

# Bootstrapping structural properties, via accessible images (Joint with Jiří Rosický)

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The beauty of this is that, with the exception of some gruesome details we suppress, everything is clean and clear.

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### Definition

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- ▶ it has at most a set of  $\lambda$ -presentable objects.
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### Note

*A general  $\lambda$ -accessible category need not be closed under arbitrary directed colimits. . .*



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### Note

*Presentability gives a notion of size. If  $\mathcal{K}$  has directed colimits the presentability rank of any object  $M$  is a successor, say  $\pi(M) = \lambda^+$ , and we define the internal size of  $M$  in  $\mathcal{K}$  by  $|M|_{\mathcal{K}} = \lambda$ .*

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*$\mathcal{K}$  an AEC:  $|M|_{\mathcal{K}} = |M|$ .  $\mathcal{K}$  an mAEC:  $|M|_{\mathcal{K}} = dc(M)$ .*

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A functor  $F : \mathcal{K} \rightarrow \mathcal{L}$  is  $\lambda$ -accessible if  $\mathcal{K}$  and  $\mathcal{L}$  are  $\lambda$ -accessible, and  $F$  preserves  $\lambda$ -directed colimits.

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“Accessible” means “ $\lambda$ -accessible for some  $\lambda$ .”

Raising the index of accessibility can be complicated:

## Definition

Given regular cardinals  $\mu \geq \lambda$ , we say that  $\mu$  is sharply larger than  $\lambda$ , denoted  $\mu \triangleright \lambda$ , if the following equivalent conditions hold:

- ▶ Every  $\lambda$ -accessible category is  $\mu$ -accessible.
- ▶ For each set  $X$ ,  $|X| < \mu$ , the poset of subsets  $P_\lambda(X)$  has a cofinal set of cardinality less than  $\mu$ .

## Notes

- ▶ For any regular  $\lambda$ ,  $\lambda \triangleright \omega$ .
- ▶ If  $\mu > 2^\lambda$ , then  $\mu^+ \triangleright \lambda^+$  iff  $\mu^\lambda = \mu$ .

Any  $\lambda$ -accessible category with arbitrary directed colimits is well  $\lambda$ -accessible:  $\mu$ -accessible for all  $\mu > \lambda$ .

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Theorem (Eklof/Mekler, '77)

*Assume  $V=L$ . For every successor  $\kappa$ , there is a nonfree abelian group  $A$  of size  $\kappa$ , all of whose subgroups of size less than  $\kappa$  are free.*



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Corollary

*Assuming  $V = L$ ,  $\mathcal{F}$  is not accessible.*

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- ▶ *The free abelian group functor  $F : \mathbf{Sets} \rightarrow \mathbf{Ab}$  is accessible, and  $\mathcal{F}$  is its image.*

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- ▶  *$\mathcal{F}$  is closed under subobjects, hence the powerful image of  $F$ .*

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A cardinal  $\kappa$  is *strongly compact* if any  $\kappa$ -complete filter can be extended to a  $\kappa$ -complete ultrafilter. We say  $\kappa$  is  *$\mu$ -strongly compact* if the ultrafilter need only be  $\mu$ -complete.

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*Let  $\mathcal{L}$  be  $\lambda$ -accessible, such that there exists a  $\mu_{\mathcal{L}}$ -strongly compact cardinal  $\kappa$ ,  $\kappa \geq \lambda$ . The powerful image of any  $\lambda$ -accessible functor to  $\mathcal{L}$  that preserves  $\mu_{\mathcal{L}}$ -presentable objects is  $\kappa$ -accessible.*



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### Note

*If  $\mathcal{L}$  is well  $\lambda$ -accessible, we can remove the  $\trianglerighteq$  condition.*

Let  $\mathbf{Pres}_\lambda(\mathcal{L})$  denote a skeleton of the full subcategory of  $\lambda$ -presentable objects in  $\mathcal{L}$ ; let  $\beta = |\mathbf{Pres}_\lambda(\mathcal{L})|$ . Take  $\gamma_{\mathcal{L}}$  the smallest cardinal such that  $\gamma_{\mathcal{L}} \geq \beta$  and  $\gamma_{\mathcal{L}} \triangleright \lambda$ .

### Note

$\gamma_{\mathcal{L}}$  will not be too large: if  $\lambda < \beta$ ,

$$\gamma_{\mathcal{L}} < (2^\beta)^+,$$

and clearly  $\gamma_{\mathcal{L}} = \lambda$  otherwise.

