial x^{i_r}}), \) and \( (K^p_{ij}, K^p_{ij1}, \ldots, K^p_{ij1 \ldots i_r}) \simeq j^r \kappa(0, e). \) We know that the invariant \( r \)-jet of the curvature function is a section of the associated bundle \( P \times (J^r \chi) \otimes_{r=1}^r (\otimes^r g^\ast) \otimes (g^\ast \wedge g^\ast) \otimes \)
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Further let $\kappa \simeq k_{ij}^{p_{ij}}\ldots i_{r+1}$ be the coordinate expression of the $r$-th invariant derivative of the curvature function $(\nabla)^r \kappa \in C^\infty (P, \otimes^r g_\mathbb{R} \otimes (g_\mathbb{R} \wedge g_\mathbb{R} \otimes g))$ (then $K_{ij}^{p_{ij}}\ldots i_{r+1}$). We also need a coordinate expression of the $(\omega(0, e))^{-1}_{g_\mathbb{R}} : g_\mathbb{R} \to T_{(0, e)}(\mathbb{R}^m \times H)$. Let $X \simeq (X^j) \in g_\mathbb{R}$. Then $\omega^{-1}(0, e)(X) \simeq \Gamma^i_j X^j \frac{\partial}{\partial x^i} + \Gamma^i_v \partial_j \frac{\partial}{\partial y^j},$ where $\tilde{\Gamma}^i_j$ is the inverse matrix to $\Gamma^i_j$.

$\nabla(\kappa)((\nabla)^r \kappa)(0, e) \simeq K_{ij}^{p_{ij}}\ldots i_{r+1} = (\omega^{-1}(0, e) \cdot \kappa_{ij}^{p_{ij}}\ldots i_{r+1})(0, e) = \frac{\partial K_{ij}^{p_{ij}}\ldots i_{r+1}}{\partial x^k} \Gamma^k_{i+1} - \lambda(\Gamma^v_j \tilde{\Gamma}^v_{i+1}) \circ (K_{ij}^{p_{ij}}, K_{ij}^{p_{ij}}\ldots i_{r+1}, K_{ij}^{p_{ij}}\ldots i_{r+1})$

where $\lambda$ is the action of $h$ on the invariant jet of the curvature function discussed in the previous section, and the term $\lambda(\Gamma^v_{i+1})$ stands for the action of the $h$-part of the vector $\omega(0, e)(\frac{\partial}{\partial x^i}) \in g$ on the invariant $r$-th jet.

_Notation._ We will write $K_r$ for the bundle $P \times \mathcal{J}^r(\lambda) (\otimes_{i=1}^r \otimes^i g^* \otimes (\Lambda^2 g^* \otimes g)).$

**Lemma 4.15.** The coordinate form of the Bianchi identity. The Bianchi identity from Lemma 1.16 can be expressed in the terms of local coordinates as

$$\sum_{\text{cyclic}} c_{qki}^p K_{ij}^q - K_{ik}^p K_{ij}^l + K_{ik}^p c_{ij}^l + K_{ij}^p = 0,$$

where $c_{qr}^p$ are structure constants of $G$.

**Proof.** The equations (2) are just transcription of Lemma 2.6 in local coordinates. \hfill $\square$

**Reductive geometries.**

Let us deal with even more specific case of reductive geometries, namely with reductive torsion-free geometries with $g_- \subset g$ an abelian ideal. Then the Bianchi and Ricci identities describe the $r$-th order curvature bundle explicitly as an affine subbundle in the bundle $C^\infty(K_r)$. 
Lemma 4.16. The functor which appoints to a principal $H$-bundle the bundle of germs of Cartan connections of type $(\mathfrak{g}, H)$ on $P$ without torsion is a gauge natural bundle. It is a subbundle of the bundle of Cartan connections.

Proof. If we fix a principal $H$-bundle $P$, then Cartan connections without torsion on $P$ correspond to the condition $\kappa_- = \kappa_0 = 0$. This condition determines a subbundle in the first jet prolongation of the bundle of Cartan connections on $P$ (in local coordinates it is in each $x \in M$ a system of linear or quadratic equations with constant coefficients). Moreover it is preserved by the action of morphisms of principal fiber bundles (for a Cartan connection $\omega$ with curvature form $\kappa$ and for $\Phi : P \to R$ we have for the curvature form $\Phi^*(\omega)$:

$$
(\kappa_{\Phi^*(\omega)})_h = (\Phi_*\kappa)_h = (\Phi_*\kappa_0)_h = \Phi_*\kappa_0 \in \mathfrak{g}^- \otimes \mathfrak{g}^- \otimes \mathfrak{h}.
$$

Consequently $(\kappa_{\Phi^*(\omega)})_- = 0$.

4.17. Lemma. The coordinate form of Ricci and Bianchi identities. We get then two identities, the first and the second Bianchi identity:

\begin{align*}
(3) & \quad K^j_{ijkl} + K^j_{klji} + K^j_{lijk} = 0, \\
(4) & \quad K^j_{jklm} + K^j_{jlmk} + K^j_{jmkl} = 0.
\end{align*}

For the Ricci identity we have

(5) \quad K^i_{jkl[m_1m_2]} = \sum_{k_1,k_2} \lambda(K^k_{k_2m_3m_4})K^i_{jkl}.

Note that unlike in (2) only the fourth subscript stands for the invariant derivation.

Proof. We just consider (2) in the torsion free geometry. The term $K^p_{ik}K^i_{ij}$ vanishes, $\mathfrak{g}_-$ is an abelian ideal and $K^p_{ik}c^i_{ij}$ vanishes as well, and the term $c^p_{iq}K^q_{ij}$ has the homogeneity degree\(^1\) 2, the term $K^p_{ijk}$ has the homogeneity degree 3, and thus both of them have to vanish. For $\mathfrak{h} \subset \text{gl}(\mathfrak{g}_-)$ then $c^p_{iq}K^q_{ij}$ yields (3) and $K^p_{ijk}$ gives (4). The (5) can be obtain easily from the non-coordinate form of the Ricci identity.

Equations defining the curvature subspaces. Let $W = \mathfrak{g}^- \otimes \mathfrak{g}^- \otimes \mathfrak{g}$ and $W_r = W \otimes \cdots \otimes W_r$, and $W^r = W \times W_1 \times \cdots \times W_r$. We have a formal curvature map $\gamma : J^1(C) \to W$ (see (1)) and its $r$-th invariant derivative $\gamma_r = \nabla^r(\gamma) : J^{r+1}(C) \to W_r$. We define the $r$-th order curvature equations $E_r$ on $W^r$ (recall our notation of indices):

\(^1\)Let $\mathfrak{g} = \sum_{i=-l}^k \mathfrak{g}_i$ be a grading of a Lie algebra $\mathfrak{g}$. Then for the multilinear mapping $f : \Pi_{r=1}^s \mathfrak{g}_r \to \mathfrak{g}_t$ we define the homogeneity degree of $f$ as the number $t - \sum_{r=1}^s i_r$.\
i) $E_0$ are the Bianchi identities:

$$K_{jkl}^i + K_{klij}^i + K_{lij}^i = 0,$$

(6)

$$K_{jklm}^i + K_{jimnk}^i + K_{jmnkl}^i = 0,$$

(7)

ii) $E_1$ are the invariant derivatives of $E_0$

iii) $E_s$, $s > 1$ are the invariant derivatives of $E_{s-1}$ and the Ricci identity for the $(r-2)$-nd invariant jet of the curvature function:

$$K_{jklm1\ldots ms-1ms}^i = \text{Polyn}(W^{r-2}),$$

where $\text{Polyn}(W^{r-2})$ are some polynomials on $W^{r-2}$.

