Foundations of categorical model theory

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The fundamental question: given a class of objects axiomatizable in first order logic, how many are there (up to isomorphism) of each cardinality? In which cardinalities is it *categorical*, i.e. has just one isomorphism class?

Note: here we consider only infinite cardinalities (else *finite model theory*. . .)

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Route 1: (Baldwin/Lachlan) If T is categorical in an uncountable λ, there is a uniform geometric structure on the uncountable models, and a uniform way of picking out a basis of each...

This led to the development of geometric stability theory...

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Route 2: Calculus of types.

A syntactic type in T is a set of formulas completely describing a potential element of models of T.

Best case scenario: minimum number of types—stability.

Worst case scenario: lots of types, intractable theory.

Morley's proof hinges on delicate balance between realizing and omitting types. Crucial ideas:

- Saturated models: realize all relevant types—"big models."
- Ehrenfeucht-Mostowski models: for linear order I, $EM(I)$ is a "lean" model built along I. This is functorial: if $I \subseteq J$, $EM(I)$ embeds in $EM(J)$. Actual construction and use wildly syntactic.

Many other important ideas, particularly when you generalize away from elementary classes.

Template: Let T be a complete first order theory in language L.

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Need to be more flexible with strong embeddings: purity, prescribed quotients...

A very popular way of getting this flexibility is moving to a more general logic. Possibilities:

- $L_{\kappa\lambda}$: conjunctions, disjunctions of fewer than κ formulas, quantification over fewer than λ variables. Special case: $L_{\omega_1\omega}$.
- $L(Q)$: add counting quantifier Qx, i.e. "there exist uncountably many."

Many of the properties that make classical model theory work disappear. Compactness, especially, only holds for $L_{\kappa\kappa}$ with κ a compact cardinal.

Generalize by forgetting logic, retaining essential properties of classes of models...

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a strong substructure relation, $\prec_{\mathcal{K}}$, that satisfies:

- $\prec_{\mathcal{K}}$ is a partial order.
- Unions of chains: if $(M_i \mid i < \delta)$ is a $\prec_{\mathcal{K}}$-increasing chain,
  1. $\bigcup_{i<\delta} M_i \in \mathcal{K}$
  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

A strong embedding $f : M \hookrightarrow_{\mathcal{K}} N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

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Note

Here we describe $\mathcal{K}$ in terms of properties of the inclusion functor $\mathcal{K} \to \mathbf{Str}(L(\mathcal{K}))$.

Shelah's Categoricity Conjecture

If an AEC $\mathcal{K}$ is λ-categorical for one sufficiently large λ, it is categorical in all sufficiently large cardinals.

Only partial results are currently known. . . .

As in classical case, calculus of types—here, *Galois types*.

Definition

A Galois type over $M \in \mathcal{K}$ *is an equivalence class of triples* (M, a, N) *with* $M \prec_{\mathcal{K}} N$ *and* $a \in N$*, under the relation* $(M, a, N) \sim (M, a', N')$ *iff there are* $f : N \to N''$ *and* $f' : N' \to N''$ *so* $f \upharpoonright M = f' \upharpoonright M$ *and* $f(a) = f'(a')$*.*

Can define stability, saturation, etc. in terms of these types.

Theorem

A model M *is* λ*-Galois saturated iff it is injective w.r.t. the class of morphisms* $N \to N'$ *with* N, N' *of size less than* λ*.*

Theorem (Grossberg/VanDieren)

If $\mathcal{K}$ is χ-tame and λ^+-categorical with $\lambda \geq \chi$, it is categorical in all $\kappa > \chi$.

Note: only gives categoricity transfer from a successor. Also, needs *tameness* of Galois types.

Theorem (Boney)

If there is a proper class of strongly compact cardinals, any $\mathcal{K}$ is χ-tame for some χ.

This is genuinely sensitive to set theory...

Terminological Note:

"Direct limit" and "directed colimit" are essentially interchangeable.

The latter term will be preferred here, as it identifies the construction as a colimit, but both are built from a system of maps indexed by a directed poset, and share the same universal diagram.

In any category of structured sets, then, the directed colimit can be identified with the familiar quotient of the disjoint union or, in the case of an increasing chain, with the union itself.

There is another reason to prefer "directed colimit"—it admits an important generalization:

Definition

For λ regular, a poset I is λ-directed if any subset $J \subseteq I$, $|J| <$ lambda, has an upper bound in I. A colimit is said to be λ-directed if the indexing poset is λ-directed.

