Weak factorization systems and stable independence

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Logic Colloquium 2019
ČVUT, Praha
16 August 2019

We highlight a surprising connection between three concepts:

1. Accessibility (of categories)
2. Stable nonforking independence
3. Cofibrantly generated weak factorization systems

We sketch this connection, with a particular focus on the category-theoretic formulation of stable independence.

Time permitting, we will apply the analysis to Ext-orthogonality classes of modules, e.g. ${}^{\perp}N$, answering a question of Baldwin/Eklof/Trlifaj: such classes are AECs just in case they have a stable independence notion.

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Here we take all monos, but could take pure, flat, etc., without really escaping the realm of model theory.

All kinds of beautiful things happen, of course, but there are costs as well. In particular:

Fact

The category $\mathbf{Ab}$ *has pushouts;* $\mathbf{Mod}(T_{ab})$ *does not.*

Easier to see: a pushout of monos in $\mathbf{Ab}$ is not a pushout in $\mathbf{Mod}(T_{ab})$—induced maps will not be mono.

We consider a more general framework, where we choose a family of morphisms $\mathcal{M}$ in a starting category $\mathcal{K}$ that is *locally presentable*.

Definition

For λ a regular cardinal, we say that a category $\mathcal{K}$ is **locally λ-presentable** *if*

1. *$\mathcal{K}$ has all colimits.*
2. *There is a set of λ-presentable objects, and every object of $\mathcal{K}$ is a λ-directed colimit thereof.*

This covers, e.g. **Set** , **Ab**, R-**Mod**, and **Str**(Σ), where presentability corresponds, roughly, to cardinality/presentation size.

Basic problem: given a locally presentable category $\mathcal{K}$ and family of $\mathcal{K}$-morphisms $\mathcal{M}$, what can we say about

$$\mathcal{K}_{\mathcal{M}}$$

the subcategory of $\mathcal{K}$ whose morphisms are precisely those in $\mathcal{M}$?

Do natural properties of $\mathcal{M}$ correspond to natural properties of $\mathcal{K}_{\mathcal{M}}$?

For a start: in the background, we assume $\mathcal{M}$ is *normal*—closed under composition, contains all isomorphisms—so $\mathcal{K}_{\mathcal{M}}$ really is a (wide) subcategory of $\mathcal{K}$.

In general, passing to $\mathcal{K}_\mathcal{M}$ expels us from the paradise of locally presentable categories, leaving us with, if we are lucky, accessibility.

Definition

*For λ a regular cardinal, we say a category $\mathcal{K}$ is λ-**accessible** if*

1. *$\mathcal{K}$ has all λ-directed colimits.*
2. *There is a set of λ-presentable objects, and every object of $\mathcal{K}$ is a λ-directed colimit thereof.*

That is, we may lose some colimits, including pushouts.

Fact

Say a category $\mathcal{C}$ is accessible with all morphisms mono (and a multi-initial object). If $\mathcal{C}$ has pushouts, it is small.

So if we engineer $\mathcal{K}_\mathcal{M}$ to be nice, we lose pushouts. Such is life.

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Version 1: Fix a theory T, monster model $\mathfrak{C}$. We say the type of a tuple $\bar{a} \in \mathfrak{C}$ over a model B does not fork over $C \subseteq B$ if the type over C has the same complexity, i.e. Morley rank.

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$$\bar{a} \mathop{\smile\!\!\!\!\!\!\mid}\limits_{C}^{(\mathfrak{C})} B$$

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Version 2: Again, given a theory T and monster model $\mathfrak{C}$, we say

$$A \underset{C}{\overset{(\mathfrak{C})}{\perp}} B$$

if the type of any $\bar{a} \in A$ over B does not fork over C. One can think of this as a kind of independence relation: A is independent from B over C.

One can think of $\perp$ as an abstract ternary relation, and axiomatize stable (or *simple*) independence directly.

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Version 3: In AECs, we can only work over models, and may not have a monster model. We end up with $\mathop{\smile\!\!\!\!\!\!|}$ as a quaternary relation

$$M_1 \overset{M_3}{\underset{M_0}{\mathop{\smile\!\!\!\!\!\!|}}} M_2$$

axiomatized as before. In particular, we are picking out a family of diagrams of strong embeddings of the form

$$\begin{array}{ccc} M_1 & \rightarrow & M_3 \\ \uparrow & \mathop{\smile\!\!\!\!\!\!|} & \uparrow \\ M_0 & \rightarrow & M_2 \end{array}$$

Idea: Do this in an arbitrary category $\mathcal{K}$.