**Definition.** The $r$-th order curvature subspace $K^r \subset W^r$ is defined by

$$E_0 = 0, \ldots, E_r = 0.$$

**Lemma 4.18.** $K^r$ is a submanifold of $W^r$.

**Proof.** The following proof is more or less precise transcription of the proof of the same statement for the linear symmetric connections from the book [KMS]. To prove $K^r$ is a submanifold we proceed by induction. For $r = 0$ we have a linear subspace. Assume $K^{r-1} \subset W^{r-1}$ is a submanifold. consider the product bundle $K^{r-1} \times W^r \subset W^r$. Equations $E_r$ consist of the following two systems:

$$K_{jkl1\ldots ir}^i = 0$$

$$K_{jkl1\ldots i_r}^i = 0$$

$$K_{jkl1\ldots [is-1is]}^i + \text{Polyn}(W^{r-2}) = 0,$$

where $\{\ldots\}$ denotes the cyclic permutation and $\text{Polyn}(W^{r-2})$ are some polynomials on $W^{r-2}$. The map defined by the left hand sides of (1) and (2) represents an affine bundle morphism from $K^{r-1} \times W^r$ to $K^{r-1} \times \mathbb{R}^N$ of constant rank, $N$ is the number of equations (1) and (2). Its kernel is then a subbundle of $K^{r-1} \times W^r$. □
In this episode we deduce the main results of this thesis, namely a characterization of natural operators of given but arbitrary order on the bundle of Cartan connections. According to Proposition 4.6 \( r \)-th order gauge natural operators between gauge natural bundles correspond to \( W^r_mG \)-equivariant maps between appropriate jet prolongations of their standard fibers. We are going to exploit this theorem thus we need to express the action of \( W^r_mG \) on the standard fiber of (the jet prolongations of) the bundle of Cartan connection.

**Lemma 5.1.** For the curvature function \( \kappa^\omega \) of the Cartan connection \( \omega \) on the principal bundle \( P \) and \( PB_m(H) \)-morphism \( \Phi \) holds

\[
\Phi^* \kappa^\omega = \kappa^{(\Phi^* \omega)} = \kappa^\omega \circ \Phi^{-1}.
\]

That is the action of \( PB_m(H) \)-morphisms on the curvature function coincides with the action of \( PB_m(H) \)-morphisms on frame forms of sections in an associated bundle \( P \times \Lambda V \).

*Proof.* It is easy to verify that the curvature function can be expressed as \( \kappa^\omega(X,Y) = [X,Y] - \omega([\omega^{-1}(X),\omega^{-1}(Y)]) \). Then

\[
\begin{align*}
\kappa^{\Phi^* \omega}(u)(X,Y) &= [X,Y] - \omega (\Phi^{-1}(T \Phi \circ \omega^{-1}(\Phi^{-1}(u))(X), T \Phi \circ \omega^{-1}(\Phi^{-1}(u))(Y))) \\
&= [X,Y] - \omega([\omega^{-1}(\Phi^{-1}(u))(X),\omega^{-1}(\Phi^{-1}(u))(Y)]) = \kappa^\omega \circ \Phi^{-1}.
\end{align*}
\]

The action of \( PB_m(G) \)-morphisms on the standard fiber of the bundle of principal connections. Explicitly we compute only the action of \( W^2_mG \) on \( J^1_0(Q) \) (\( Q \) is the bundle of principal connections). We more or less copy the calculations made in [KMS, Section 52]. Knowing this action we will be able to compute the action of \( PB_m(H) \)-morphisms on the standard fiber of Cartan connections quite easily.

For the standard fiber \( S \) of the bundle of principal connections we use the identification \( S = J^1_0(\mathbb{R}^m \times G)/G \simeq J^1_0(\mathbb{R}^m, G)_e \simeq \mathbb{R}^{m*} \otimes g \). In this identification we take \( \Gamma(x,e) \in \mathbb{R}^{m*} \otimes g \) to represent a connection \( \Gamma \) in the fiber over \( x \). In coordinates we get \( \Gamma_p \), the Christoffel symbols (for a fixed base \( e_p \) in \( g \)); then the connection form \( \omega \)
of the connection $\Gamma$ is related to the coordinates $\Gamma^p_i$ in the standard fiber via the left Maurer-Cartan form $\omega_G$ of $G$ by the formula $\Gamma^p_i dx^i = \omega_G(e) - \omega(x, e)$.

Coordinates on $W^1_m(G)$: for a $\mathcal{PB}_m(G)$-isomorphism $\Phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G$; $\Phi(x, g) = (f(x), \varphi(x) \cdot g)$, $f(0) = 0$, $\varphi : \mathbb{R}^m \to G$, $\varphi(x) = \Phi(x, e)$, the corresponding element in $W^1_m(G)$ has the coordinates $a = \varphi(0) \in G$, $a^p_i \sim j^1_i(a^{-1} \cdot \varphi(x)) \in \mathbb{R}^{m \times \mathfrak{g}}$, $a^i_j \sim j^1_i f \in G_m$.

The action of $\Phi$ on $\Gamma(x) \simeq j^1_x s \in \mathcal{Q}(\mathbb{R}^m \times G)$, is given by the formula $\mathcal{Q}\Phi(j_1x s) = j^1_{\Phi(x)}(y \mapsto (r_0(\varphi(x))^{-1} \circ \Phi \circ s \circ f^{-1}(y))) = j^1_{f(x)}(y \mapsto (\text{conj}(\varphi(x)) \circ \mu \circ (l_0(\varphi(x))^{-1} \circ \varphi, s) \circ f^{-1}(y)))$, where $r_a$ is the right multiplication by an element $a$ in $G$, $l_a$ is the left multiplication by $a$ in $G$, and $\mu$ is the multiplication in $G$. This yields then in coordinates (we overline the coordinates changed by the action of $\Phi$):

$$\Gamma^p_i(f(x)) = A^p_i(\varphi(x))(\Gamma^q_j(x) + a^q_j(x))\overline{a}^j_i,$$