This is an important distinction: **Ban** (as a concrete category) is not closed under direct limits, but is closed under ω_1-directed colimits. This prevents it from being an AEC, but it is an accessible category...

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. Basic terminology:

Definition

An object N in a category $\mathbf{C}$ is finitely presentable (ω-presentable) if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves directed colimits.

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Definition

A category $\mathbf{C}$ is finitely accessible (ω-accessible) if

- it has at most a set of finitely presentable objects,
- it is closed under directed colimits, and
- every object is a directed colimit of finitely presentable objects.

For general regular cardinal λ:

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Example: As noted, **Ban** lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.

Definition

For any object M in an accessible category $\mathcal{K}$, we define its **presentability rank**, *$\pi(M)$, to be the least λ such that M is λ-presentable.*

Fact

In any accessible category with arbitrary directed colimits, $\pi(M)$ is always a successor, say $\pi(M) = \lambda^+$. In this case we say λ is the **size** *of M.*

Theorem

If $\mathcal{K}$ is an AEC, $M \in \mathcal{K}$ is of size λ iff $|M| = \lambda$. By DLS, it follows that any AEC $\mathcal{K}$ is LS($\mathcal{K}$)$^+$-accessible.

In a general accessible category, objects are not structured sets. To define Galois types, though, we do need to introduce sets and elements into the picture.

We do this via a functor $U : \mathcal{K} \to \mathbf{Sets}$, which assigns

- to each $M \in \mathcal{K}$ a set $U(M)$, and
- to each $\mathcal{K}$-map $f : M \to N$ a set map $U(f) : U(M) \to U(N)$

To ensure good behavior, we insist that this functor

- Is faithful: If $f \neq g$ in $\mathcal{K}$, then $U(f) \neq U(g)$.
- Preserves directed colimits: the image of any colimit in $\mathcal{K}$ is the colimit of the corresponding diagram of sets.

We say $(\mathcal{K}, U)$ is an accessible category with *concrete directed colimits*.

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits. A Galois type is an equivalence class of pairs (f, a), where $f : M \to N$ and $a \in U(N)$.
Pairs (f_0, a_0) and (f_1, a_1) are equivalent if there are morphisms $h_0 : N_0 \to N$ and $h_1 : N_1 \to N$ such that $h_0 f_0 = h_1 f_1$ and $U(h_0)(a_0) = U(h_1)(a_1)$.

If $\mathcal{K}$ has the amalgamation property (which, of course, is purely diagrammatic), this is an equivalence relation.

No surprises: this is a straightforward generalization of the definition for AECs.

By the way: you would not lose much in thinking of this U as a *forgetful functor* (or *underlying object functor*), in the usual sense. There are peculiarities, however.

Note

The size of an object M in $\mathcal{K}$ need not correspond to $|U(M)|$. In principle, they could disagree for arbitrarily large M.

This poses little problem for the theory, but one might ask how it can be avoided.

Fact

If U reflects split epimorphisms, it preserves sufficiently large sizes.

We can achieve the same through a stronger, but more familiar condition on U:

Definition

*We say $U : \mathcal{K} \to$ **Sets** is **coherent** if, given any set map $f : U(M) \to U(N)$ and $\mathcal{K}$-map $g : N \to N'$, if $U(g) \circ f = Uh$ for some $h : M \to N'$, then there is $\bar{f} : M \to N$ with $U(\bar{f}) = f$.*

Definition

We say that an accessible category with concrete directed colimits, $(\mathcal{K}, U)$, is coherent if U is coherent.

This guarantees sizes behave well, and seems to be indispensable in the element-by-element construction of morphisms...

Big Picture

Accessible categories

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- Foundations: Makkai/Paré, Adámek/Rosický

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Accessible categories with directed colimits

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...**AECs!**

In AECs, the following looms large:

Theorem (Shelah's Presentation Theorem)

For any AEC $\mathcal{K}$ in signature L, there is a signature $L' \supseteq L$, a first order theory T' in L', and a set of T'-types Γ such that if $\mathcal{K}'$ is the class of L'-structures

$$\{M' \mid M' \models T', M' \text{ omits } \Gamma\}$$

then $\mathcal{K}' \restriction L = \mathcal{K}$. Moreover, if $M' \subseteq_{L'} N'$ in $\mathcal{K}'$, $M' \restriction_L \prec_{\mathcal{K}} N' \restriction_L$.

