Categorial Abstract Model Theory

Michael Lieberman
Kalamazoo College

Louise Hay Logic Seminar

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- A generalization of Boney's theorem on tameness under large cardinals.
- A robust version of the Ehrenfeucht-Mostowski construction.

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- A robust version of the Ehrenfeucht-Mostowski construction.

In each case, the coherence axiom for AECs is seen to be dispensable.

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- (Lawvere/Tierney; Makkai/Reyes, 1977) Functorial semantics for general first order theories—theories as topoi, models as structure preserving functors.
- (Makkai/Paré, 1989) Theories set aside, instead consider categories that have essential properties of categories of models—accessible categories.

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With coherence, these are nearly AECs...

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A strong embedding $f : M \hookrightarrow_{\mathcal{K}} N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

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Note

Here we describe $\mathcal{K}$ in terms of properties of the inclusion functor $\mathcal{K} \to \mathbf{Str}(L(\mathcal{K}))$.

Terminological Note:

“Direct limit” and “directed colimit” are essentially interchangeable.

The latter term is preferred in the sense that it identifies the construction as a colimit, but both are built from a system of maps indexed by a directed poset, and share the same universal diagram.

In any category of structured sets, then, the directed colimit can be identified with the familiar quotient of the disjoint union or, in the case of an increasing chain, with the union itself.

There is another reason to prefer "directed colimit:" it admits an important generalization:

Definition

For λ regular, a poset I is λ-directed if any subset $J \subseteq I$, $|J| <$ lambda, has an upper bound in I. A colimit is said to be λ-directed if the indexing poset is λ-directed.

This is an important distinction: **Ban** is not closed under direct limits, but is closed under ω_1-directed colimits. This prevents it from being an AEC, but not an accessible category...

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. Basic terminology:

Definition

An object N in a category $\mathbf{C}$ is finitely presentable (ω-presentable) if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves directed colimits.

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Definition

A category $\mathbf{C}$ is finitely accessible (ω-accessible) if

- it has at most a set of finitely presentable objects,
- it is closed under directed colimits, and
- every object is a directed colimit of finitely presentable objects.

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Example: **Ban** lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.

This gives a notion of size:

Definition

For any object M in an accessible category $\mathcal{K}$, we define its **presentability rank**, *$\pi(M)$ to be the least cardinal λ such that M is λ-presentable.*

Fact

In well-behaved accessible categories, $\pi(M)$ is always a successor, say $\pi(M) = \lambda^+$. In this case we say λ is the **size** *of M.*

Theorem

If $\mathcal{K}$ is an AEC, $M \in \mathcal{K}$ is of size λ iff $|M| = \lambda$. By DLS, it follows that any AEC $\mathcal{K}$ is LS($\mathcal{K}$)$^+$-accessible.

In a general accessible category, objects are not structured sets. To define Galois types, though, we do need to introduce sets and elements into the picture.

We do this via a functor $U : \mathcal{K} \to \mathbf{Sets}$, which assigns

- to each $M \in \mathcal{K}$ a set $U(M)$, and
- to each $\mathcal{K}$-map $f : M \to N$ a set map $U(f) : U(M) \to U(N)$

To ensure good behavior, we insist that this functor

- Is faithful: If $f \neq g$ in $\mathcal{K}$, then $U(f) \neq U(g)$.
- Preserves directed colimits: the image of any colimit in $\mathcal{K}$ is the colimit of the corresponding diagram of sets.

You would not lose much in thinking of this U as a *forgetful functor* (or *underlying object functor*), in the usual sense. There are peculiarities, however.

Note

The size of an object M in $\mathcal{K}$ need not correspond to $|U(M)|$. In principle, they could disagree for arbitrarily large M.

This poses little problem for the theory, but one might ask how it can be avoided.

Fact

If U reflects split epimorphisms, it preserves sufficiently large sizes.

