TOPOLOGY AND RANK FUNCTIONS FOR GALOIS TYPES IN AECS

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Problem: elementary classes—classes of models of such theories—do not exhaust the interesting classes of mathematical objects.

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Generalize by abandoning syntax, and considering abstract classes of structures that retain only the essential properties common to such classes of models.

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    2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
    3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$

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A $\mathcal{K}$-embedding $f : M \hookrightarrow_{\mathcal{K}} N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

Example 1: Let $\mathcal{K}$ be the class of models of a first order theory T, and $\prec_{\mathcal{K}}$ the elementary submodel relation. Then $\mathcal{K}$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0 + |L(T)|$.

One can think of AECs as the category-theoretic hulls of elementary classes—abandoning syntax, but retaining certain basic properties of the elementary submodel relation.

Example 2: Let ϕ be a sentence of $L_{\kappa,\omega}$, $\mathcal{A}$ a fragment containing ϕ. The class $\mathcal{K} = \mathrm{Mod}(\phi)$, with $\prec_{\mathcal{K}}$ elementary embedding with respect to L^*, is an AEC ($\mathrm{LS}(\mathcal{K}) = |\mathcal{A}|$). With suitable $\prec_{\mathcal{K}}$, can do the same with models of sentences in $L(Q)$, $L_{\omega_1,\omega}(Q)$, etc.

We assume the following additional properties:

Definition (Joint Embedding Property)

For any $M_1, M_2 \in \mathcal{K}$, there is $N \in \mathcal{K}$ with embeddings $f_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $f_2 : M_2 \hookrightarrow_{\mathcal{K}} N$.

Definition (Amalgamation Property)

If $M \prec_{\mathcal{K}} M_1$ and $M \prec_{\mathcal{K}} M_2$ (with all three models in $\mathcal{K}$), there is an $N \in \mathcal{K}$ and embeddings $g_1 : M_1 \hookrightarrow_{\mathcal{K}} N$ and $g_2 : M_2 \hookrightarrow_{\mathcal{K}} N$ that agree on M.

Having discarded syntax, we consider a new notion of type: Galois types. In AECs with sufficient amalgamation, there is a monster model $\mathfrak{C}$ in $\mathcal{K}$, and the Galois types have a simple description:

Definition

For $a \in \mathfrak{C}$, $M \in \mathcal{K}$, the *Galois type of a over M* is defined to be the orbit of a under automorphisms of $\mathfrak{C}$ that fix M. The set of all types over M is denoted by ga-S(M).

Example: If $\mathcal{K}$ an EC and $\prec_{\mathcal{K}}$ elementary submodel, the Galois types over $M \in \mathcal{K}$ are precisely the complete types over M:

$$\text{ga-tp}(a/M) = \text{ga-tp}(b/M) \quad \text{iff} \quad \text{tp}(a/M) = \text{tp}(b/M)$$

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2. For any M, $a \in \mathfrak{C}$, and $N \prec_{\mathcal{K}} M$, the restriction of ga-tp(a/M) to N, denoted ga-tp$(a/M) \upharpoonright N$, is the orbit of a under $\text{Aut}_N(\mathfrak{C})$.

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3. Let $N \prec_{\mathcal{K}} M$ and $p \in \text{ga-S}(N)$. We say that M realizes p if there is $a \in M$ such that ga-tp$(a/M) \restriction N = p$. Or: the orbit in $\mathfrak{C}$ corresponding to p meets M.

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4. We say that $M \in \mathcal{K}$ is λ-Galois saturated if for every $p \in \text{ga-S}(N)$, $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$, p is realized in M.

Question

What can we say about the stability spectra of AECs? Can we use vaguely classical techniques to address the problem? Topology? Rank functions?

In this talk, we give:

- A way of topologizing the sets ga-S(M) in such a way that topological properties of the resulting spaces correspond to semantic properties of M and $\mathcal{K}$.
- A closely related notion of rank (actually, a family of ranks) which shows some promise in analyzing the stability spectra of reasonably well-behaved AECs.