where $A^p_i(a)$ is a coordinate expression of the adjoint representation of $G$, $\overline{a}^j_i$ is the inverse matrix to $a^i_j$, that’s $\overline{a}^j_i \sim j^1_i(f^{-1})$. Especially for $\Gamma(0) \sim j^1_0(s) \in S$ we get

$$\Gamma^p_i = A^p_i(a)(\Gamma^q_j + a^q_j(0))\overline{a}^j_i.$$

On $W^1_mH$ we have coordinates $(a^i_j, a, a^p_i)$ and these coordinates describe the embedding of $W^1_mH$ into $W^1_mG$ (in $W^1_mG$ has the element $(a^i_j, a, a^p_i) \in W^1_mH$ coordinates $(a^i_j, a, a^p_i, a^p_m = 0)$, where we write $a^i_j \simeq (a^i_j, a^p_i)$ for the $\mathfrak{h}$ and $\mathfrak{g}_-$-part of the coordinates in $W^1_mG$). The action of $W^1_mH$ on Cartan connections is then given by the same formula (one can easily see that it really preserves Cartan connections, that is the regularity of the matrix $\Gamma^p_m$).

The action of $W^2_mH$ on $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$. On $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ we have coordinates $\Gamma^p_i$ and $\Gamma^p_{jk} = \partial \Gamma^p_i / \partial x^k$.

On $W^2_m(G)$ we have the coordinates $(a^i_j, a^i_j \simeq \partial a^i_j / \partial x^k, a, a^p_i, a^p_i \simeq \partial a^p_i / \partial x^k)$ (recall the coordinates on $W^1_m(G)$).

The action of $W^2_m(G)$ on $J^1_0(\mathcal{Q}(\mathbb{R}^m \times G))$ is then given by the formula

$$(j^2_0\Phi)(\Gamma) = j^1_b(\mathcal{Q}\Phi \circ \Gamma \circ f^{-1}) \sim j^1_0(\mathcal{Q}\Phi \circ j^1_s \circ f^{-1}).$$

In coordinates we get

$$\Gamma^p_i = \frac{\partial \Gamma^p_i}{\partial x^j} = \frac{\partial}{\partial x^j} (A^p_i(\varphi \circ f^{-1}(x))(\Gamma^q_k(f^{-1}(x)) + a^q_k(f^{-1}(x)))\overline{a}^k_i)$$

$$= A^p_i(a)\Gamma^q_k a^k\overline{a}^k_i + A^p_i(a)\overline{a}^k_i\overline{a}^k_i a^q_j + A^p_i(a)\Gamma^q_k a^k_\overline{a}^k_i a^q_j + A^p_i(a)\Gamma^q_k a^k\overline{a}^k_i a^q_j$$

$$+ A^p_i(a)\overline{a}^k_i a^q_j \overline{a}^{k\_j} + A^p_i(a)(\Gamma^q_k + a^q_k)\overline{a}^k_i,$$

where $A^p_i(a)$ is a coordinate expression of the adjoint representation of $G$, $\overline{a}^i_j$ is the inverse matrix to $a^i_j$, that’s $\overline{a}^i_j \sim j^1_i(f^{-1})$, $A^p_{qr}(a) = \frac{\partial}{\partial x^r} A^p_q(a)$ are some functions on $G$ and since $A^p_q$ is a coordinate expression of the adjoint representation we have $A^p_q(e) = c^p_qr$. 

The coordinates of the image of the canonical inclusion of $W^2_m H$ into $W^2_m G$ are $(a^1_i, a^2_{jk}, a, a^m_k, a^m_i = 0, a^m_{ij} = 0)$. And the action of $W^2_m H$ on $J^1_h(C)$ is again given by the same formulas.

**The action of the kernel of the projection $W^2_m H \to G^1_m \times H$ on the curvature, Christoffels and their symmetrisations:**

\[
\begin{align*}
\Gamma^v_i &= \Gamma^v_i + a^v_i \\
\Gamma^m_i &= \Gamma^m_i \\
\Gamma^m_{ij} &= A^m_{q r}(e) \Gamma^q_{i (j} a^r_{i)} + A^m_{q r}(e) a^q_{i (j} a^r_{i)} + \left( \Gamma^k_k + a^k_i \right) a^k_{ij} \\
\Gamma^m_{ij} &= \Gamma^m_{ij} + A^m_{q r}(e) \Gamma^q_{i (j} a^r_{i)} + A^m_{q r}(e) a^q_{i (j} a^r_{i)} + \Gamma^m_{k} a^k_{ij} \\
\Gamma^m_{ij} &= K^m_{ij}.
\end{align*}
\]

The last equality can be justified by the tensoriality of the curvature or it can be verified by direct computation (we substitute (1) into the coordinate expression of the curvature: $K^p_{ij} = \Gamma^p_{ij} + c^p_{qr} \Gamma^q_{ij} \Gamma^r_{jk}$).

**The action of $W^r_m H$ on the $r$-th invariant jet of the curvature function.** Let us recall we use the value of the jet in $(0, e)$ (in the local trivialization) to represent the values of the curvature function in the fiber. For the $\Phi : \mathbb{R}^m \times H \to \mathbb{R}^m \times H$ we use the notation $\Phi \simeq (f, \varphi)$, $f$ the base map, $\varphi : \mathbb{R}^m \to H$ (see above). Then we have

\[
(\Phi \circ j^r_{\omega} \kappa^\omega)(0, e) = j^r_{\Phi(\omega)}(0, e)
\]

\[
\begin{align*}
&= j^r_{\Phi(\omega)}(0, e) = j^r_{\Phi(\omega)}(0, e) \\
&= j^r_{\varphi}(0, \varphi^{-1}(0)) = J^r(\lambda)(\varphi(0))(j^r_{\varphi}(0, e)),
\end{align*}
\]

where $\lambda : H \to \text{GL}(\mathbb{g}^+ \otimes \mathbb{g}^+ \otimes \mathbb{g})$ is just tensor product of the Ad representation (see earlier) and $J^r(\lambda)$ is the $r$-th jet prolongation of the representation $\lambda$. Especially we can see that the group $W^r_m H = G^r_m \times T^r_m H$ acts on the invariant $r$-jet of curvature function only by the values in $H$ ($H \subset W^r_m H$, see later for the explicit form of the injection in reductive, locally effective geometries). On the other hand we will show that the kernel of the projection $W^r_m H = G^r_m \times T^r_m H \to G^1_m H$ acts transitively on the symmetrisations of Christoffel symbols of Cartan connections.

