Extensions of ZFC through the lens of accessible categories

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Accessible categories and their connections
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The dream of a categorical algebra with no meaningful dependence on set theory is somewhat feasible in the following popular context:

Definition

Let λ be a regular cardinal. We say that a category $\mathcal{K}$ is *locally λ-presentable* if it

- contains a set S of λ-presentable objects (up to iso),
- any object of $\mathcal{K}$ is a λ-directed colimit of objects in S,
- and $\mathcal{K}$ has all (small) colimits, i.e. it is *cocomplete*.

Here we can nearly get away with thinking of cardinals simply as parameters drawn from a well-ordered collection—up to some basic bookkeeping: regular versus singular, successor versus limit.

Things work very smoothly in this case, e.g.

Fact

Trivially, if $\mathcal{K}$ is locally λ-presentable, it is locally μ-presentable for all regular $\mu \geq \lambda$.

But locally presentable categories are a paradise in which few can live. An equivalent definition:

Definition

A category $\mathcal{K}$ is locally λ-presentable if it

- is λ-accessible, and
- has all (small) limits, i.e. it is *complete*.

For model theorists, say, who tend to like concrete monomorphisms of structures, this means (among many, many other things) that pullbacks exist, hence we must be closed under intersections...

A little abstract model theory:

Definition

Let L be a finitary vocabulary. An *abstract elementary class* in L consists of a class of L-structures $\mathcal{K}$ and an isomorphism-closed strong substructure relation $\prec$ with properties that include:

- Tarski-Vaught: $(\mathcal{K}, \prec)$ is closed under unions of $\prec$-chains.
- Löwenheim-Skolem: There is an infinite cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$ and subset A of M, there is $N \in \mathcal{K}$ with $A \subseteq N \prec M$, and $|N| \leq |A| + \mathrm{LS}(\mathcal{K})$.

Considering $\mathcal{K}$ with $\prec$-embeddings, we obtain a $\mathrm{LS}(\mathcal{K})^+$-accessible category with (concrete) directed colimits.

A little abstract model theory:

Definition

Let L be a finitary vocabulary. A *metric AEC* (or *mAEC*) in L consists of a class of complete metric L-structures $\mathcal{K}$ and an isomorphism-closed strong substructure relation $\prec$ with properties that include:

- Tarski-Vaught: $(\mathcal{K}, \prec)$ is closed under metric completions of unions of $\prec$-chains.
- Löwenheim-Skolem: There is an infinite cardinal $\mathrm{LS}^d(\mathcal{K})$ such that for any $M \in \mathcal{K}$ and subset A of M, there is $N \in \mathcal{K}$ with $A \subseteq N \prec M$, and $\mathrm{dc}(N) \leq |A| + \mathrm{LS}^d(\mathcal{K})$.

Considering $\mathcal{K}$ with $\prec$-embeddings, we obtain an $\mathrm{LS}^d(\mathcal{K})^+$-accessible category with (not necessarily concrete) directed colimits and concrete $\aleph_1$-directed colimits.

A little abstract model theory:

Definition ([BGL$^+$16])

Let L be a μ-ary vocabulary. A *μ-AEC* in L consists of a class of L-structures $\mathcal{K}$ and an isomorphism-closed strong substructure relation $\prec$ with properties that include:

- Tarski-Vaught: $(\mathcal{K}, \prec)$ is closed under μ-directed unions of $\prec$-inclusions.
- Löwenheim-Skolem: There is an infinite cardinal $\lambda = \lambda^{<\mu} \geq \mu$ such that for any $M \in \mathcal{K}$ and subset A of M, there is $N \in \mathcal{K}$ with $A \subseteq N \prec M$, and $|N| \leq |A|^{<\mu} + \lambda$.

Fact

Any λ-accessible category with all morphisms monomorphisms is equivalent to a λ-AEC. Any μ-AEC with LS-number λ is equivalent to a λ^+-accessible category with all morphisms monomorphisms.

This gives a useful synthesis of model-theoretic and accessible category perspectives. Foundation for more:

Definition

A category is *locally λ-polypresentable* if it is λ-presentable with wide pullbacks.

