TOPOLOGICAL AND CATEGORY-THEORETIC ASPECTS OF ABSTRACT ELEMENTARY CLASSES

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An outline of the content of this talk:

Abstract Elementary Classes:

- Topologizing Sets of Types
- Rank Functions
- Stability Spectra

Accessible Categories:

- Connections to AECs
- Implications for Stability
- A Structure Theorem for Categorical AECs

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Problem: elementary classes—classes of models of such theories—do not exhaust the interesting classes of mathematical objects.

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Generalize by abandoning syntax, and considering abstract classes of structures that retain only the essential properties common to such classes of models.

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  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
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- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

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A strong embedding $f : M \hookrightarrow_{\mathcal{K}} N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

Example 1: Let $\mathcal{K}$ be the class of models of a first order theory T, and $\prec_{\mathcal{K}}$ the elementary submodel relation. Then $\mathcal{K}$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0 + |L(T)|$.

One can think of AECs as the category-theoretic hulls of elementary classes—abandoning syntax, but retaining certain basic properties of the elementary submodel relation.

Example 2: Let ϕ be a sentence of $L_{\infty,\omega}$, $\mathcal{A}$ a fragment containing ϕ. The class $\mathcal{K} = \mathrm{Mod}(\phi)$, with $\prec_{\mathcal{K}}$ elementary embedding with respect to $\mathcal{A}$, is an AEC ($\mathrm{LS}(\mathcal{K}) = |\mathcal{A}|$). With suitable $\prec_{\mathcal{K}}$, can do the same with models of sentences in $L(Q)$, $L_{\omega_1,\omega}(Q)$, etc.

We assume the following additional properties:

Definition (Joint Embedding Property)

For any $M_1, M_2 \in \mathcal{K}$, there is $N \in \mathcal{K}$ with embeddings $f_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $f_2 : M_2 \hookrightarrow_{\mathcal{K}} N$.

Definition (Amalgamation Property)

If $M \prec_{\mathcal{K}} M_1$ and $M \prec_{\mathcal{K}} M_2$ (with all three models in $\mathcal{K}$), there is an $N \in \mathcal{K}$ and embeddings $g_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $g_2 : M_2 \hookrightarrow_{\mathcal{K}} N$ that agree on M.

Having discarded syntax, we consider a new notion of type: Galois types. In AECs with sufficient amalgamation, there is a monster model $\mathfrak{C}$ in $\mathcal{K}$, and the Galois types have a simple description:

Definition

For $a \in \mathfrak{C}$, $M \in \mathcal{K}$, the *Galois type of a over M* is defined to be the orbit of a under automorphisms of $\mathfrak{C}$ that fix M. The set of all types over M is denoted by ga-S(M).

Example: If $\mathcal{K}$ an EC and $\prec_{\mathcal{K}}$ elementary submodel, the Galois types over $M \in \mathcal{K}$ are precisely the complete types over M:

$$\text{ga-tp}(a/M) = \text{ga-tp}(b/M) \quad \text{iff} \quad \text{tp}(a/M) = \text{tp}(b/M)$$

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3. We say that $M \in \mathcal{K}$ is λ-Galois saturated if for every type p over $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$, p is realized in M, i.e. the orbit corresponding to p meets M.

Question

What can we say about the stability spectra of AECs? Can we use vaguely classical techniques to address the problem? Topology? Rank functions?

We give:

- A way of topologizing the sets ga-S(M) in such a way that topological properties of the resulting spaces correspond to semantic properties of M and $\mathcal{K}$.
- A closely related notion of rank (actually, a family of ranks) which is useful in analyzing the stability spectra of reasonably well-behaved AECs.

Let $\mathcal{K}$ be an AEC with monster model $\mathfrak{C}$. Let $\lambda \geq \mathrm{LS}(\mathcal{K})$ and $M \in \mathcal{K}$.

Definition (X_M^λ)

For each $N \prec_{\mathcal{K}} M$ with $|N| \leq \lambda$ and type $p \in \text{ga-S}(N)$, let

$$U_{p,N} = \{q \in \text{ga-S}(M) \,:\, q \upharpoonright N = p\}$$

The sets $U_{p,N}$ form a basis for a topology on ga-S(M). We denote by X_M^λ the set ga-S(M) endowed with this topology.

Note

The $U_{p,N}$ are, in fact, clopen. Types over small submodels play a role analogous to formulas in topologizing spaces of syntactic types.

One virtue of this setup is that it provides a dictionary between model-theoretic properties of $\mathcal{K}$ and models $M \in \mathcal{K}$ and topological properties of the spaces X_M^λ. Recall:

Definition

An AEC $\mathcal{K}$ is said to be χ-tame if for any $M \in \mathcal{K}$, if $q, q' \in \text{ga-S}(M)$ are distinct, then there is submodel $N \prec_{\mathcal{K}} M$ with $|N| \leq \chi$ such that $q \restriction N \neq q' \restriction N$.