Applying the invariant derivation formula we can see that in the coordinate expression of the first invariant derivation of the curvature function, the highest order canonical coordinates on $J^2_h(C)$ appears as $\Gamma^p_{[ij]k}$, the antisymmetrisation in first two subscripts. Inductively we deduce that in the coordinate expression of $K^p_{ij1 \ldots ir}$, the $r$-th invariant derivation of the curvature function $\kappa$, the $\Gamma^p_{ij1 \ldots ir}$ are present just as antisymmetrisations in the first two subscripts. That is the $r$-th invariant jet $(K^p_{ij}, K^p_{i1j1}, \ldots, K^p_{ij1 \ldots ir})$, the symmetrisations $\Gamma^p_{(ij)}, \Gamma^p_{(ij1)}, \ldots, \Gamma^p_{(ij1 \ldots ir)}$ of
the derivations of the Christoffels symbols and the Christoffels symbols $\Gamma^p_i$ determine uniquely an $r$-th jet in the $J^r_0 C$, the $r$-th jet prolongation of the standard fiber of the bundle of the Cartan connections.

From the above computations it is also clear that the kernel of the projection $W^r_m H \to G^1_m \times H$ acts transitively on the symmetrisations and on the $\Gamma^r_v$, the $h$ part of the Cartan connection. Thus on the set level we are already ready to exploit the orbit reduction theorem and we get

**Proposition 5.2.** All $W^{r+1}_m(H)$-equivariant maps from $J^r_0 C$ to any $G^1_m \times H$ space (it is also a $W^{r+1}_m(H)$ space – we define the action of the kernel of the projection $W^{r+1}_m H \to G^1_m \times H$ to be the trivial one) can be factorized through the formal $(r-1)$-st invariant jet of the curvature function mapping $J^r_0 C \to \bigoplus_{i=0}^{r-1} (\otimes^i g^* \otimes g)$ and $g_-$ part of the connection $\omega_- : J^r_0 C \to g_- \otimes g_-$. All maps here are considered as maps between sets without any additional structure. □

We can formulate the previous result in the language of natural sheafs. There is a smooth structure on the natural sheaf $K^r_r$ of $r$-th invariant jet prolongation of curvature function: we know $K^r_r \subset C^\infty(K^r_r)$ and there is a canonical smooth structure on sections of $K^r_r$ (smooth maps are smoothly parametrized maps, see [KM]). Smooth local maps $\psi : \mathbb{R}^\alpha \to K^r_r$ are then those maps which are also smooth as maps to $C^\infty(K^r_r)$.

There is also the second smooth structure on the stalk of $K^r_r$ as it is a factor space of $J^r_0 C$. It remains unanswered to what extent these two structures coincide. Nevertheless smooth maps with respect to the second structure are also smooth with respect to the first one.

For the formulation of what we have proved we use the notion of the $(r, s)$ order of the gauge natural bundle.

**Orders of gauge natural bundles.** Let $Y, Z$ be fibered manifolds and $f, g : Y \to Z$ be $\mathcal{F}\mathcal{M}$-morphisms, $q \geq r, s \geq r$ be integers. We say that $f, g$ determine the same $(r, s, q)$-jet at $y \in Y$, if

$$j^r_y f = j^r_y g, \quad j^s_y (f|_{Y_x}) = j^s_y (g|_{Y_x}), \quad j^q_y (B_f) = j^q_y (B_g),$$

where $x$ is the projection of $y$ onto the base manifold of $Y$. We denote the equivalence class of such morphisms as $j^r_y \circ \mathcal{F} f$. Now for the $\mathcal{P}\mathcal{B}_m(H)$-morphisms $\Phi, \Psi : P \to P'$ the equality $j^r_y \circ \mathcal{F} \Phi = j^s_x \circ \mathcal{F} \Psi$ holds for any point in the fiber of $y$, and we write $j^r_x \circ \mathcal{F} \Phi = j^s_x \circ \mathcal{F} \Psi$, $x$ the projection of $y$ to the base of $P$.

Further let $Z : \mathcal{P}\mathcal{B}_m(H) \to \mathcal{F}\mathcal{M}$ be a gauge natural bundle. We say that $Z$ is of order $(r, s)$ if for any $\Phi, \Psi : P \to P', \mathcal{P}\mathcal{B}_m(H)$-morphisms, and any $x \in BP$ the equality $j^r_x \circ \mathcal{F} \Phi = j^s_x \circ \mathcal{F} \Psi$ implies $Z\Phi|Z_x P = Z\Psi|Z_x P$. 

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**Theorem 5.3.** Let $Z$ be a tensor gauge natural bundle (bundle of the order $(1,0)$). Then any $r$-th order gauge natural operator $D: C \to Z$ can be factorized through the $(r-1)$-st invariant jet of the curvature function and $g_-$-part of the connections, that is $D = E \circ (j_{\omega,\kappa,\omega,-}^{r-1}) : J^r(C) \to K_{r-1}$. □

**Remark.** Let us summarize what we have proved: there is a natural sheaf in the sheaf of sections of $K_r$ where the values of $r$-th invariant jets of curvature functions of Cartan connections on $P$ take place. We know this sheaf is limited by Ricci and Bianchi identities and their invariant derivatives. We further know, that this sheaf is $W_{r,m}^r(H)$-equivariant and that all gauge natural operators of the order $r$ from $CP$ to some bundle of the order $(1,0)$ have to factor through it.

The role of $\omega_-$ can be well interpreted in the reductive geometries (see later).

Thus we have reduced the problem of finding gauge natural operators of the order $r$ on the bundle of Cartan connections to the problem of searching $W_{r+1}^r(H)$-equivariant maps from a $W_{r+1}^r(H)$-equivariant subspace of $P \times J^r(C) \oplus \bigwedge_{i=0}^r (\otimes^i g_+ \otimes V)$ (has to be specified from case to case) to a standard fiber of the bundle to which the natural operators aim. Further the group $W_{r+1}^r(H)$ acts on the curvatures nontrivially only by values in $H$. Unfortunately then such $W_{r}^r(H)$-equivariant maps can give rise to zero operators.

Further there is a question whether all smooth operators on $K_r$ comes from the smooth maps from the stalk of $K_r$ (with the smooth structure as factor space of $J^1_0(C)$) to the standard fiber of the bundle in question.

There is a hope, that the subbundle in $C^\infty(K_r)$ given by Bianchi and Ricci identities is $K_r$. In that case there would be a bijective correspondence between $C^1_{m} \times H$-equivariant maps from the algebraic submanifold of the standard fiber of $K_r$ given by the Ricci and Bianchi identities and $r$-th order natural operators on the bundle of Cartan connections.

This is true for the bundle of torsion free affine connections: the Ricci and Bianchi identities have exactly the same form (after some identifications) as those for symmetric linear connections. Thus the proof of the theorem 51.16 from [KMS] can be modified for the torsion free affine connections.