Let $\mathcal{K}$ be an AEC with monster model $\mathfrak{C}$. Let $\lambda \geq \mathrm{LS}(\mathcal{K})$ and $M \in \mathcal{K}$.

Definition (X_M^λ)

For each $N \prec_{\mathcal{K}} M$ with $|N| \leq \lambda$ and type $p \in \text{ga-S}(N)$, let

$$U_{p,N} = \{q \in \text{ga-S}(M) \, : \, q \upharpoonright N = p\}$$

The sets $U_{p,N}$ form a basis for a topology on ga-S(M). We denote by X_M^λ the set ga-S(M) endowed with this topology.

Note

The $U_{p,N}$ are, in fact, clopen. Types over small submodels play a role analogous to formulas in topologizing spaces of syntactic types.

Remark

The assignment $(M, \lambda) \mapsto X_M^\lambda$ is functorial in both arguments:

- *For any $\mu > \lambda$, the set-theoretic identity map $Id_{\mu,\lambda} : X_M^\mu \to X_M^\lambda$ is continuous.*
- *For $M, M' \in \mathcal{K}$ and $f : M \to M'$ a $\mathcal{K}$-embedding, the induced map from ga-$S(M')$ to ga-$S(M)$ is a continuous surjection from $X_{M'}^\lambda$ to X_M^λ.*

In particular, for each $M \in \mathcal{K}$, we obtain a well-behaved spectrum of spaces, with topological properties passing up and down the line.

An easy connection between semantic and topological properties:

Note

A type of the form ga-tp(a/M) with $a \in M$ is always isolated in X_M^λ. There is no reason to think that these are the only isolated types in X_M^λ.

Theorem

If M is λ-saturated, isolated points are dense in X_M^μ for all $\mu < \lambda$.

There is a partial converse, if not a full one:

Theorem

If $\{\text{ga-tp}(a/M) : a \in M\}$ is a dense subset of X_M^μ for all $\mu < \lambda$, M is λ-saturated.

Definition

An AEC $\mathcal{K}$ is said to be χ-tame if for any $M \in \mathcal{K}$, if $q, q' \in \text{ga-S}(M)$ are distinct, then there is submodel $N \prec_{\mathcal{K}} M$ with $|N| \leq \chi$ such that $q \restriction N \neq q' \restriction N$.

Intuition: if we regard types over small models as formulas, tameness means that types are determined entirely by their constituent formulas.

A critically important property. Automatic for ECs, but may fail in certain contexts (Baldwin/Shelah and Baldwin/Kolesnikov build non-tame classes from Abelian groups), and does not hold in a general AEC. Required for all existing stability and categoricity transfer results for AECs.

The centerpiece:

Theorem (Tameness As Separation Principle)

The AEC $\mathcal{K}$ is χ-tame iff for all $M \in \mathcal{K}$, X_M^χ is Hausdorff.

Since each space X_M^λ has a basis of clopens (i.e. the $U_{p,N}$) we have a bit more:

Proposition

$\mathcal{K}$ is χ-tame iff X_M^χ is totally disconnected for every $M \in \mathcal{K}$.

Moreover,

Proposition

If $\mathcal{K}$ is χ-tame, X_M^μ is totally disconnected for every $M \in \mathcal{K}$ and $\mu \geq \chi$.

Naturally, compactness is too much to hope for. In particular,

Proposition

Let $\mathcal{K}$ be an arbitrary AEC with monster model, $M \in \mathcal{K}$, and $\lambda \geq LS(\mathcal{K})$. Then X_M^λ is not compact.

We might hope for some weaker form of compactness, but this proves incompatible with our desire for tameness. The critical complication results from the following:

Fact

For any $M \in \mathcal{K}$, the intersection of any λ many open sets in X_M^λ is open.

If $\mathcal{K}$ is sufficiently tame to guarantee Hausdorffness, this leads to near-discreteness:

Remark

Any subset $S \subseteq X_M^\lambda$ with $|S| = \lambda$ is discrete and closed.