Definition

An independence notion $\downarrow$ on $\mathcal{K}$ is a family of commutative squares in $\mathcal{K}$ (suitably closed). We say that $\downarrow$ is **weakly stable** *if it satisfies*

1. *Existence: Any span $M_1 \leftarrow M_0 \rightarrow M_2$ can be completed to an independent square.*
2. *Uniqueness: there is only one independent square for each span, up to equivalence.*
3. *Transitivity: horizontal and vertical compositions of independent squares are independent.*

Fact

If $\downarrow$ is weakly stable, these squares satisfy the usual cancellation property of pushouts.

To get the analogue of stability, we must impose a locality condition—accessibility now appears.

Consider the category $\mathcal{K}_{\downarrow}$:

- Objects: $f : M \to N$ in $\mathcal{K}$.
- Morphisms: A morphism from $f : M \to N$ to $f' : M' \to N'$ is a $\downarrow$-independent square

$$\begin{array}{ccc} M' & \longrightarrow & N' \\ \uparrow & \downarrow & \uparrow \\ M & \longrightarrow & N \end{array}$$

Definition

We say that $\downarrow$ *is* λ-**stable** *if* $\mathcal{K}_{\downarrow}$ *is* λ*-accessible, and* **stable** *if it is* λ*-stable for some* λ*.*

Returning to the basic framework, i.e. $\mathcal{K}$ a category, $\mathcal{M}$ a class of morphisms, there is a natural candidate for stable independence:

Definition

We say a square

$$\begin{array}{ccc} M_1 & \rightarrow & M_3 \\ \uparrow & & \uparrow \\ M_0 & \rightarrow & M_2 \end{array}$$

*in $\mathcal{K}$ is $\mathcal{M}$-***effective** *if*

1. *all morphisms are in $\mathcal{M}$,*
2. *the pushout of $M_1 \leftarrow M_0 \rightarrow M_2$ exists, and*
3. *the induced map from the pushout to M_3 is in $\mathcal{M}$.*

If $\mathcal{M} = \{\text{regular monos}\}$, these are the *effective unions* of Barr.

To force these squares to form a nice independence relation, we need a few additional properties:

Definition

Let $\mathcal{K}$ be a category.

1. *We say that $\mathcal{M}$ is* **coherent** *if whenever $gf \in \mathcal{M}$ and $g \in \mathcal{M}$, $f \in \mathcal{M}$.*
2. *We say that $\mathcal{M}$ is a* **coclan** *if pushouts of morphisms in $\mathcal{M}$ exist, and $\mathcal{M}$ is closed under pushouts.*
3. *We say $\mathcal{M}$ is* **almost nice** *if it is a coherent coclan, and* **nice** *if, in addition, it is closed under retracts.*

Proposition

If $\mathcal{M}$ is almost nice, the $\mathcal{M}$-effective squares give a weakly stable independence notion on $\mathcal{K}_{\mathcal{M}}$.

We now veer sharply in the direction of algebraic topology. Recall:

Note

In **Top**, *CW-complexes are built inductively by gluing on new cells along their boundaries, $S^{n-1} \to D^n$. The corresponding morphisms are constructed in similar fashion...*

Gluing corresponds to pushing out along some $S^{n-1} \to D^n$.

The inductive construction corresponds to transfinite composition.

So we are concerned with the maps *cellularly generated* by the set $\{S^{n-1} \to D^n \,:\, n \in \omega\}$.

Being generated in this way from a **set** of morphisms is an important smallness condition...

Definition

Let X be a family of morphisms in a category $\mathcal{K}$. Recall:

1. $\mathrm{Po}(X)$ *is the closure of X under pushouts.*
2. $\mathrm{Tc}(X)$ *is the closure under transfinite composition.*
3. $\mathrm{Rt}(X)$ *is the closure under retracts.*
4. $\mathrm{cell}(X) = \mathrm{Tc}(\mathrm{Po}(X))$
5. $\mathrm{cof}(X) = \mathrm{Rt}(\mathrm{cell}(X))$

Under certain circumstances, we can dispense with retracts.

Definition

We say that a set of morphisms $\mathcal{M}$ in $\mathcal{K}$ is **cofibrantly generated** *if $\mathcal{M} = \mathrm{cof}(X)$, X a* **set** *of morphisms.*

Theorem

Let $\mathcal{K}$ be locally presentable, $\mathcal{M}$ nice and $\aleph_0$-continuous. The following are equivalent:

1. *$\mathcal{K}_{\mathcal{M}}$ has a stable independence notion.*
2. *$\mathcal{M}$-effective squares form a stable independence notion on $\mathcal{K}_{\mathcal{M}}$.*
3. *$\mathcal{M}$ is cofibrantly generated.*

Proof.