Proposition ([LRVb])

Any locally λ-polypresentable category with all morphisms mono is equivalent to a λ-AEC **admitting intersections**.

Definition

A category is *locally λ-multipresentable* if it is λ-accessible with connected limits.

Proposition ([LRVb])

Any locally λ-multipresentable category with all morphisms mono is equivalent to a **universal** *λ-AEC.*

Theorem ([BR12])

If $\mathcal{K}$ is a λ-accessible category with directed colimits, it is μ-accessible for all regular $\mu \geq \lambda$, i.e. it is **well-λ-accessible***.*

Question

Given a λ-accessible category, is it μ-accessible for regular $\mu \geq \lambda$?

Definition

For regular $\mu > \lambda$, we say that $\mu \trianglerighteq \lambda$ if the following equivalent conditions hold:

1. Every λ-accessible category is μ-accessible.
2. For any set X with $|X| < \mu$, the set $P_\lambda(X)$ consisting of all subsets of X of size λ has a cofinal subset of size $< \mu$.
3. In any λ-directed poset, any subset of size $< \mu$ is contained in a λ-directed subset of size $< \mu$.

None of this is very intuitive...

Examples ([MP89],[AR94])

1. *For any (uncountable) regular μ, $\mu \triangleright \omega$.*
2. *For any regular μ, $\mu^+ \triangleright \mu$.*
3. *For regular $\mu \geq \lambda$, $(2^\mu)^+ \triangleright \lambda$.*
4. *If $\mu \geq \lambda$ and for all cardinals $\alpha < \lambda$ and $\beta < \mu$, $\beta^\alpha < \mu$, then $\mu \triangleright \lambda$.*

Definition

We say λ is *μ-closed* if $\theta^{<\mu} < \lambda$ for all $\theta < \lambda$; we say it is *almost μ-closed* if $\theta^{<\mu} \leq \lambda$.

Proposition ([LR17b]/Ref,[LRVa])

Let $\lambda > \mu$ regular. If λ is μ-closed, $\lambda \triangleright \mu$. If $\lambda > 2^{<\mu}$ and $\lambda \triangleright \mu$, λ is μ-closed.

Definition

The Generalized Continuum Hypothesis (GCH) states that for all infinite λ, $2^\lambda = \lambda^+$.

We don't much like assuming GCH, but it makes exponentiation easy: if $\lambda > \mu$,

$$\lambda^\mu = \begin{cases} \lambda^+ & \text{cf}(\lambda) \leq \mu \\ \lambda & \text{cf}(\lambda) > \mu \end{cases} \qquad \lambda^{<\mu} = \begin{cases} \lambda^+ & \lambda = \kappa^+, \text{cf}(\kappa) \leq \mu \\ \lambda & \text{else} \end{cases}$$

So, under GCH, any $\lambda > \mu$ is closed as long as it is not the successor of a cardinal of small cofinality.

Proposition

Under GCH, any μ-accessible category is λ-accessible for **pretty much every** $\lambda > \mu$.

This is cracking a walnut with a sledgehammer, though: we can get away with much weaker assumptions.

Definition

1. The Singular Cardinal Hypothesis (SCH) states that $2^\lambda = \lambda^+$ if λ is a singular strong limit cardinal ($2^\mu < \lambda$ for all $\mu < \lambda$).
2. We say that $\mathsf{SCH}_{\mu,\lambda}$ holds if there is a set of almost μ-closed cardinals unbounded in λ.
3. For any θ, we say $\mathsf{SCH}_{\mu,\geq\theta}$ holds if $\mathsf{SCH}_{\mu,\lambda}$ holds for all $\lambda \geq \theta$.

Notes ([LRVa])

1. *Relevant instances $SCH_{\mu,\lambda}$ follow from SCH (hence GCH).*
2. *SCH holds iff $SCH_{\mu,\geq 2^{>\mu}}$ holds for all regular μ.*
3. *If κ is a (ω_1-)strongly compact cardinal, then $SCH_{\mu,\theta}$ holds for all $\theta \geq \kappa$ and regular $\mu \leq \theta$.*

If $\text{SCH}_{\mu,\geq\theta}$ holds, then cardinal exponential behaves just nicely above θ, so:

Proposition

Assuming $SCH_{\mu,\geq\theta}$, a μ-accessible category is λ-accessible for any $\lambda \geq \theta$ that is not the successor of a cardinal of cofinality $\leq \mu$.