That is, the reduct $\restriction_L : \mathcal{K}' \to \mathcal{K}$ is functorial. More: $\restriction_L$ is faithful, surjective on objects, and preserves directed colimits.

We can package Shelah's Presentation Theorem as asserting the existence of such a well-behaved covering of any AEC $\mathcal{K}$:

$$\mathcal{K}' \xrightarrow{\restriction_L} \mathcal{K}$$

There are several things to note:

- This result is essential for the computation of Hanf numbers, used in the construction of the EM-functor for AECs.
- The proof makes essential use of coherence.
- The expansion L' and set Γ are a little *ad hoc*.

Theorem (Alternative Presentation Theorem, L/Rosický)

If $\mathcal{K}$ is an accessible category with directed colimits, there is a finitely accessible category $\mathcal{K}'$ and a functor $F : \mathcal{K}' \to \mathcal{K}$ that is faithful and surjective on objects, and preserves directed colimits:

$$\mathcal{K}' \xrightarrow{F} \mathcal{K}$$

Note:

- No coherence is required.
- This $\mathcal{K}'$ is actually nicer: $L^*_{\kappa\omega}$, not $L_{\kappa\omega}$.

But is it useful in the same way as Shelah's Presentation Theorem?

Theorem (Makkai/Paré)

If $\mathcal{K}'$ is finitely accessible, there is a faithful, directed colimit preserving functor $E : \mathbf{Lin} \to \mathcal{K}'$.

So we get a composition $\mathbf{Lin} \to \mathcal{K}' \to \mathcal{K}\ldots$

Corollary

If $\mathcal{K}$ is a large accessible category with directed colimits, there is a faithful functor $EM : \mathbf{Lin} \to \mathcal{K}$ that preserves directed colimits.

This is a genuine *EM*-functor—partial results in [L/Rosický], more to come.

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One might ask where coherence is absolutely essential.

In trying to build maps element-by-element, we run into such situations—without it there is no reason to think the set maps being constructed actually arise from maps in the category.

In particular, we can't prove the technical result

Lemma

An object M is Galois-saturated iff it is model-homogeneous.

This is essential in the proof of the uniqueness of saturated models, and in the transfer of categoricity.

If we do assume coherence? The EM-functor works beautifully, much of classification theory for AECs seems to generalize...

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- Fragment of classification theory...

... **AECs!**

Continuous logic—and various offshoots—developed as a way of applying model theory to structures arising in analysis (e.g. Banach spaces). Here structures are built on complete metric spaces, rather than discrete sets.

Recently, people have begun to develop a classification theory for metric AECs (mAECs), which are an amalgam of AECs and the program of continuous logic.

The definition of an mAEC is identical to the definition of an AEC except in two important respects:

- The class is closed under directed colimits, but they need not be concrete—for unions of short chains, we may need to take the completion.
- The role of cardinality is played by *density character*, the size of the smallest dense subset.

In a forthcoming paper we show:

Theorem

Let $\mathcal{K}$ be an mAEC with $LS^d(\mathcal{K}) = \lambda$. Then $\mathcal{K}$ is μ-accessible for all $\mu > \lambda$, and has arbitrary directed colimits. Moreover, an object is of categorical size μ iff it has density character μ.

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- Presentation theorem, EM-functor. **[mAECs!]**

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. . . **AECs!**

We get an awful lot for free, starting with a presentation theorem and EM-functor:

Theorem (L/Rosický)

For any mAEC $\mathcal{K}$, there is a finitely accessible category $\mathcal{K}'$ and a faithful functor

$$F : \mathcal{K}' \to \mathcal{K}$$

that preserves directed colimits. Moreover, there is a faithful functor $EM : \mathbf{Lin} \to \mathcal{K}$ that preserves sufficiently large sizes.

This is a noteworthy improvement on existing results—Hirvonen constructs an EM-functor only for countable, homogeneous mAECs.

Points to a much broader project, once just a crazy dream: Let $\mathcal{K}$ be accessible with directed colimits.

- AECs: abstract model theory in sense of **Sets**,

$$\mathcal{K} \stackrel{U}{\to} \mathbf{Sets}$$

- mAECs: abstract model theory in sense of **Met**,

$$\mathcal{K} \stackrel{U}{\to} \mathbf{Met}$$

- Big picture: abstract model theory in sense of a general finitely accessible category $\mathcal{A}$,

$$\mathcal{K} \stackrel{U}{\to} \mathcal{A}$$

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