Math 9260
Model Theory
Final Exam Guidelines

The exam will resemble the midterm in structure: first examples and nonexamples of various properties (with their precise definitions), then sketches of proofs of theorems covered in the course (e.g. countable atomic models are homogeneous) or on the midterm (e.g. model complete theories are universal existential), then at least one problem that you haven't seen (e.g. if a set and its complement are both type-definable, then it is definable). The following is intended as an aid in review, but should not be taken to be comprehensive—if you see some important topic does not appear below, that means I have simply forgotten to include it.

Part I: Definitions and Examples. For each of the following, you should be able to give a precise definition, and present examples and nonexamples.

Basic terminology: Signatures, languages, substructure, elementary substructure, (elementary) embedding, elementary equivalence, isomorphism.

Definability and types: Definable sets. Indiscernible sequences. Realizing and omitting types, isolated and principal types.

Varieties of models: Saturated, atomic, universal, prime, homogeneous.

Properties of theories: Joint embedding, amalgamation, universal, universal-existential, with q.e., o-minimal, strongly minimal, c.c., small.

Categoricity: countably, uncountably, and totally categorical. Possibilities for $I(T, \aleph_0)$, T c.c.

Constructions: Compactness, unions of chains, back and forth, (M, N)-maps, partial isomorphisms, Skolemization, EM-models.

Miscellaneous: Monster model, successor and limit cardinals, regular and singular cardinals, cofinality, inaccessible cardinals.

Examples to consider: $T_{DLO}, T_{ACF_p}, T_{ACF}, T_{RCF}$, $\langle \mathbb{R}, +, \cdot, 0, 1\rangle$, $\langle \omega, <\rangle$, $\langle \omega, S, 0\rangle$, $\langle \omega, +, \cdot, 0, 1\rangle$.

Part II: As on the midterm, I will here ask you prove a couple of theorems from class. I may choose anything reasonable, meaning that I will not ask for, say, the proof of Erdős-Rado (although I may ask about the Sierpiński example), or the complete proof of Ryll-Nardzewski (although I may ask you to prove some of the many lemmas).

Part III: Please be prepared to give solutions to the following problems, if requested.

1. Let T be a complete theory, $A \subseteq |M|$, $M \models T$. We say A is *type definable* if there is a T-type $\pi(x)$ such that

$$A = \{a \in |M| \mid a \models \pi(x)\}$$

Show that if A and its complement are type definable, A is definable (i.e. by a formula).

2. We say that a model $M \models T$ is *minimal* if it contains no proper elementary submodel. Give minimal models of T_{ACF} and T_{RCF}. Give a theory with a prime model that is not minimal.

3. Consider the structure $\langle \mathbb{Z}, S \rangle$, where $S(x) = x + 1$ for all $x \in \mathbb{Z}$. Show that this structure has quantifier elimination, and is strongly minimal. Show that $\langle \mathbb{N}, S \rangle$ does not have quantifier elimination.

4. Show that $\langle \mathbb{N}, < \rangle$ has quantifier elimination in the expanded language that includes a unary function symbol S for successor and constant symbol 0. Show that every definable $X \subseteq \mathbb{N}$ is either finite or cofinite, but that $\langle \mathbb{N}, < \rangle$ isn't strongly minimal.