The move to instances of SCH is not an idle one, or a product of the usual pathology of generalization—by the previous note, they follow not just from GCH, but from a large cardinal axiom.

That is, if our additional axioms contract or expand the universe, we still get well-accessibility, almost. More about this later...

Fact/Definition

Let $\mathcal{K}$ be an accessible category. Any object M is λ-presentable for some regular λ. The presentability rank of M, *denoted $r_{\mathcal{K}}(M)$, is the least such λ.*

In examples, this tracks the right notion of size.

Examples

For sufficiently large λ,

- **Grp**: $r_{\mathbf{Grp}}(G) = \lambda^+$ *iff G is λ-presented.*
- *$\mathcal{K}$ an AEC:* $r_{\mathcal{K}}(M) = \lambda^+$ *iff* $|M| = \lambda$.
- *$\mathcal{K}$ an mAEC:* $r_{\mathcal{K}}(M) = \lambda^+$ *iff* $dc(M) = \lambda$.

Definition

Define the internal size of M in $\mathcal{K}$ to be

$$|M|_{\mathcal{K}} = \begin{cases} \lambda & \text{if } \mathrm{r}_{\mathcal{K}}(M) = \lambda^+ \\ \mathrm{r}_{\mathcal{K}}(M) & \text{else} \end{cases}$$

Questions

Is presentability rank always a successor? Always the successor of the natural notion of size?

There is no known counterexample to the first question: an accessible category $\mathcal{K}$ and object M such that $\mathrm{r}_{\mathcal{K}}(M)$ is a regular **limit** cardinal, i.e. a *weakly inaccessible cardinal*.

Fact ([BR12])

Let $\mathcal{K}$ be λ-accessible with directed colimits. For any M in $\mathcal{K}$ with $r_{\mathcal{K}}(M) > \lambda$, $r_{\mathcal{K}}(M)$ is a successor.

They show that the same holds for λ-accessible $\mathcal{K}$, under GCH. Boils down to cardinal arithmetic again...

Fact ([LRVa])

Assuming a suitable instance of SCH, the ranks of objects in any λ-accessible category are successors.

We consider an internal analogue of Löwenheim-Skolem.

Definition

We say that a category $\mathcal{K}$ is *LS-accessible* if there is some λ such that for all $\lambda' \geq \lambda$, $\mathcal{K}$ contains an object of internal size λ. We say that $\mathcal{K}$ is *weakly LS-accessible* if this holds only for regular λ'.

Question

Is every large accessible category LS-accessible?

No counterexample is known.

Proposition ([LR14])

If $\mathcal{K}$ is a large accessible category with directed colimits and all morphisms mono, it is LS-accessible.

Purely category-theoretic arguments give weak LS-accessibility in the locally multi- and polypresentable cases, [LRVa]. Via μ-AECs?

The question remains how internal size relates to, say, cardinality.

Example

*Assume $SCH_{\mu,\geq\theta}$. Let $\mathcal{K}$ be a μ-AEC, $U:\mathcal{K}\to$ **Sets** the forgetful functor. Given M in $\mathcal{K}$, $|UM|=\lambda\geq\theta+LS(\mathcal{K})$, what are the possibilities for $|M|_{\mathcal{K}}$?*

$$|M|_{\mathcal{K}}=\begin{cases}\lambda & \lambda\neq\lambda_0^+, cf(\lambda_0)<\mu\\ \lambda \text{ or } \lambda_0 & \text{else}\end{cases}$$

It is remarkably easy to construct μ-AECs with gaps in cardinalities λ with $\mathrm{cf}(\lambda)<\mu$, but internal size is much smoother—these gaps may simply disappear. Hence the difficulty with counterexamples.

Fact ([LRVa])

Assume $SCH_{\mu,\geq\theta}$. Let $\mathcal{K}$ be a μ-AEC admitting intersections with arbitrarily large models. Then $\mathcal{K}$ is LS-accessible.