In case  $\mathcal{K}$  is well  $\lambda$ -accessible, we may simply take  $\gamma_{\mathcal{L}} = \max(\beta, \lambda)$ , noting the parallel between this cardinal and the corresponding bound for AECs, i.e.  $I(\mathcal{K}, \lambda) + \lambda$ .

Finally, we define  $\mu_{\mathcal{L}} = (\gamma_{\mathcal{L}}^{<\gamma_{\mathcal{L}}})^+$ . By design,  $\mu_{\mathcal{L}} \triangleright \lambda$ .

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For our purposes,  $\mathcal{A}$  and  $\mathcal{A}'$  will be finite categories, and we can identify the categories of diagrams with  $\mathcal{K}^{\mathcal{A}}$  and  $\mathcal{K}^{\mathcal{A}'}$ . The forgetful functor

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If the (powerful) image of  $F$  is accessible, completability of an  $\mathcal{A}$ -diagram is determined by its small sub- $\mathcal{A}$ -diagrams, hence we can bootstrap full completability from completability in the small...

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In terms of diagrams:



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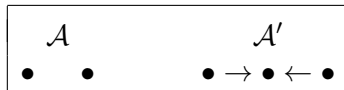
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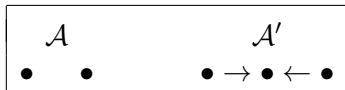
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Let  $F_J : \mathcal{K}^{\mathcal{A}'} \rightarrow \mathcal{K}^{\mathcal{A}}$  be the forgetful functor that retains only the outermost objects.



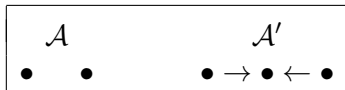




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## Notes

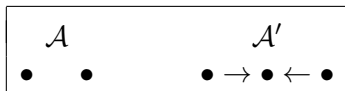
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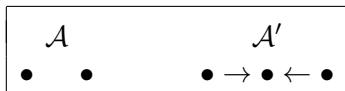
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3. The image of  $F_J$  is closed under subobjects, hence powerful.
4. As colimits are computed componentwise in  $\mathcal{K}^{\mathcal{A}}$ ,  $\mathcal{K}^{\mathcal{A}'}$ ,  $F_J$  preserves everything. Hence  $F_J$  is as accessible as  $\mathcal{K}^{\mathcal{A}}$  and  $\mathcal{K}^{\mathcal{A}'}$  are.

## Proposition

*If  $\mathcal{K}$  is  $\lambda$ -accessible, so are  $\mathcal{K}^{\mathcal{A}}$  and  $\mathcal{K}^{\mathcal{A}'}$ . In either case, the  $\lambda$ -presentables are precisely the diagrams in which all objects are  $\lambda$ -presentable. If  $\mathcal{K}$  is well  $\lambda$ -accessible, so are  $\mathcal{K}^{\mathcal{A}}$  and  $\mathcal{K}^{\mathcal{A}'}$ .*

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So  $F_J$  satisfies the conditions of B-T/R, and thus we are ready to prove the main theorem.

## Theorem

*Let  $\mathcal{K}$  be  $\lambda$ -accessible. If  $\kappa$  is a  $\mu_{\mathcal{K}}$ -strongly compact cardinal and  $\kappa \geq \lambda$ , then if  $\mathcal{K}$  has the  $< \kappa$ -JEP, it has the JEP.*

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Thus every pair of objects in  $\mathcal{K}$  is jointly embeddable. □

In case  $\mathcal{K}$  is well accessible, we can dispense with the sharp inequality:

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Baldwin/Boney give a syntax-heavy argument for an analogue of this result for AECs.

The theorem above is a two-fold improvement, encompassing metric AECs and generalizations, and offering a tighter characterization of the compactness required of  $\kappa$ .

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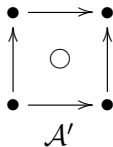
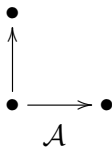
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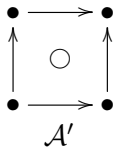
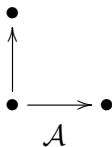
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We say  $\mathcal{K}$  has the AP if the above holds for all  $\kappa$ .

Diagrammatically, again:



Let  $F_A : \mathcal{K}^{\mathcal{A}'} \rightarrow \mathcal{K}^{\mathcal{A}}$  be the obvious forgetful functor.