(1) $\Rightarrow$ (2): By canonicity—clean category-theoretic proof of this. □

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Proof.

(2) $\Rightarrow$ (3): Take λ such that $\mathcal{K}_{\mathcal{M},\downarrow}$ and $\mathcal{K}$ are λ-accessible, consider

$$\mathcal{M}_\lambda = \mathcal{M} \cap \mathbf{Pres}_\lambda(\mathbf{C})^{\to}.$$

One can show that $\mathcal{M} = \mathrm{cof}(\mathcal{M}_\lambda)$. ☐

Theorem

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Proof.

(3) $\Rightarrow$ (1): Say $\mathcal{M} = \mathrm{cof}(X)$, and λ such that everything is λ-accessible, domains and codomains of morphisms in X are λ-presentable. Show class $\mathcal{M}^*$ of λ-directed colimits of maps in $\mathcal{M}_\lambda$ (in $\mathcal{K}_{\mathcal{M},\downarrow}$) is exactly $\mathcal{M}$. Need elimination of retracts, Makkai/Rosický/Vokřínek. □

Definition

A **weak factorization system** *(or WFS) in a category* $\mathcal{K}$ *consists of a pair of classes of morphisms* $(\mathcal{M}, \mathcal{N})$ *such that:*

1. *Any morphism* h *of* $\mathcal{K}$ *can be written as* $h = gf$*, where* $f \in \mathcal{M}$ *and* $g \in \mathcal{N}$*.*
2. $\mathcal{M} = {}^{\square}\mathcal{N}$ *and* $\mathcal{N} = \mathcal{M}^{\square}$*.*

Paradigmatic example: (monos, epis) in **Set**.

Why? They underlie model structures, for one.

Fact

If $(\mathcal{M}, \mathcal{N})$ *is a coherent WFS—that is,* $\mathcal{M}$ *is coherent—then* $\mathcal{M}$ *is nice and* $\aleph_0$*-continuous.*

Corollary

If $(\mathcal{M}, \mathcal{N})$ is a coherent weak factorization system on locally presentable $\mathcal{K}$, the following are equivalent:

1. *$\mathcal{K}_{\mathcal{M}}$ has stable independence.*
2. *$\mathcal{M}$ is cofibrantly generated.*

Note (Quillen's small object argument)

If $\mathcal{K}$ is locally presentable, $\mathcal{M}$ cofibrantly generated, then $(\mathcal{M}, \mathcal{M}^{\square})$ is a WFS on $\mathcal{K}$.

So, modulo coherence—see forthcoming work of Simon Henry—subcategories $\mathcal{K}_{\mathcal{M}}$ with stable independence are in correspondence with cofibrantly generated WFSs.

Fix a category of R-modules and homomorphisms, $\mathcal{C} = R$-**Mod**, and class of R-modules K. Define

$$^{\perp}K = \{A : \mathrm{Ext}^i(A, N) \text{ for all } 1 \leq i < \omega \text{ and } N \in K\}$$

That is, $^{\perp}K$ consists of all modules that do not admit nontrivial extensions by modules in K.

1. If $K = \{N\}$, we write $^{\perp}N$.
2. If K is the class of pure injective modules, $^{\perp}K$ is the class of flat modules.

As morphisms, we take the class of R-**Mod**-monomorphisms

$$\mathcal{M}_K = \{f : A \to B : B/f[A] \in {}^{\perp}K\}$$

What can we say about $\mathcal{C}_{\mathcal{M}_K}$? In particular, when is it an AEC?

Baldwin/Eklof/Trlifaj answer this question for the case ${}^{\perp}N$.

Assuming the class is closed under directed colimits, the only trick is satisfying the DLS axiom. This holds just in case the class *admits refinements*.

Definition

We say ${}^{\perp}N$ admits refinements if any $A \in {}^{\perp}N$ is the colimit of a continuous increasing chain of submodules $\langle A_i : i < \alpha \rangle$, with small quotients A_{i+1}/A_i, all of which belong to ${}^{\perp}N$.

But this is simply cofibrant generation of $\mathcal{M}_N$ in disguise...

Theorem

For any N (pure-injective, say), TFAE:

1. $^{\perp}N$ *is an AEC.*
2. $^{\perp}N$ *has refinements.*
3. $\mathcal{M}_N$ *is cofibrantly generated by set of morphisms with domains and codomains in* $^{\perp}N$.
4. $^{\perp}N$ *has stable independence.*

In particular, this answers the question of Baldwin/Eklof/Trlifaj concerning when such AECs are stable: always.

This generalizes to *flat-like categories* of modules, see §6 of L/Rosický/Vasey.

Lots of other examples and applications there...