We toy with a few large cardinal (or anti-large cardinal) axioms:

- $V = L$, the axiom of constructibility.
- Strongly inaccessible cardinals.
- Measurable cardinals.
- Almost strongly compact cardinals/μ-strongly compact cardinals.
- Strongly compact cardinals.
- Vopěnka cardinals.
- Huge cardinals.

For each of the large cardinal axioms, you may wish to add "There exists a proper class of..."

Definition (Vopěnka's Principle)

We say that Vopěnka's Principle holds if the following equivalent conditions hold:

1. No locally presentable category contains a large discrete full subcategory.
2. For any class $\mathcal{C}$ of first-order structures in a finitary signature L, there are distinct $M, N \in \mathcal{C}$ with an elementary embedding $i : M \to N$.

(This can be turned, rather artificially, into an explicit large cardinal principle.)

This is a category-theoretic magic wand: see Chapter 6 of [AR94].

Funny story: this was meant as a kind of satire on large cardinal principles, but proved to be independent.

We concentrate on certain flavors of compactness.

Definition

A cardinal κ is *strongly compact* if the following equivalent conditions hold:

1. Every κ-complete filter extends to a κ-complete ultrafilter.
2. The infinitary logic $L_{\kappa\kappa}$ is compact: any inconsistent set of formulas contains an inconsistent subset of size $< \kappa$.

We say that κ is *μ-strongly compact* if every κ-complete filter extends to a μ-complete ultrafilter.

We say κ is *almost strongly compact* if it is μ-strongly compact for all $\mu < \kappa$.

What do these have to do with anything? Well, instances of SCH.

Let **Ab** denote the category of abelian groups, and $\mathcal{F}$ the full subcategory of free abelian groups.

Ab is beautifully accessible, but is $\mathcal{F}$?

Theorem ([EM90])

Assume $V = L$. Then $\mathcal{F}$ is **not** *accessible.*

Theorem ([EM90])

Assume there is a strongly compact cardinal. Then $\mathcal{F}$ **is** *accessible.*

Notes

- *The free abelian group functor F :* **Sets** $\to$ **Ab** *is finitely accessible, and $\mathcal{F}$ is its image.*
- *$\mathcal{F}$ is closed under subobjects, hence the* powerful image *of F.*

Theorem ([MP89])

Assume there is a proper class of strongly compact cardinals. Then the powerful image of any accessible functor is accessible.

Theorem ([BTR16])

Let $\mathcal{L}$ be **(well-)**λ*-accessible, such that there exists a* $\mu_{\mathcal{L}}$*-strongly compact cardinal* κ*. The powerful image of any* λ*-accessible functor to* $\mathcal{L}$ *that preserves* $\mu_{\mathcal{L}}$*-presentable objects is* κ*-accessible.*

One can give an explicit bound on this $\mu_{\mathcal{L}}$.

Corollary

If there is a μ-strongly compact cardinal for every μ, then the powerful image of any accessible functor is accessible.

Note

The following statements are equivalent:

- *There is a μ-strongly compact cardinal for every μ.*
- *There is a proper class of almost strongly compact cardinals.*

So a proper class of almost strongly compact cardinals suffices.

Question

Is this an improvement?

Answer

Consistently yes, consistently no?

Given an abstract class of structures $\mathcal{K}$, we often ask: can every diagram of shape $\mathcal{A}$ be completed to a diagram of shape $\mathcal{A}'$?

Example

We say that an accessible category $\mathcal{K}$ has the $<\kappa$-JEP if for any κ-presentable $M_0, M_1 \in \mathcal{K}$, there are $f_i : M_i \to N$ for $i = 1, 2$. We say that $\mathcal{K}$ has the JEP if this holds for arbitrary $M_0, M_1 \in \mathcal{K}$.

In terms of diagrams:

$$\begin{array}{ccc} \bullet \quad \bullet & \qquad & \bullet \to \bullet \leftarrow \bullet \\ \mathcal{A} & & \mathcal{A}' \end{array}$$

The collection of jointly embeddable pairs corresponds to the image of the forgetful functor

$$F_J : \mathcal{K}^{\mathcal{A}'} \to \mathcal{K}^{\mathcal{A}}.$$

Roughly, if the (powerful) image of F_J is accessible, completability of an $\mathcal{A}$-diagram is determined by the completability of its small sub-$\mathcal{A}$-diagrams. In short:

Proposition ([BB17],[LR17a])

Let $\mathcal{K}$ be well-λ-accessible. If κ is a $\mu_{\mathcal{K}}$-strongly compact cardinal, then if $\mathcal{K}$ has the $< \kappa$-JEP, it has the JEP.