Intuition: if we regard types over small models as formulas, tameness means that types are determined entirely by their constituent formulas.

The centerpiece:

Theorem (Tameness As Separation Principle)

The AEC $\mathcal{K}$ is χ-tame iff for all $M \in \mathcal{K}$, X_M^χ is Hausdorff.

Since each space X_M^λ has a basis of clopens (i.e. the $U_{p,N}$) we have a bit more:

Proposition

$\mathcal{K}$ is χ-tame iff X_M^χ is totally disconnected for every $M \in \mathcal{K}$.

Moreover,

Proposition

If $\mathcal{K}$ is χ-tame, X_M^μ is totally disconnected for every $M \in \mathcal{K}$ and $\mu \geq \chi$.

Naturally, compactness is too much to hope for. In particular,

Proposition

Let $\mathcal{K}$ be an arbitrary AEC with monster model, $M \in \mathcal{K}$, and $\lambda \geq LS(\mathcal{K})$. Then X_M^λ is not compact.

We might hope for some weaker form of compactness, but this proves incompatible with our desire for tameness. The critical complication results from the following:

Fact

For any $M \in \mathcal{K}$, the intersection of any λ many open sets in X_M^λ is open.

If $\mathcal{K}$ is sufficiently tame to guarantee Hausdorffness, this leads to near-discreteness:

Remark

Any subset $S \subseteq X_M^\lambda$ with $|S| = \lambda$ is discrete and closed.

As a consequence:

Proposition

Let $\mathcal{K}$ be χ-tame. Then for any $M \in \mathcal{K}$ and any $\lambda \geq \chi$, the space X_M^λ is not countably compact.

Another important consequence of sufficient tameness:

Proposition

Let $\mathcal{K}$ be χ-tame. Then for any $M \in \mathcal{K}$ and any $\lambda \geq \chi$, a type $q \in X_M^\lambda$ is an accumulation point of $S \subseteq X_M^\lambda$ only if every neighborhood of q contains more than λ elements of S.

In light of this fact, we define a related, slightly Morley-like rank:

Definition (RM^λ)

For $\lambda \geq \mathrm{LS}(\mathcal{K})$, we define RM^λ by the following induction: for any $q \in \text{ga-S}(M)$ with $|M| \leq \lambda$,

- $\mathrm{RM}^\lambda[q] \geq 0$.
- $\mathrm{RM}^\lambda[q] \geq \alpha$ for limit α if $\mathrm{RM}^\lambda[q] \geq \beta$ for all $\beta < \alpha$.
- $\mathrm{RM}^\lambda[q] \geq \alpha + 1$ if there exists a structure $M' \succ_{\mathcal{K}} M$ such that q has strictly more than λ many extensions to types q' over M' with $\mathrm{RM}^\lambda[q'] \geq \alpha$.

For types q over M of arbitrary size, we define

$$\mathrm{RM}^\lambda[q] = \min\{\mathrm{RM}^\lambda[q \restriction N] : N \prec_{\mathcal{K}} M, |N| \leq \lambda\}.$$

The ranks RM^λ are: monotonic, invariant under automorphisms of $\mathfrak{C}$, decreasing in λ. They are larger (typewise) than the appropriate Cantor-Bendixson ranks: for any $q \in \text{ga-S}(M)$, $\mathrm{RM}^\lambda[q]$ is greater than or equal to the CB-rank of q in X_M^λ.

Definition (λ-t.t.)

We say that $\mathcal{K}$ is λ-totally transcendental if for every $M \in \mathcal{K}$ and $q \in \text{ga-S}(M)$, $\mathrm{RM}^\lambda[q]$ is an ordinal.

No guarantee that types have unique extensions of same RM^λ rank, but:

Proposition (Quasi-unique Extension)

Let $M \prec_{\mathcal{K}} M'$, $q \in ga\text{-}S(M)$, and say that $RM^\lambda[q] = \alpha$. Given any rank α extension q' of q to a type over M', there is an intermediate structure M'', $M \prec_{\mathcal{K}} M'' \prec_{\mathcal{K}} M'$, $|M''| \leq |M| + \lambda$, and a rank α extension $p \in ga\text{-}S(M'')$ of q with $q' \in ga\text{-}S(M')$ as its unique rank α extension.

Connections with Galois stability, in case $\mathcal{K}$ is tame:

Theorem

If $\mathcal{K}$ is λ-stable where λ satisfies $\lambda^{\aleph_0} > \lambda$, then $\mathcal{K}$ is λ-t.t.