**Reductive geometries.**

Let us again suppose that $g_-$ is an abelian ideal. The representation $\text{ad}$ provides us a homomorphism $\mathfrak{h} \to \text{gl}(g_-)$, and let us further suppose that it is injective, i.e. $\mathfrak{h} \subset \text{gl}(g_-)$. In other words we will deal with locally effective models only (see [S]). Further $\omega = \omega_+ \oplus \omega_-$ and it is known that the form $\omega_-$ on a principal $H$-bundle $P$ over $M$ enables us to identify $P$ with a cover of a subbundle in $P^1 M$ the bundle of linear frames on $M$ (see [K], Chapter 2, also for next lemmas; the notation in [K] is however a little different). For a comfort of a reader let us quickly recall this identification: let us fix an isomorphism $g_- \simeq \mathbb{R}^m$ (as vector spaces, $m$ is the dimension of $M$) and let $X \in T_x(M), u \in P$. Then the coordinates of $X$ in the frame $u$ are given by
where $\gamma(u)(X)$ is a horizontal lift of $X$ to $u$ (because $\omega_-$ is a horizontal form, we can use any horizontal lift). Let us denote by $i$ the inclusion of $P$ into $P^1M$.

Since the $\omega_-$ has the same equivariant properties as the solder form on $P^1M$ does, this identification is really a morphism of principal bundles.

**Remark.** Let us spend a few words on the principal bundle structure of the linear frame bundle $P^1M$ of a manifold $M$. Let $v \in P^1M$ be a linear frame. It can be identified with the mapping $v : T_xM \rightarrow \mathbb{R}^m$ which assigns to a vector $X \in T_xM$ its coordinates in the base $v$. The group $\text{Ad}(H) \subset \text{GL}(\mathfrak{g}_-) = \text{GL}(\mathbb{R}^m)$ then acts on $P^1M$ in the following way: for $h \in H$ we have $(vh)(X) = \text{Ad}(h^{-1})(v(X))$. A linear frame $v$ can be as well considered as a mapping $\mathbb{R}^m \rightarrow T_xM$ and then $A \in \text{GL}(\mathbb{R}^m)$ acts as $vh = v \circ h$; we will use either identification according to a situation.

**Lemma 5.4.** The inclusion $i$ of $P$ into $P^1M$ is a morphism of principal fiber bundles.

**Proof.** Let $u \in P$, then

$$i(uh)(X) = \omega_-(\gamma(u)(X)) = \text{Ad}(h^{-1})\omega_-(u)(X) = ((i(u))h)(X),$$

where we used the "first" principal bundle structure of the linear frame bundle from the previous remark.

**Lemma 5.5.** For the embedding $i : P \rightarrow P^1M$ we have

$$i_*(\omega_-) = \theta,$$

where $\theta$ is the solder form on $P^1M$, $\theta(X_u) = u(T\pi(X))$ ($u : T_xM \rightarrow \mathbb{R}^m$ a linear frame, $\pi : P^1M \rightarrow M$ the projection on the base).

**Proof.**

$$(i_*(\omega_-))(X_u) = \omega_-(T(i^{-1})X_u) = \omega_-(\pi(T(i^{-1})X_u)) = \omega_-(T\pi(X_u)) = u(T\pi(X)),$$

where in the second equality we have used the identification $w_- : TP \rightarrow \mathbb{R}^m \cong TM \rightarrow (P \times_{\text{Ad}} \mathbb{R}^m)$ and further we know that $i : P \rightarrow P^1M$ is fibered over identity.

Further we know $i_*(\omega_h)$ is a connection on the $H$-structure. Let us compute how do the Christoffel symbols of $\omega_h$ and $i_*(\omega_h)$ differ:
Lemma 5.6. Let $\omega = \Gamma_j^i dx^j + \Gamma_{jk}^i dx^k + \Gamma_{jkl}^i dy_l^i$ be a coordinate expression of a Cartan connection $\omega$ in local trivialization $P = \mathbb{R}^m \times H$ ($x^i$ are coordinates on $\mathbb{R}^m$, $y_j^i$ on $H$), that is $\Gamma_j^i$ be the Christoffel symbols of $\omega_h$, $\omega_- = \Gamma_j^i dx^j$. Then the Christoffel symbols of $i_*(\omega_h)$ are $\Gamma_{j;k}^i + \Gamma_{ik}^j \Gamma_j^k$ (in the coordinates induced on $P^1 M$ by $x^i$ on $M$).

Proof. Let $X = X^i \frac{\partial}{\partial x^i} = \frac{\partial}{\partial t}|_0 x(t) \in T_x M$ be a vector on the base. Then the horizontal lift $\gamma(X)$ of $X$ with respect to principal connection $\omega_h$ has the coordinates

$$\gamma(X) = X^i \frac{\partial}{\partial x^i} - \Gamma_{jk}^i X^k \frac{\partial}{\partial y_j^i} = \frac{\partial}{\partial t}|_0 (x(t), y(t)).$$

Then the horizontal lift of $X$ with respect to a linear connection $i_*(\omega_h)$ is $(Ti)(\gamma(X))$:

$$Ti(\gamma(X)) = \frac{\partial}{\partial t}|_0 i(x(t), y(t))$$

$$= \frac{\partial}{\partial t}|_0 (x(t), \Gamma_j^i(x(t), y(t)))$$

$$= (X^i, \Gamma_{j;k}^i X^k + \Gamma_{ik}^j \Gamma_j^k X^k),$$

where we have used the following identification: we have $\omega_- : TP \to \mathbb{R}^m$ is a horizontal form satisfying $\omega_-(uh)(Tr_h(Y)) = Ad(h^{-1})\omega_-(u)(X)$, that is we can consider the form $\omega_-$ as a form on $TM$ with values in $(P \times_{Ad} \mathbb{R}^m)$, that is a mapping $\omega_-(X) : P \to \mathbb{R}^m$ which is Ad-equivariant and for fix $u \in P$ it is linear in $X$, that is $\omega_- : P \to TM^* \otimes \mathbb{R}^m$. Then the derivation of $\omega_-$ in the direction $Z \in \mathfrak{h}$ that is in the direction of a fiber of $P$ is according to Lemma 1.8., $-ad(Z) \circ (\omega_-)$, which for $ad(h) \subset gl(\mathbb{R}^m)$ gives in coordinates $-Z_k^i \Gamma_j^k$.

The vertical part of the lift $Ti(\gamma(X))$ then gives the Christoffel symbols of the linear connection $i_*(\omega_h)$. □

Thus $\omega_h$ is a connection on a $G$-structure corresponding to the group $H$. Further we have the following characterization of the morphisms preserving the form $\omega_-$:

Lemma 5.7. Let $\omega$ be a Cartan connection on the principal $H$-bundle $P$ over the base manifold $M$. Then the automorphisms of $P$ preserving the form $\omega_-$ are exactly morphisms of the form $i^*(P^1 f)$, where $f$ is a morphism of the $H$-structure $i(P) \subset P^1 M$.