The spaces X_M^λ are uniform, hence completely regular. So:

Proposition

Let $\mathcal{K}$ be χ-tame. Then for any $M \in \mathcal{K}$ and any $\lambda \geq \chi$, the space X_M^λ is not countably compact.

Another important consequence of sufficient tameness:

Proposition

Let $\mathcal{K}$ be χ-tame. Then for any $M \in \mathcal{K}$ and any $\lambda \geq \chi$, a type $q \in X_M^\lambda$ is an accumulation point of $S \subseteq X_M^\lambda$ only if every neighborhood of q contains more than λ elements of S.

The latter will be worth remembering in the discussion of ranks.

In light of this fact, we define a related, slightly Morley-like rank:

Definition (RM^λ)

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- $\mathrm{RM}^\lambda[q] \geq \alpha + 1$ if there exists a structure $M' \succ_{\mathcal{K}} M$ such that q has strictly more than λ many extensions to types q' over M' with $\mathrm{RM}^\lambda[q'] \geq \alpha$.

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For types q over M of arbitrary size, we define

$$\mathrm{RM}^\lambda[q] = \min\{\mathrm{RM}^\lambda[q \restriction N] : N \prec_{\mathcal{K}} M, |N| \leq \lambda\}.$$

The ranks RM^λ are: monotonic, invariant under automorphisms of $\mathfrak{C}$, and decreasing in λ. Are larger (typewise) than Cantor-Bendixson ranks in spaces X_M^λ.

Definition (λ-t.t.)

We say that $\mathcal{K}$ is λ-totally transcendental if for every $M \in \mathcal{K}$ and $q \in \text{ga-S}(M)$, $\mathrm{RM}^\lambda[q]$ is an ordinal.

Note

A sample consequence: If $\mathcal{K}$ is λ-t.t., isolated points are dense in X_M^μ for any $\mu \geq \lambda$.

No guarantee that types have unique extensions of same RM^λ rank, but:

Proposition (Quasi-unique Extension)

Let $M \prec_{\mathcal{K}} M'$, $q \in ga\text{-}S(M)$, and say that $RM^\lambda[q] = \alpha$. Given any rank α extension q' of q to a type over M', there is an intermediate structure M'', $M \prec_{\mathcal{K}} M'' \prec_{\mathcal{K}} M'$, $|M''| \leq |M| + \lambda$, and a rank α extension $p \in ga\text{-}S(M'')$ of q with $q' \in ga\text{-}S(M')$ as its unique rank α extension.

Connections with Galois stability, in case $\mathcal{K}$ is tame:

Theorem

If $\mathcal{K}$ is λ-stable where λ satisfies $\lambda^{\aleph_0} > \lambda$, then $\mathcal{K}$ is λ-t.t.

Proof (Sketch): Assume $\mathcal{K}$ isn't λ-t.t, and deduce failure of λ-stability.

Note

There exists an ordinal α such that for all $M \in \mathcal{K}$, $q \in$ ga-S(M), $RM^\lambda[q] = \infty$ iff $RM^\lambda[q] \geq \alpha$.

If $\mathcal{K}$ not λ-t.t., there is $q \in$ ga-S(M) for some $M \in \mathcal{K}$ with $\mathrm{RM}^\lambda[q] = \infty \geq \alpha + 1$. This rank is witnessed by a restriction to some $N \prec_{\mathcal{K}} M$, $|N| = \lambda$: $p = q \restriction N$ has rank $\geq \alpha + 1$.

We construct an extension N_ω of N of size λ over which p has $\lambda^{\aleph_0}$ distinct extensions, hence more than λ types over N_ω.

Construction:

Stage 1: There is an extension of N over which p has extensions $\{q_i \mid i < \lambda\}$ of rank $\geq \alpha$. Exists size λ extension N_1 of N over which restrictions are distinct: $\{p_i = q_i \upharpoonright N \mid i < \lambda\}$. All rank $\geq \alpha$.