We can achieve the same through a stronger, but more familiar condition on U:

Definition

We say $U : \mathcal{K} \to \mathbf{Sets}$ *is* **coherent** *if, given any set map* $f : U(M) \to U(N)$ *and* $\mathcal{K}$*-map* $g : N \to N'$*, if* $U(g) \circ f = Uh$ *for some* $h : M \to N'$*, then there is* $\bar{f} : M \to N$ *with* $U(\bar{f}) = f$*.*

Definition

We say that an accessible category with concrete directed colimits, $(\mathcal{K}, U)$*, is coherent if* U *is coherent.*

Given an AEC (or accessible category of structures), this is equivalent to the usual notion. Notice, though, that we need not refer to an ambient category of structures, or signature.

In AECs, the following looms large:

Theorem (Shelah's Presentation Theorem)

For any AEC $\mathcal{K}$ in signature L, there is a signature $L' \supseteq L$, a first order theory T' in L', and a set of T'-types Γ such that

$$\mathcal{K} = \{M \upharpoonright L \,|\, M \models T', M \text{ omits } \Gamma\}$$

There are several things to note:

- This result is essential for the computation of Hanf numbers, used in the construction of the EM-functor for AECs.
- The proof makes essential use of coherence.
- The expansion L' and set Γ are ad hoc, unrelated to the structure of the AEC itself.

In fact, any accessible category of structures (roughly speaking) admits a syntactic presentation:

Theorem (Makkai/Pare, 4.5.1)

If $\mathcal{K}$ is a κ-accessible category of structures in signature L, it is equivalent to the category of models of a basic (infinitary) sentence σ.

asic ingredients: A structure A in in $\mathcal{K}$ if and only if each map $f : C \to A$ with C κ-presentable in factors through a κ-presentable object D in $\mathcal{K}$. Morphisms out of a structure can be coded via atomic diagrams...

The result is an enormous sentence, but one in the natural signature of $\mathcal{K}$ and which captures the way objects are assembled from smaller ones.

We do not insist that our accessible categories consist of structured sets, but even our weak underlying object functor U is enough to give an analogue:

Lemma

For any accessible category with concrete directed colimits, $(\mathcal{K}, U)$, there is a canonical signature $\Sigma_{\mathcal{K}}$ such that $\mathcal{K}$ is equivalent to a full subcategory of $\mathbf{Str}(\Sigma_{\mathcal{K}})$.

This signature contains finitary relation symbols corresponding to well-behaved subfunctors of U^n, and function symbols corresponding to natural transformations $U^n \to U$. In this way, any category of interest to us has a representation of the form just described.

This is provided largely as a matter of interest...

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits. A Galois type is an equivalence class of pairs (f, a), where $f : M \to N$ and $a \in U(N)$.
Pairs (f_0, a_0) and (f_1, a_1) are equivalent if there are morphisms $h_0 : N_0 \to N$ and $h_1 : N_1 \to N$ such that $h_0 f_0 = h_1 f_1$ and $U(h_0)(a_0) = U(h_1)(a_1)$.

If $\mathcal{K}$ has the amalgamation property (which, of course, is purely diagrammatic), this is an equivalence relation.

No surprises there: this is a straightforward generalization of the definition for AECs.

In an AEC, Galois types are said to be tame if they are determined by restriction to small submodels of their domains. The situation here is the same:

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits and κ regular. We say that $\mathcal{K}$ is κ-tame if for two non-equivalent types (f, a) and (g, b) there is a morphism $h : X \to M$ with X κ-presentable such that the types (fh, a) and (gh, b) are not equivalent.

$\mathcal{K}$ is called tame if it is κ-tame for some regular cardinal κ.

Will Boney showed recently that under the assumption that there is a proper class of strongly compact cardinals—henceforth (C)—every AEC is tame.

Theorem

Assuming (C), any accessible category with directed colimits is tame.

Proof (Idea): Consider the following categories of configurations:

- $\mathcal{L}_2 : (f_0, f_1, a_0, a_1)$, with $f_i : M \to N_i$, $a_i \in U(N_i)$.
- $\mathcal{L}_1 : (f_0, f_1, a_0, a_1, h_0, h_1)$, with the h_i witnessing equivalence.

