Bootstrapping structural properties, via accessible images

(Joint with Jiří Rosický)

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The beauty of this is that, with the exception of some gruesome details we suppress, everything is clean and clear.

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A general λ-accessible category need not be closed under arbitrary directed colimits...

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Presentability gives a notion of size. If $\mathcal{K}$ has directed colimits the presentability rank of any object M is a successor, say $\pi(M) = \lambda^+$, and we define the internal size of M in $\mathcal{K}$ *by $|M|_{\mathcal{K}} = \lambda$.*

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$\mathcal{K}$ an AEC: $|M|_{\mathcal{K}} = |M|$. $\mathcal{K}$ an mAEC: $|M|_{\mathcal{K}} = dc(M)$.

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A functor $F : \mathcal{K} \to \mathcal{L}$ is λ-*accessible* if $\mathcal{K}$ and $\mathcal{L}$ are λ-accessible, and F preserves λ-directed colimits.

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"Accessible" means "λ-accessible for some λ."

Raising the index of accessibility can be complicated:

Definition

Given regular cardinals $\mu \geq \lambda$, we say that μ is sharply larger than λ, denoted $\mu \trianglerighteq \lambda$, if the following equivalent conditions hold:

- *Every λ-accessible category is μ-accessible.*
- *For each set X, $|X| < \mu$, the poset of subsets $P_\lambda(X)$ has a cofinal set of cardinality less than μ.*

Notes

- *For any regular λ, $\lambda \trianglerighteq \omega$.*
- *If $\mu > 2^\lambda$, then $\mu^+ \trianglerighteq \lambda^+$ iff $\mu^\lambda = \mu$.*

Any λ-accessible category with arbitrary directed colimits is *well λ-accessible*: μ-accessible for all $\mu > \lambda$.

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Corollary

Assuming $V = L$, $\mathcal{F}$ is not accessible.

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- *The free abelian group functor* $F : \mathbf{Sets} \to \mathbf{Ab}$ *is accessible, and* $\mathcal{F}$ *is its image.*

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- *The free abelian group functor* $F : \textbf{Sets} \to \textbf{Ab}$ *is accessible, and $\mathcal{F}$ is its image.*
- *$\mathcal{F}$ is closed under subobjects, hence the* powerful image *of F.*

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Theorem (Brooke-Taylor/Rosický)

Let $\mathcal{L}$ be λ-accessible, such that there exists a $\mu_{\mathcal{L}}$-strongly compact cardinal κ, $\kappa \trianglerighteq \lambda$. The powerful image of any λ-accessible functor to $\mathcal{L}$ that preserves $\mu_{\mathcal{L}}$-presentable objects is κ-accessible.

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Note

If $\mathcal{L}$ is well λ-accessible, we can remove the $\trianglerighteq$ condition.

Let $\mathbf{Pres}_\lambda(\mathcal{L})$ denote a skeleton of the full subcategory of λ-presentable objects in $\mathcal{L}$; let $\beta = |\mathbf{Pres}_\lambda(\mathcal{L})|$. Take $\gamma_\mathcal{L}$ the smallest cardinal such that $\gamma_\mathcal{L} \geq \beta$ and $\gamma_\mathcal{L} \trianglerighteq \lambda$.

Note

$\gamma_\mathcal{L}$ will not be too large: if $\lambda < \beta$,

$$\gamma_\mathcal{L} < (2^\beta)^+,$$

and clearly $\gamma_\mathcal{L} = \lambda$ otherwise.

In case $\mathcal{K}$ is well λ-accessible, we may simply take $\gamma_\mathcal{L} = \max(\beta, \lambda)$, noting the parallel between this cardinal and the corresponding bound for AECs, i.e. $I(\mathcal{K}, \lambda) + \lambda$.

Finally, we define $\mu_\mathcal{L} = (\gamma_\mathcal{L}^{<\gamma_\mathcal{L}})^+$. By design, $\mu_\mathcal{L} \trianglerighteq \lambda$.

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For our purposes, $\mathcal{A}$ and $\mathcal{A}'$ will be finite categories, and we can identify the categories of diagrams with $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$. The forgetful functor

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If the (powerful) image of F is accessible, completability of an $\mathcal{A}$-diagram is determined by its small sub-$\mathcal{A}$-diagrams, hence we can bootstrap full completability from completability in the small...

Definition

We say that an accessible category $\mathcal{K}$ has the $< \kappa$-*JEP* if for any κ-presentable $M_0, M_1 \in \mathcal{K}$, there are $f_i : M_i \to N$ for $i = 1, 2$. We say that $\mathcal{K}$ has the *JEP* if this holds for arbitrary $M_0, M_1 \in \mathcal{K}$.

Definition

We say that an accessible category $\mathcal{K}$ has the $<\kappa$-*JEP* if for any κ-presentable $M_0, M_1 \in \mathcal{K}$, there are $f_i : M_i \to N$ for $i = 1, 2$. We say that $\mathcal{K}$ has the *JEP* if this holds for arbitrary $M_0, M_1 \in \mathcal{K}$.

In terms of diagrams:

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Let $F_J : \mathcal{K}^{\mathcal{A}'} \to \mathcal{K}^{\mathcal{A}}$ be the forgetful functor that retains only the outermost objects.

$\mathcal{A}$	$\mathcal{A}'$
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1. *$\mathcal{K}$ has the $< \kappa$-JEP just in case* $\mathbf{Pres}_\kappa(\mathcal{K})^{\mathcal{A}}$ *is contained in the image of F_J.*

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4. *As colimits are computed componentwise in $\mathcal{K}^{\mathcal{A}}$, $\mathcal{K}^{\mathcal{A}'}$, F_J preserves everything. Hence F_J is as accessible as $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$ are.*

Proposition

If $\mathcal{K}$ is λ-accessible, so are $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$. In either case, the λ-presentables are precisely the diagrams in which all objects are λ-presentable. If $\mathcal{K}$ is well λ-accessible, so are $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$.