Stage 2: Repeat process for each p_i, obtaining extension N_2 of N_1 over which we have family $\{p_{ij} \mid i, j < \lambda\}$, where p_{ij} extends p_i for all i, j. All rank $\geq \alpha$.

...

Stage n: Same procedure. End up with extension N_n, types $\{p_\sigma \mid \sigma \in \lambda^n\}$, where p_σ extends $p_{\sigma \upharpoonright m}$ for all $m \leq n$. All rank $\geq \alpha$.

...

Stage ω: Set $N_\omega = \bigcup_{i<\lambda} N_i$. For each $\tau \in \lambda^\omega$, there is $p_\tau \in$ ga-S(N_ω) with $p_\tau \upharpoonright N_n = p_{\tau \upharpoonright n}$. Distinct, $\lambda^{\aleph_0}$ many.

Theorem

If $\mathcal{K}$ is λ-totally transcendental, and $M \in \mathcal{K}$ with $cf(|M|) > \lambda$, then $|ga\text{-}S(M)| \leq |M| \cdot \sup\{|ga\text{-}S(N)| \mid N \prec_{\mathcal{K}} M, |N| < |M|\}$.

Proof (Sketch): Say $|M| = \kappa$. Take a filtration of M of length κ:

$$M_0 \prec_{\mathcal{K}} M_1 \prec_{\mathcal{K}} \ldots \prec_{\mathcal{K}} M_i \prec_{\mathcal{K}} \ldots$$

with $i < \kappa$, $|M_i| < |M|$ and $M = \bigcup_{i<\kappa} M_i$. Let $q \in \text{ga-S}(M)$. We have $\text{RM}^\lambda[q] = \beta$ for some ordinal β, and there must be $N \prec_{\mathcal{K}} M$ with $|N| = \lambda$ such that $p = q \restriction N$ has rank β.

Expand to $N' \succ_{\mathcal{K}} N$ so q is unique rank β extension of $q \restriction N'$.
$N' \prec_{\mathcal{K}} M_i$ for some i, so q is unique rank β extension of $q \restriction M_i$.

So no more types over M than over all of the submodels M_i.

Theorem

Let $\mathcal{K}$ be λ-stable with $\lambda^{\aleph_0} > \lambda$, and let κ satisfy cf(κ) $> \lambda$. If there is an interval $[\mu, \kappa)$ such that every M $\in \mathcal{K}_{[\mu,\kappa)}$ satisfies $|$ga-S(M)$| \leq \kappa$, then $\mathcal{K}$ is κ-stable.

A very nice special case:

Corollary

If $\mathcal{K}$ is $\aleph_0$-stable, and κ is of uncountable cofinality, then if $\mathcal{K}$ is stable in every cardinality below κ, it is κ-stable as well.

This was established in (Baldwin-Kueker-Van Dieren, 2004), using the machinery of splitting. Our method has produced a vastly more general result.

We can also say something in case $\mathcal{K}$ is only weakly χ-tame (the defining condition of tameness holds only for saturated models):

Theorem

If $\mathcal{K}$ is λ-t.t., and κ is regular with $\kappa > \lambda$, if there is an interval $[\mu, \kappa)$ on which $\mathcal{K}$ is stable, then $\mathcal{K}$ is κ-stable.

Note: we must now assume t.t. explicitly, as the implication from stability to t.t. holds only for tame AECs.

Question: Can we find a sufficient condition for total transcendence when $\mathcal{K}$ is only weakly tame? Not tame at all?

Note: the stronger stability assumption in the last result serves only to guarantee the existence of enough saturated models that the weakening of tameness is not an issue.

A purely category-theoretic notion due to Rosický—weak κ-stability—will do just as well.

As it happens, AECs are accessible categories, and any accessible category is weakly κ-stable in infinitely many, arbitrarily large cardinals.

The result: the beginnings of a stability spectrum for weakly tame AECs.