Proof. Let $\Phi : P \to P$, $\Phi^*(\omega_-) = \omega_-$. For technical reasons let us indicate the inclusion $i : P \to P^1 M$ described above as $i_{\omega_-}$. Then $i_{\omega_-} = i_{\Phi^*(\omega_-)} = i$ and $i_*(\Phi) : i(P) \to i(P)$ is according to previous lemma a $P^1 M$ morphism, even a morphism of the $H$-structure $i(P) \subset P^1 M$ and therefore it is of the form $P^1 f$, where $f : M \to M$. □
Lemma 5.8. Let $\omega$ be a Cartan connection on a principal bundle $P$ defining a torsion free geometry. Then we have for the coordinate expression of $\omega$ (in proper coordinates):

$$\Gamma^i_{[jk]} = \Gamma^i_{[j;k]} = 0,$$

that is the antisymmetrisation of the Christoffel symbols of the principal connection $\omega_h$ as well as the antisymmetrisations of the derivations of the form $\omega_-$ are zero.

Proof. The nullity of the torsion form is preserved by the push-forward of the embedding $i : P \to P^1M$. The torsion of the linear connection $i_*(\omega_h)$ on $P^1M$ is then given by the antisymmetrisation of its Christoffel symbols. Then the requirement of the zero torsion gives in coordinates (according to Lemma 5.6):

$$\Gamma^i_{j;k} + \Gamma^i_{l;k}\Gamma^l_j = \Gamma^i_{k;j} + \Gamma^i_{l;j}\Gamma^l_k.$$

As we have discussed before, we can choose the coordinates in such a manner that $\Gamma^i_j$ would be the identity matrix and $\Gamma^i_{j;k} = 0$. In this coordinates then we get immediately $\Gamma^i_{[jk]} = \Gamma^i_{[j;k]} = 0$. □

Remark 1. A morphism of linear frame bundles is uniquely determined by its base map, thus it looses sense to speak about the orders of bundles as about two numbers and we describe the order of the natural bundle by a single non-negative integer only. Further we will speak about natural bundles on a given $G$-structure (an $H$ principal subbundle $P$ in $P^1M$). These will be the bundles invariant under the actions of morphisms $f$ of the base manifold preserving the given $G$-structure ($P^1f(P) \subset P$).

Remark 2. We know that a Cartan connection on a principal bundle $P$ in a reductive geometry splits to $\omega = \omega_h + \omega_-$, where $\omega_h$ is a principal connection on $P$. Then $i_*(\omega_h)$, where the $i : P \to P^1M$ is the embedding described above, can be uniquely extended to a linear connection belonging to the $H$-structure, and with the help of the previous lemma we are able to reformulate Theorem 5.3 for the $H$-structures.

Theorem 5.9. Let $H \subset \text{GL}(m, \mathbb{R})$ be a linear subgroup. Then any $r$-th order natural operator $D$ on the bundle of linear connections belonging to a given $H$-structure $P \subset P^1M$ with values in the first order natural bundle $Z$ on the $H$-structure factorizes through up to the $r$-th order invariant (covariant) derivatives of the curvature and torsion operators.

We present two proofs. In the first proof we show the theorem is the consequence of Theorem 5.3. The second proof is the straight calculation.

First proof. Theorem 5.3 talks about the gauge natural operators over the category $\mathcal{PB}_m(H)$. We will introduce new, in a way equivalent notion of naturality: let $\mathcal{PB}^0_m(H)$ be a category of principal bundles whose objects are $H$-principal bundles with $m$-dimensional manifolds together with a horizontal form $\rho : TP \to g_-$ with the invariant property $(r_h)^*\rho = \text{Ad}(h^{-1})\rho$ and such that $\rho(T_uP) = g_-$ for any $u \in P$. We will
write $P_\rho$ for the objects. Morphisms between $P_\rho$ and $P'_\sigma$ are (local) principal bundle morphisms $\Phi : P \to P'$ such that $\Phi^*(\sigma) = \rho$. Let us notice, that for a Cartan connection $\omega$ on an $H$-principal bundle $P$ the object $P_{\omega_-}$ is from $\mathcal{PB}^\theta_m(H)$, and on $\mathcal{PB}^\theta_m(H)$ there is a counterpart of the gauge natural bundle of Cartan connections: a functor $C_\theta$ which appoints to each object $P_\rho$ a bundle of principal connections on $P$.

Now let $D : C \to Z$ be a gauge natural operator between the bundle of Cartan connections and a gauge natural bundle $Z$, i.e. $D(\Phi^*(\omega)) = \Phi^*(D(\omega))$ for a Cartan connection $\omega$ on $P$ and a $\mathcal{PB}^\theta_m(H)$-morphism $\Phi$. Then we define $D_\theta : C_\theta \to Z$, $D_{P_{\omega_-}}(\omega_h) = D_{\omega_-}(\omega_h) = D(\omega_h \oplus \omega_-)$, where $\omega_h$ is an arbitrary principal connection on $P$ and $\omega_-$ a horizontal form with the needed properties. And for Cartan connections $\omega$ on $P$ and $\omega'$ on $P'$ such that $\Phi^*(\omega') = \omega$, $\Phi : P \to P'$ we have:

$$D_{\omega_-}(\Phi^*(\omega'_h)) = D_{\omega_-}(\omega_h) = D(\omega_h \oplus \omega_-)$$

$$= D(\Phi^*(\omega')) = \Phi^*(D(\omega')) = \Phi^*(D_{\omega_-}(\omega'_h)), $$

and the operator $D_\theta$ is natural in $\mathcal{PB}^\theta_m(H)$.

Conversely let $D_\theta : C_\theta \to Z$ be an operator commuting with the action of $\mathcal{PB}^\theta_m(H)$-morphisms. Then we have

$$D(\Phi^*(\omega')) = D(\omega) = D_{\omega_-}(\omega_h) = D_{\omega_-}(\Phi^*(\omega'_h))$$

$$= \Phi^*(D_{\omega_-}(\omega_h)) = \Phi^*(D(\omega)),$$

and $D$ is a gauge natural operator.

Moreover the naturality in $\mathcal{PB}^\theta_m(H)$ is equivalent to the naturality in the sense of $G$-structures, that is we call an operator on a bundle over the bundle of linear frames $(P^1 M)$ natural if it commutes with the actions (pullbacks and pushouts) of morphisms of the form $P^1 f$, $f : M \to M$ a local diffeomorphism.

Let $D$ be a natural operator over $\mathcal{PB}^\theta_m(H)$ on $C_\theta$. Then it is trivially a natural operator on linear connections on $P^1(M)$ (we just restrict the set of morphisms).