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Hence if $\mathcal{K}$ is λ-accessible, the functor F_J, which we know preserves colimits, has λ-accessible domain and codomain; that is, it is λ-accessible. Moreover:

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So F_J satisfies the conditions of B-T/R, and thus we are ready to prove the main theorem.

Theorem

Let $\mathcal{K}$ be λ-accessible. If κ is a $\mu_{\mathcal{K}}$-strongly compact cardinal and $\kappa \trianglerighteq \lambda$, then if $\mathcal{K}$ has the $<\kappa$-JEP, it has the JEP.

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Proof.

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By B-T/R, the powerful image of F_J is κ-accessible—as noted, the image itself is powerful. $\square$

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Consider a pair $(M_0, M_1) \in \mathcal{K}$. Since $\mathcal{K}^{\mathcal{A}}$ is λ-accessible, it is also κ-accessible, meaning that (M_0, M_1) is a κ-directed colimit of pairs of κ-presentables. If $\mathcal{K}$ has the $< \kappa$-JEP, all pairs of κ-presentables are in the image of F_J. As the image of F_J is κ-accessible, it is closed under κ-directed colimits. That is, (M_0, M_1) is in the image of F_J. $\square$

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Thus every pair of objects in $\mathcal{K}$ is jointly embeddable. □

In case $\mathcal{K}$ is well accessible, we can dispense with the sharp inequality:

Theorem

Let $\mathcal{K}$ be well λ-accessible. If κ is a $\mu_{\mathcal{K}}$-strongly compact cardinal, then if $\mathcal{K}$ has the $< \kappa$-JEP, it has the JEP.

Baldwin/Boney give a syntax-heavy argument for an analogue of this result for AECs.

The theorem above is a two-fold improvement, encompassing metric AECs and generalizations, and offering a tighter characterization of the compactness required of κ.

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In fact, the argument runs exactly as before, giving:

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Theorem

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This again generalizes the analogue for AECs in Baldwin/Boney.

Other applications:

In an AEC $\mathcal{K}$, $U : \mathcal{K} \to$ **Sets**, pairs $(f_i : M \to N_i, a_i \in U(N_i))$ for $i = 0, 1$ determine the same *Galois type* if there is an amalgam $g_i : N_i \to N$ of the form:

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$$\begin{tikzcd} a_0 \arrow[rr, mapsto] & & U(g_0)(a_0) \arrow[dr, phantom, "="] & \\ UN_0 \arrow[rr, "U(g_0)"] & & UN & U(g_1)(a_1) \\ N_0 \arrow[r, dashed, "g_0"'] & N & & \\ M \arrow[u, "f_0"] \arrow[r, "f_1"] \arrow[ur, phantom, "\bigcirc"] & N_1 \arrow[u, "g_1"'] & UN_1 \arrow[uu, "U(g_1)"'] & a_1 \arrow[uu, mapsto] \end{tikzcd}$$

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κ*-tameness* follows from κ-accessibility (in sufficiently compact κ) of the image of the forgetful functor to category of "pointed spans."

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In an mAEC $\mathcal{K}$, $U : \mathcal{K} \to \mathbf{Met}$, the Galois types of pairs $(f_i : M \to N_i, a_i \in U(N_i))$, $i = 0, 1$ are *within* ϵ if there is an amalgam $g_i : N_i \to N$ with:

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$$\begin{tikzcd}
N_0 \arrow[r, "g_0"'] & N \\
M \arrow[u, "f_0"'] \arrow[r, "f_1"] \arrow[ur, phantom, "\bigcirc"] & N_1 \arrow[u, "g_1"]
\end{tikzcd}
\qquad
\begin{tikzcd}
 & & & 0 \arrow[ddll, mapsto] \\
 & & & 2_\epsilon \arrow[ddll, dashed, "h"'] \\
a_0 \arrow[r, mapsto] & U(g_0)(a_0) & & 1 \arrow[dl, mapsto] \\
UN_0 \arrow[r, "U(g_0)"] & UN & U(g_1)(a_1) & \\
 & UN_1 \arrow[u, "U(g_1)"'] & a_1 \arrow[u, mapsto] &
\end{tikzcd}$$

Other applications:

In an mAEC $\mathcal{K}$, $U : \mathcal{K} \to \mathbf{Met}$, the Galois types of pairs $(f_i : M \to N_i, a_i \in U(N_i))$, $i = 0, 1$ are *within* ϵ if there is an amalgam $g_i : N_i \to N$ with:

$$
\begin{tikzcd}
 & & & & 0 \arrow[dll, mapsto] \\
 & & & & 2_\epsilon \arrow[ddll, dashed, "h"'] \\
a_0 \arrow[rr, mapsto] & & U(g_0)(a_0) & & 1 \arrow[dl, mapsto] \\
UN_0 \arrow[rr, "U(g_0)"] & & UN & U(g_1)(a_1) & \\
N_0 \arrow[r, "g_0"'] \arrow[ddr, phantom, "\bigcirc"] & N & & & \\
 & & & & \\
M \arrow[uu, "f_0"] \arrow[r, "f_1"] & N_1 \arrow[uu, "g_1"'] & UN_1 \arrow[uuu, "U(g_1)"'] & a_1 \arrow[uuu, mapsto] &
\end{tikzcd}
$$

κ-d-tameness: determined (for all $\epsilon > 0$) over κ-sized structures...