Let $G : \mathcal{L}_1 \to \mathcal{L}_2$ be the forgetful functor. Both categories are accessible, as is G. It is a matter of some subtlety to prove (following Makkai/Pare, 5.5.1) that, assuming (C), the full image of G in $\mathcal{L}_2$ is κ-accessible for a compact cardinal κ. Proving closure under κ-directed colimits involves a delicate compactness argument—$L_{\kappa,\kappa}$ must be large enough to capture the relevant atomic diagrams, among other things.

If you believe that $G(\mathcal{L}_1)$, the subcategory consisting of equivalent pairs, is κ-accessible, the rest is easy:

Consider (f_0, f_1, a_0, a_1), where $(f_0 u, a_0)$ and $(f_1 u, a_1)$ are equivalent for all $u : X \to M$, X κ-presentable. Then $(f_0 u, f_1 u, a_0, a_1)$ belongs to $G(\mathcal{L}_1)$ for all such u, and since (f_0, f_1, a_0, a_1) is their κ-directed colimit, it belongs to $G(\mathcal{L}_1)$ as well. That is, (f_0, a_0) and (f_1, a_1) are equivalent.

Thus $\mathcal{K}$ is κ-tame. $\square$

Notice that this proof made no use of coherence—the apparent connection between tameness and large cardinals goes well beyond AECs.

As a last illustration, we consider the appearance of EM-models in accessible categories with concrete directed colimits.

Theorem

Any large accessible category with directed colimits $\mathcal{K}$ whose morphisms are monomorphisms admits a faithful functor $E : \mathbf{Lin} \to \mathcal{K}$, where **Lin** *is the category of linear orders and order embeddings.*

Specifically, E assigns to each linear order I an object $E(I)$ in $\mathcal{K}$, and to each order embedding $\sigma : I \to J$ a $\mathcal{K}$-map $E(\sigma) : E(I) \to E(J)$. This looks, formally, like the EM-functors from more familiar contexts—but does it retain the same useful properties?

Proposition

E : $\mathbf{Lin} \to \mathcal{K}$ preserves sufficiently large sizes.

That is, for sufficiently large I, the size of $E(I)$ in $\mathcal{K}$ will be precisely $|I|$. Moreover,

Theorem

If $(\mathcal{K}, U)$ is coherent (weaker: U preserves split epimorphisms), $|E(I)| = |I|$ for sufficiently large I.

In the paper that introduced the use of EM-models in AECs, Baldwin gave an argument for Galois stability below a categoricity cardinal. It can be replicated in this context...

Theorem

Let $(\mathcal{K}, U)$ be coherent (weaker: U preserves split epimorphisms), with amalgamation and joint embedding. If $\mathcal{K}$ is λ-categorical, then it is Galois-stable in all sufficiently large $\mu < \lambda$.

Definition

We say that an object M is brimful if for any substructure N_0 of smaller size, there is an intermediate subobject $\bar{N}$ that is appropriately universal over N_0.

Lemma

Let a $(\mathcal{K}, U)$ be coherent (weaker: U preserves split epimorphisms). If I is brimful, $E(I)$ is brimful.

In the original argument, coherence is invoked directly—here it is used only to ensure that sizes mean what we would expect.

Proof (Theorem): As in Baldwin. One first shows the unique object M of size λ is μ-stable: $\lambda^{<\omega}$ is brimful, so $E(I)$ is brimful and of size λ, hence M itself is brimful. It is easy to see that, as such, it can realize at most μ types over any subobject of size μ.

It follows that the category is μ-stable: by amalgamation and joint embedding, any object of size μ realizing too many types embeds in M, producing a subobject of M realizing too many types—this is impossible. $\square$

This is a good sign. One might ask, though, where coherence is absolutely essential.

In trying to replicate arguments that involve building maps element-by-element, we run into situations where coherence is absolutely essential—else there is no reason to think the set map constructed actually arises from a map in the category. In particular, we can't prove

Theorem

An object M is Galois-saturated iff it is model-homogeneous.

This, of course, is essential in the proof of the uniqueness of saturated models, and in the transfer of categoricity...

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