Math 9260
Model Theory
Midterm Exam

The exam is due by noon on Monday, 13.11, in my office (2022). I will also accept electronic submissions, provided they are easily readable. As a reminder, you are not to consult your notes, classmates, books, or the internet about this exam until after the deadline on Monday.

Part I: Examples. In each case, give an example with the properties indicated, if one exists. Justify your answer. If none exist, give an argument establishing that conclusion.

1. A theory that is: countably categorical, but not uncountably categorical.

2. A theory that is: uncountably categorical, but not countably categorical.

3. A theory T that is: complete, but for some sentence ϕ in $L(T)$, neither $\phi \in T$ nor $\neg\phi \in T$.

4. A complete theory that has models only of cardinality 5, 7, and 13.

5. Structures M and N in the same signature L such that $M \equiv_L N$ but $M \ncong_L N$.

6. Structures M and N in the same signature L such that $M \subseteq_L N$and $M \cong_L N$ but $M \not\preceq_L N$.

Part II: Please do two of the following problems:

7. Let T be a complete theory. State the joint embedding property and prove that it is satisfied by the models of T.

8. State and prove the (strong) Upward Löwenheim-Skolem Theorem.

9. Let $T = Th_L(M)$, $A \subseteq |M|$, $\kappa = |A| + |T|$, and $\lambda = 2^\kappa$. Show that there is $N \prec M$, $||N|| \leq \lambda$, $A \subseteq |N|$, and for all $B \subseteq |N|$ with $|B| \leq \kappa$, N realizes all types in $S(B, M)$.

10. Show that $\langle \mathbb{Q}, < \rangle$ is, up to isomorphism, the only countable model of T_{DLO}.

Part III: Please do two of the following problems:

11. Suppose that M is a *saturated model* of $T = Th(M)$: for any $A \subseteq |M|$ with $|A| < ||M||$, any complete type over A is realized in M. Let $\bar{a}, \bar{b} \in |M|$. Show that $tp(\bar{a}/\emptyset, M) = tp(\bar{b}/\emptyset, M)$ iff there is an automorphism f of M such that $f(\bar{a}) = \bar{b}$.

12. Let $M := \langle \mathbb{C}, +, \cdot, 0, 1, \exp \rangle$ be the field of complex numbers with exponentiation ($\exp(x) = e^{2\pi i x}$). Show that there is $N \succ M$ of size $2^{\aleph_0}$ such that the set $\{a \in |N| \mid \exp(a) = 1\}$ has size $2^{\aleph_0}$.

13. Let T be a countable complete theory. Show that $|S(\emptyset)| = \lambda > \aleph_1$ implies $I(\aleph_1, T) > \lambda$.

Part IV: Please do one of the following extended problems.

(A) We say that a theory T is axiomatized by a set of sentences Γ if for any $L(T)$-structure M, $M \models T$ iff $M \models \Gamma$. We call a theory T *universal* if T can be axiomatized by universal sentences, i.e. sentences of the form $\forall \bar{x} \phi(\bar{x})$, where $\phi(\bar{x})$ is quantifier-free (for example, the theory of fields). We say that T is a *universal-existential theory* (or a $\forall\exists$-theory) if it can be axiomatized by $\forall\exists$-sentences, i.e. sentences of the form $\forall \bar{x} \exists \bar{y} \phi(\bar{x}, \bar{y})$, where $\phi(\bar{x}, \bar{y})$ is quantifier-free (for example, the theory of dense linear orders without endpoints).

Problem (A): Show that T is universal iff T is closed under submodels: for any $M \models T$, if $N \subseteq_{L(T)} M$ then $N \models T$. (If you want, you may also show that T is $\forall\exists$ iff T is closed under unions of chains.)

(B) Topological interpretation of the compactness theorem: Let T be a theory and consider $S_n = S^n(\emptyset)$, the set of all complete n-types of T. We endow S_n with a topology by declaring a set $X \subset S_n$ to be closed if there is a set of formulas $\pi(x_1, \ldots, x_n)$ such that $X = \{p \in S_n \mid \pi \subseteq p\}$.

Problem (B): Show that S_n becomes a topological space, and that it is compact, Hausdorff, and totally disconnected (i.e. has a basis of clopen sets).