In the other direction,

Theorem

If $\mathcal{K}$ is λ-t.t., and $M \in \mathcal{K}$ with $cf(|M|) > \lambda$, then $|ga\text{-}S(M)| \leq |M| \cdot \sup\{|ga\text{-}S(N)| \mid N \prec_{\mathcal{K}} M, |N| < |M|\}$.

Roughly: the number of types over M is no worse than the number of types over submodels of strictly smaller size.

Theorem

Let $\mathcal{K}$ be λ-stable with $\lambda^{\aleph_0} > \lambda$, and let κ satisfy cf(κ) $> \lambda$. If there is an interval $[\mu, \kappa)$ such that every M $\in \mathcal{K}_{[\mu,\kappa)}$ satisfies $|$ga-S(M)$| \leq \kappa$, then $\mathcal{K}$ is κ-stable.

A very nice special case:

Corollary

If $\mathcal{K}$ is $\aleph_0$-stable, and κ is of uncountable cofinality, then if $\mathcal{K}$ is stable in every cardinality below κ, it is κ-stable as well.

This was established in (Baldwin-Kueker-VanDieren, 2004), using the machinery of splitting. Our method has produced a vastly more general result.

If $\mathcal{K}$ is only weakly χ-tame (the defining condition of tameness holds only for saturated models), the analysis still works, provided the class contains enough saturated models.

Theorem

If $\mathcal{K}$ is weakly χ-tame and μ-t.t., and κ is such that cf(κ) $> \mu$ and each M of size κ has a saturated extension also of size κ, then $\mathcal{K}$ is κ-stable.

The property of having enough saturated models crops up, oddly enough, in the context of accessible categories.

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- (Rosický, 1997) Accessible categories with directed colimits, considers exceedingly model-theoretic notions.

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. To be precise:

Definition

An object N in a category $\mathbf{C}$ is λ-presentable if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves λ-directed colimits.

Definition

A category $\mathbf{C}$ is λ-accessible if

- it has at most a set of λ-presentables
- it is closed under λ-directed colimits
- every object is a λ-directed colimit of λ-presentables

The Downward Löwenheim-Skolem Property ensures that models in an AEC are generated as directed unions of their submodels of size LS($\mathcal{K}$). As it happens,

Theorem

As a category, an AEC $\mathcal{K}$ is LS($\mathcal{K}$)$^+$-accessible (actually, μ-accessible for all regular $\mu \geq$ LS($\mathcal{K}$)$^+$), and the μ-presentable objects are precisely the models of size less than μ. Moreover, $\mathcal{K}$ is closed under directed colimits.

Rosický considers categories of this form, in which context he defines a number of category-theoretic analogues of notions from model theory. Most notably: weak κ-stability.

Weak κ-stability is precisely what we need to guarantee the existence of saturated extensions of size κ. So:

Theorem

If $\mathcal{K}$ is weakly χ-tame and μ-t.t., and κ is such that cf(κ) $> \mu$, if $\mathcal{K}$ is weakly κ-stable, then $\mathcal{K}$ is κ-stable.

Curiously, weak stability occurs in any category of the form under consideration.

For example,

Proposition

If $LS(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-t.t. and weakly $\aleph_0$-tame, then for any $\mu > |\mathcal{K}_{\aleph_0}^{mor}|$ with $\aleph_1 \trianglelefteq \mu$, $\mathcal{K}$ is $\mu^{<\mu}$-stable.

The notion of sharp inequality, $\trianglelefteq$, is well treated in texts on accessible categories. Suffice to say, there are many (and arbitrarily large) cardinals μ with $\aleph_1 \trianglelefteq \mu$.

For example, if $\mathcal{K}$ is $\aleph_0$-categorical, $|\mathcal{K}_{\aleph_0}^{\mathrm{mor}}| \leq 2^{\aleph_0}$. Then $\mathcal{K}$ is stable in

$$[(2^{\aleph_k})^{+(n+1)}]^{(2^{\aleph_k})^{+n}} \text{ for } n < \omega \text{ and } 1 \leq k < \omega,$$

and so on.

By an easy exercise in category theory,

Theorem

If $\mathcal{K}$ is λ-categorical, the sub-AEC $\mathcal{K}_{\geq\lambda}$ consisting of models of size at least λ is equivalent to a category of sets with actions of M, the monoid of endomorphisms of the unique structure of size λ, say C.

The assignment is:

$$N \in \mathcal{K}_{\geq\lambda} \quad \mapsto \quad \mathrm{Hom}_{\mathcal{K}}(C, N)$$

where $M = \mathrm{Hom}_{\mathcal{K}}(C, C)$ acts by precomposition.

This amounts to an astonishing transformation of a very abstract entity—an AEC—into a category of relatively simple algebraic objects.