Conversely let $D$ be a natural operator on linear connections over $P^1(M)$. Then we extend the operator over the whole $\mathcal{PB}^\theta_m(H)$ as follows: $D_{\omega_-}(\omega_h) = \Phi^* D_\theta(\Phi_*(\omega_h)) = \Phi^* D(\Phi_*(\omega_h))$, where $\theta$ is the canonical solder form on $P^1(M)$ and $\Phi : P \to P^1(M)$ is the principal bundle morphism over identity on $M$ such that $\Phi_*(\omega_-) = \theta$ (under these conditions it is unique). Let us verify that this extension is natural: let $\omega'$ be a Cartan connection on $P'$, $\Psi : P' \to P$, $\Psi_*(\omega') = \omega$, and let $\Phi' : P' \to P^1(M)$ be the unique morphism such that $\Phi'_*(\omega_-') = \theta$, then

$$D_{\omega_-}(\Psi_*(\omega'_h)) = \Phi^* D(\Phi_*(\Psi_*(\omega'_h)))$$

$$= \Phi^* D(\Phi_*(\Phi'_h))^* \Phi_*(\omega'_h))$$

$$= \Phi^* \Phi_*(\Phi'_h) D(\Phi'_*(\omega'_h)) = \Psi_* D_{\omega_-'}(\omega'_h). \quad \square$$
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Second Proof. Theorem 5.3 gives us that the \( r \)-th order gauge natural operators from the bundle of the Cartan connections of type \((\mathfrak{h}, \mathfrak{h} \oplus \mathbb{R}^m)\) to a gauge natural bundle over the category \(\mathcal{PB}_m(H)\) factorize through the \( r \)-th invariant jet of the curvature function and the \(\omega_-\)-part of the Cartan connections.

In the statement of this theorem we restrict both the set of morphisms to which we require the invariance and the domain of natural operator. Thus it is theoretically possible that the set of natural operators invariant under these new conditions could be different.

We have to make some considerations in local coordinates again. Let us first describe the action of jets of the morphisms in question as a subgroup of \(W^1_m H\). Let \( g : \mathbb{R}^m \to \mathbb{R}^m \), \( g(0) = 0 \) be a local diffeomorphism. Then we have \( P^1 g : P^1 \mathbb{R}^m \to P^1 \mathbb{R}^m \), further \( (P^1 \mathbb{R}^m)_0 = \text{inv}J_0^1(\mathbb{R}^m, \mathbb{R}^m)_0 = G^1_m \) and \( P^1 \mathbb{R}^m \) has the canonical structure of the trivial bundle \( \mathbb{R}^m \times G^1_m \). Thus \( j^1_{(0, e)} P^1 g \) lies in \( W^1_m (G^1_m) \).

Recall that for \( \Phi : \mathbb{R}^m \times G \to \mathbb{R}^m \times G \) we write \( \Phi = (f, \varphi) \), \( \Phi(x, g) = (f(x), \varphi(x)g) \) and on \( W^1_m (G) \) we have the coordinates \( a_j^i = (\frac{\partial}{\partial x^j} f^i)(0), a = \varphi(0), a^p_i = (\frac{\partial}{\partial x^2}(\varphi(0))^{-1} \varphi^p)(0) \), and on \( W^2_m (G) \) then we add the coordinates \( a^p_{ij} = (\frac{\partial}{\partial x^2} \varphi(0))^{-1} \varphi^p)(0) \). Then we have

\[
(P^1 g)(x, j^1_{0, e} h) = (g(x), j^1_{0, e}(h \circ g(x))),
\]

\( h : \mathbb{R}^m \to \mathbb{R}^m \), \( h(0) = 0 \), and for 1-jet in \((0, e)\) we get in coordinates

\[
j^1_{(0, e)} g \simeq (b^i_j = \frac{\partial}{\partial x^j} g^i(0), b^i_j = \text{pr}_2((P^1 g)(0, e)), \tilde{b}^i_{jk} = (\frac{\partial}{\partial x^j} \partial x^k} \text{pr}_2((P^1 g)(0, e))))^{-1} g^i(0)).
\]

Since both in \( G^1_m \) and \( W^1_m G^1_m \) the group operation is defined by jet composition and \( P^1 \) is a functor we get the inclusion \( i : G^2_m \to W^1_m G^1_m, i(b^i_j, b^i_{jk}) = (b^i_j, b^i_{jk}, b^i_{jk}) \). And if we consider the equation (7) also for higher jets we obtain the inclusion \( i : G^m \to W^1_m G^1_m, i(a^i_{jk}, \ldots, a^i_{jk}) = (a^i_{jk}, a^i_{jk}, \ldots, a^i_{jk}) \).

Now for an \( H \)-principal bundle \( P \), Cartan connections with values in \( g \) with a fixed \(\omega_-\) form a subbundle in \( CP \). In coordinates it is given by \(\Gamma^m \) constant and \(\Gamma^m \) arbitrary. Now we write \(\Gamma^m_{j;k} \) instead of \(\Gamma^m \) since \(\Gamma^m \) \( \in \mathfrak{h} \oplus \mathbb{R}^m \) and \(\mathfrak{h} \subset \text{gl}(\mathbb{R}^m) \).

The equations (2) can thus be split in two equations:

\[
\Gamma^m_{(j;k)} = \Gamma^m_{(j;k)} + a^i_{kj},
\]

\[
\Gamma^m_{[j;k]} = \Gamma^m_{[j;k]},
\]
that is the kernel of the composition of the inclusion \(i : G_{m}^{1} \rightarrow W_{m}^{1}G_{m}^{1}\) with the projection of \(W_{m}^{1}G_{m}^{1} \rightarrow G_{m}^{1} \times G_{m}^{1}\) acts transitively on the symmetric part \(\Gamma_{i}^{m}(j,k)\) of the Christoffel symbols and leaves the antisymmetrisations (that is the formal torsion) intact. It leaves \(\Gamma_{i}^{m}\), that is the \(g_{-}\)-part of the connection intact as well. The situation is similar on the first jet prolongation of the standard fiber of the bundle of Cartan connections: the equations (4) and (5) gives us that the kernel of the projection of \(i(G_{m}^{3})\) on \(G_{m}^{1} \times G_{m}^{1}\), that is in coordinates \((\delta_{ij}, \delta_{ij}, a_{jkl}^{k}, a_{ijkl})\), acts transitively on the symmetrisations of the two jets of the \(h\)-part of the elements of Cartan connections (that is in coordinates on \(\Gamma_{i}^{m}(j;kl)\)) and acts trivially on the curvature (in (5) we act transitively by the term \(\Gamma_{i}^{m}a_{kij}\), in (4) then by the term \(a_{ij}\), which corresponds to \(a_{i}^{k}\) in the ”reductive” notation). Further derivations of (4) and (5) give us that \(\ker(\pi_{r} \circ i(G_{m}^{r+1}))\), \(\pi_{r} : W_{m}^{r}G_{m}^{1} \rightarrow G_{m}^{1} \times G_{m}^{1}\) acts transitively on \(\Gamma_{i}^{m}(j;1...r)\) and the tensor character of \(r\)-th invariant jets of the curvature function implies it leaves the invariant (covariant) derivatives of the curvature intact.