Notes

1. *$\mathcal{K}$ has the $< \kappa$-JEP just in case $\mathbf{Pres}_\kappa(\mathcal{K})^{\mathcal{A}}$ is contained in the image of F_J.*
2. *$\mathcal{K}$ has the JEP just in case F_J is surjective.*
3. *The image of F_J is closed under subobjects, hence powerful.*
4. *As colimits are computed componentwise in $\mathcal{K}^{\mathcal{A}}$, $\mathcal{K}^{\mathcal{A}'}$, F_J preserves everything. Hence F_J is as accessible as $\mathcal{K}^{\mathcal{A}}$ and $\mathcal{K}^{\mathcal{A}'}$ are.*

There's a little work to do, to check that the conditions of the Brooke-Taylor/Rosický theorem apply—but very little.

Proof.

By B-T/R, the powerful image–in this case, just the image—if F_J is κ-accessible.

Consider a pair $(M_0, M_1) \in \mathcal{K}$. Since $\mathcal{K}^{\mathcal{A}}$ is λ-accessible, it is also κ-accessible, meaning that (M_0, M_1) is a κ-directed colimit of pairs of κ-presentables. If $\mathcal{K}$ has the $< \kappa$-JEP, all pairs of κ-presentables are in the image of F_J. As the image of F_J is κ-accessible, it is closed under κ-directed colimits. That is, (M_0, M_1) is in the image of F_J.

Thus every pair of objects in $\mathcal{K}$ is jointly embeddable. □

With only minor changes, this argument can be applied much more broadly, e.g.

Definition

We say that $\mathcal{K}$ has the $< \kappa$-*AP* if for all cospans $M_0 \stackrel{f_0}{\leftarrow} M \stackrel{f_1}{\rightarrow} M_1$, there are $g_i : M_i \rightarrow N$ such that

$$g_0 f_0 = g_1 f_1.$$

We say $\mathcal{K}$ has the AP if the above holds for all κ.

Theorem

Let $\mathcal{K}$ be well-λ-accessible. If κ is a $\mu_{\mathcal{K}}$-strongly compact cardinal, then if $\mathcal{K}$ has the $< \kappa$-AP, it has the AP.

With sufficient care, the same can be done for the disjoint versions of the JEP and AP.

With still more care, one can derive the tameness of AECs—an important locality property of *Galois* (or *orbital*) types. This forms part of a deep equivalence:

Theorem

The following are equivalent:

1. *There is a proper class of almost strongly compact cardinals.*
2. *The powerful image of every accessible functor is accessible.*
3. *Every AEC is tame.*

Proof.

$(1 \Rightarrow 2)$ [BTR16]. $(2 \Rightarrow 3)$ [LR14]. $(3 \Rightarrow 1)$ [BU17]. □

So, in particular, accessibility of powerful images is a large cardinal principle.

I would like to suggest that *co-wellpoweredness*, which is currently sandwiched between two large cardinal axioms, may warrant further examination.

Definition

We say that a category $\mathcal{K}$ is co-wellpowered if any object of $\mathcal{K}$ has at most a set of quotients.

Any locally presentable category is co-wellpowered ([AR94]), but things are more complicated in the accessible case.

Theorem ([MP89])

If there is a proper class of strongly compact cardinals, every accessible category is co-wellpowered.

Theorem ([AR94])

If every accessible category is co-well-powered, there is a proper class of measurable cardinals.

Seems difficult to improve on [MP89] in the style of [BTR16]. Another route?

Definition

A cardinal κ is *measurable* if it satisfies the following equivalent conditions:

1. There is a nonprincipal κ-complete ultrafilter on κ.
2. If a theory T in $L_{\kappa\kappa}$ is the union of an increasing chain of satisfiable theories, then T is satisfiable ([Bon]).

The latter chain-compactness condition **must** be useful...

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