## Notes

1.  $\mathcal{K}$  has the  $< \kappa$ -AP just in case  $\mathbf{Pres}_\kappa(\mathcal{K})^A$  is contained in the image of  $F_A$ .
2.  $\mathcal{K}$  has the AP just in case  $F_A$  is surjective.
3. The image of  $F_A$  is closed under subobjects, hence powerful.
4. As colimits are computed componentwise in  $\mathcal{K}^A$ ,  $\mathcal{K}^{A'}$ ,  $F_A$  preserves everything. Hence  $F_A$  is as accessible as  $\mathcal{K}^A$  and  $\mathcal{K}^{A'}$  are.

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## Proposition

If  $\mathcal{K}$  is  $\lambda$ -accessible, so are  $\mathcal{K}^A$  and  $\mathcal{K}^{A'}$ . In either case, the  $\lambda$ -presentables are precisely the diagrams in which all objects are  $\lambda$ -presentable. If  $\mathcal{K}$  is well  $\lambda$ -accessible, so are  $\mathcal{K}^A$  and  $\mathcal{K}^{A'}$ .

In fact, the argument runs exactly as before, giving:

### Theorem

*Let  $\mathcal{K}$  be  $\lambda$ -accessible. If  $\kappa$  is a  $\mu_{\mathcal{K}}$ -strongly compact cardinal and  $\kappa \geq \lambda$ , then if  $\mathcal{K}$  has the  $< \kappa$ -AP, it has the AP.*

### Theorem

*Let  $\mathcal{K}$  be well  $\lambda$ -accessible. If  $\kappa$  is a  $\mu_{\mathcal{K}}$ -strongly compact cardinal, then if  $\mathcal{K}$  has the  $< \kappa$ -AP, it has the AP.*

This again generalizes the analogue for AECs in Baldwin/Boney.

## Other applications:

In an AEC  $\mathcal{K}$ ,  $U : \mathcal{K} \rightarrow \mathbf{Sets}$ , pairs  $(f_i : M \rightarrow N_i, a_i \in U(N_i))$  for  $i = 0, 1$  determine the same *Galois type* if there is an amalgam  $g_i : N_i \rightarrow N$  of the form:

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$$\begin{array}{ccccc}
 a_0 & \xrightarrow{\quad} & U(g_0)(a_0) & & \\
 UN_0 & \xrightarrow{U(g_0)} & UN & = & U(g_1)(a_1) \\
 N_0 & \overset{g_0}{\dashrightarrow} & N & & \\
 \uparrow f_0 & \circlearrowleft & \uparrow g_1 & & \uparrow \\
 M & \xrightarrow{f_1} & N_1 & & a_1 \\
 & & \uparrow U(g_1) & & \uparrow
 \end{array}$$

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 \end{array}$$

$\kappa$ -tameness follows from  $\kappa$ -accessibility (in sufficiently compact  $\kappa$ ) of the image of the forgetful functor to category of “pointed spans.”

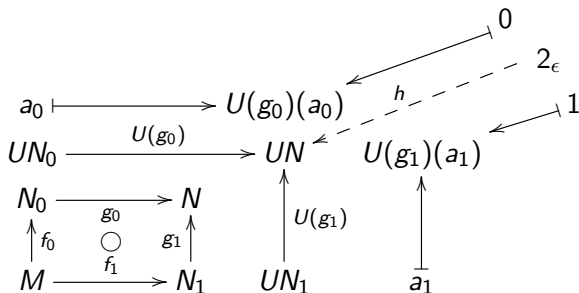


## Other applications:

In an mAEC  $\mathcal{K}$ ,  $U : \mathcal{K} \rightarrow \mathbf{Met}$ , the Galois types of pairs  $(f_i : M \rightarrow N_i, a_i \in U(N_i))$ ,  $i = 0, 1$  are *within*  $\epsilon$  if there is an amalgam  $g_j : N_j \rightarrow N$  with:

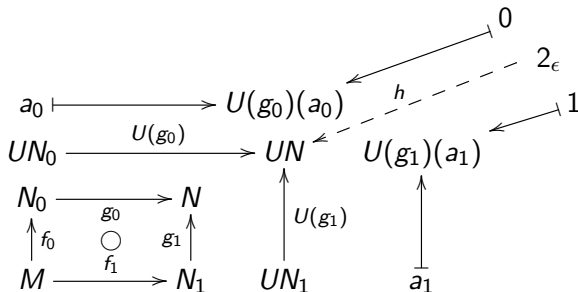
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$\kappa$ - $d$ -tameness: determined (for all  $\epsilon > 0$ ) over  $\kappa$ -sized structures...