Thus the orbit reduction theorem says that any \(i(G_{m}^{r+1})\)-map from the \(J_{0}^{r}C\) to \(G_{m}^{1} \times G_{m}^{1}\)-space factors through the antisymmetrisation of the \(h\)-part of the Christofel symbols, up to \(r\)-th order covariant derivatives of the curvature and through up to \(r\)-th derivatives of the \(g_{-}\)-part of the Christoffel symbols and this is in the language of natural operators the statement of the theorem. □

**Remark.** The action \(\mathcal{J}^{1}(\lambda)\) of \(h\) on the invariant jet of the curvature function \(\kappa\) can be in reductive geometries simplified. Namely we have

\[
\mathcal{J}^{1}(Z)(v, \varphi) = (\lambda(Z)(v), \lambda(Z) \circ \varphi - \varphi \circ \text{ad}_{-}(Z)),
\]

and this is the ”tensor” action corresponding to the following action of the group \(H\):

\[
\mathcal{J}^{1}(h)(v, \varphi) = (\lambda(h)(v), \lambda(h)(\varphi) \circ \text{Ad}(h^{-1})),
\]

which shows also the invariance of the invariant derivative itself. This also gives the higher prolongations of the action: for \(\varphi \in \otimes^{r}\mathfrak{g}^{n} \otimes V\) and \(h \in H\) we have

\[
\mathcal{J}^{r}(\lambda)(\varphi) = \lambda(h)(\varphi) \circ \otimes^{r}(\text{Ad}(h^{-1})).
\]

**Torsion free geometries**

A Cartan connection (geometry) is **torsion free** if its curvature function takes values in the subalgebra \(\mathfrak{h} \subset \mathfrak{g}\) only.

Lemma 4.17 says that Ricci and Bianchi identities define an affine subbundle in \(P \times_{\lambda} (\Lambda^{2}\mathfrak{g}^{*} \otimes \mathfrak{g})\) and their invariant derivatives (up to appropriate order) define an affine subbundle in \(P \times_{\mathcal{J}^{r}(\lambda)} \otimes^{r}\mathfrak{g}^{n} \otimes (\Lambda^{2}\mathfrak{g}^{*} \otimes \mathfrak{g})\) (which we denote by \(K^{r}\)). We have \(\sum_{i=0}^{r} K^{r} = K_{r}\).
Corollary 5.10. Let $H \subset \text{GL}(m, \mathbb{R})$ be a linear subgroup. Then any $r$-th order natural operator $D$ on the bundle of linear connections belonging to a given $H$-structure $P \subset P^1M$ with values in a first order natural bundle $Z$ on the $H$-structure factorizes through up to the $(r-1)$-st order invariant (covariant) derivatives of the curvature, i.e. there exists a natural transformation $\tau : K_{r-1} \rightarrow Z$ such that $D = \tau \circ j^{\omega}_{r-1}\gamma$, $\gamma$ is the curvature operator. □

Remark. Either of the theorems of this chapter suggest the possible way of finding natural operators in Cartan geometries (or in a class of Cartan geometries). But it is still a long way to the ”real” operators, typically one has to use some of representation theory. The representation $J^r(\lambda)$ of $W^r_mH$ is complicated to that extent that it is out of the scope of this thesis to discuss the representation in general case. This is where the title of this episode comes from.
List of symbols

Some of the letters of Latin or Greek alphabets are used in a particular meaning in this text. Though they are not proper symbols, they are included in the following list.

B the base functor \( B : \mathcal{FM} \to \mathcal{M} \).
\( \mathcal{FM} \) the category of fibered manifolds and fiber respecting mappings
C the bundle of Cartan connections (a functor from \( \mathcal{PB}_m(H) \) to \( \mathcal{FM} \))
G a Lie group
\( \mathfrak{g} \) the Lie algebra of \( G \)
H a Lie group, \( H \subset G \) a closed subgroup
\( \mathfrak{h} \) the Lie algebra of \( H \)
\( G^r_m = \text{inv} J^r_0(\mathbb{R}^m,\mathbb{R}^m) \) the \( r \)-th jet group in dimension \( m \) (with the jet composition)
i the insertion operator
\( \text{inv} J^r(M,N) \) the bundle of invertible \( r \)-jets of \( M \) into \( N \)
\( J^rY \) the bundle of \( r \)-jets of local sections of a fibered manifold \( Y \)
\( J^r(M,N) \) the bundle of \( r \)-jets of smooth functions from \( M \) to \( N \)
\( J^r_x(M,N)_y \) the \( r \)-th jets in \( x \in M \) of smooth functions from \( M \) to \( N \) with the target in \( y \in N \)
\( j^r_x f \) \( r \)-jet of a mapping \( f \) at \( x \)
\( j^{(r,s,q)}_u \Phi \) \( (r,s,q) \)-jet of a principal bundle morphism \( \Phi : P \to P \) at \( u \in P \)
\( j^r_x \Phi \) fibered \( r \)-jet of a principal bundle morphism \( \Phi : P \to P \) at \( x \in BP \), the base manifold of \( P \)
l\( g \) left multiplication with the element \( g \in G \) on a Lie group \( G \)
\( \mathcal{MF}_m \) the category of \( m \)-dimensional manifolds and local diffeomorphisms
\( \mathcal{PB}_m(H) \) the category of principal \( G \)-bundles over \( m \)-dimensional manifolds and their local isomorphisms
\( P \times H V \) an associated vector bundle to the \( H \)-principal bundle \( P \) via the action \( \lambda : H \to V \)
\( P^rM = \text{inv} J^r_0(\mathbb{R}^{\dim M},M) \) the \( r \)-th order frame bundle of a manifold \( M \)
r\( a : P \to P \) right principal action of a Lie group \( G \) (\( a \in G \)) on a principal \( G \)-bundle \( P \)
\( W^rP = \{ j^r\Phi(0,e) ; \Phi \in \text{Mor}(\mathcal{PB}_m(H)), \Phi : \mathbb{R}^m \times G \to P \} \) the \( r \)-th principal prolongation of a principal bundle \( P \)
\( W^r_mG = J^r_0(\mathbb{R}^m \times G,\mathbb{R}^m \times G)_0 \) the \((m,r)\)-principal prolongation of a Lie group \( G \)
\( \lambda : H \to \text{GL}(V) \) a representation of the Lie group \( H \) on the vector space \( V \), i.e. the homomorphism of the groups \( H \) and \( \text{GL}(V) \)

\( \lambda' : \mathfrak{h} \to \text{gl}(V) \) the linear representation of the Lie algebra \( \mathfrak{h} \) obtained from \( \lambda : H \to \text{GL}(V) \) by the derivation at the identity.

\( \nabla_X^\omega \) the invariant derivation with respect to a Cartan connection \( \omega \) in the direction of \( X \in \mathfrak{g} \)

\( \Omega^k(P,V) \) the space of \( k \)-forms on \( P \) with values in \( V \)

\( \zeta_X(u) = Tr_u(X) \) the fundamental vector field on a principal fiber bundle \( P \) corresponding to a vector \( X \in \mathfrak{g} \)
REFERENCES


