Set-theoretic pathologies in accessible categories

(Joint with Jiří Rosický and Sebastien Vasey)

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Vopěnka	$\Leftrightarrow$	Any subfunctor of an accessible functor is accessible (Adámek/Rosický)
Strongly compact	$\Rightarrow$	Any accessible category is cowellpowered (Makkai/Paré)
Almost strongly compact	$\Leftrightarrow$	Powerful images of accessible functors are accessible (Boney/Unger, et al)
	...	
Boundedly many measurables	$\Rightarrow$	Exists a non-cowellpowered accessible category (Adámek/Rosický)
	...	
$V = L$		Failure of eventual categoricity?

Under GCH*, we can say a great deal about internal sizes versus cardinalities, and gather compelling evidence for the former as the better way of phrasing test questions: categoricity, existence, etc.

Makkai/Paré: Let ZC denote Zermelo-Fraenkel with choice, but without replacement. Let $\mathcal{K}$ be the category of well-founded models of ZC, with elementary embeddings.

Facts

1. *Mostowski Collapse: Any object of $\mathcal{K}$ is uniquely iso to unique transitive standard model: $\widehat{M} = (M, \in|_M)$, M transitive.*
2. *In particular, $\mathcal{K}$ contains $\widehat{V_\alpha} = (V_\alpha, \in|_{V_\alpha})$, for limit α.*
3. *$\mathcal{K}$ is accessible: category of models of ZC plus $L_{\omega_1\omega_1}$-sentence*

$$\forall (x_i)_{i\in\omega} \vee_{i\in\omega} \neg[x_{i+1} \in x_i]$$

4. *Directly: $\mathcal{K}$ is $\aleph_1$-accessible, with $M \in \mathcal{K}$ $\aleph_1$-presentable iff countable.*

Form $\mathcal{K}^*$ by formally adjoining an initial object I. This new category is still $\aleph_1$-accessible.

Claim

If there are no measurables, I has a proper class of quotients, namely those represented by the $I \to \widehat{V_\alpha}$, α limit.

It suffices to show that there is at most one $f : \widehat{V_\alpha} \to N$ for $N \in \mathcal{K}$, hence that the maps $I \to \widehat{V_\alpha}$ are trivially surjective, and that the full models form a proper class of pairwise nonisomorphic objects.

Set theory: If $f : \widehat{V_\alpha} \to N$ is such that $f(\beta) = \beta$ for all ordinals β, $f(x) = x$ for all $x \in V_\alpha$. So if f is not simply the inclusion, $f(\beta) \neq \beta$ for some ordinal β. The least such must be a measurable cardinal. But there aren't any of those...

Proposition (Adámek/Rosický)

If there are boundedly many measurables, there is a non-cowellpowered accessible category.

Let $\mathcal{K}$ be the accessible category with the same objects, and with λ-elementary embeddings, where λ is larger than any measurable cardinal. Again, formally adjoin initial object 1.

By λ-elementarity, any $\mathcal{K}$-morphism $f : \widehat{V_\alpha} \to N$ preserves ordinals $\beta < \lambda$, meaning that the first ordinal it moves must be a measurable above λ. Doesn't work.

Question

Does the existence of a proper class of measurables imply cowellpoweredness of accessible categories?

The central organizing concern of abstract model theory is the following:

Conjecture (Shelah)

Let $\mathcal{K}$ be an abstract elementary class (AEC). If $\mathcal{K}$ is λ-categorical for some sufficiently large λ, it is μ-categorical for all sufficiently large μ.

Convincing approximations exist, particularly assuming, e.g. strongly compact cardinals. But at what level of generality does it fail? Is the following true?

Conjecture (Beke/Rosický—who don't believe it either.)

Let $\mathcal{K}$ be an accessible category. If $\mathcal{K}$ is categorical in sufficiently large internal size λ (λ-IS-categorical), it is μ-IS-categorical for all sufficiently large μ.

Intuition: Look for a counterexample here, with $V = L$.

Any accessible category $\mathcal{K}$ comes with a notion of size:

Definition

An object K in $\mathcal{K}$ is λ-*presentable* (λ regular) if $\mathrm{Hom}_{\mathcal{K}}(K, -)$ preserves λ-directed colimits. The *presentability rank* of K, $\pi_{\mathcal{K}}(K)$, is the least λ such that K is λ-presentable.

In **Sets**, $\pi(X) = |X|^+$. In an AEC, $\pi(M) = |M|^+$. In a metric AEC (mAEC), $\pi(M) = dc(M)^+$. A pattern here...

Fact (Beke/Rosický)

Let $\mathcal{K}$ be accessible, $M \in \mathcal{K}$. If

1. *GCH* holds, or*
2. *$\mathcal{K}$ has directed colimits and all morphisms mono,*

then $\pi(M) = \lambda^+$ for some λ. In either case, we define the internal size of M in $\mathcal{K}$, denoted $|M|_{\mathcal{K}}$, to be λ.

Tossed out in the opening paragraph of Shelah:1019 is the following example (this version due to L/Rosický/Vasey):

Let $\mathcal{K}$ be the category of well-founded models of Kripke-Platek set theory (no powerset, restricted separation and replacement) plus V=L, with morphisms the elementary embeddings.

Fact

1. *By Condensation, any M in $\mathcal{K}$ is uniquely isomorphic to a unique*
$$\widehat{L_\alpha} = (L_\alpha, \in|_{L_\alpha}).$$
2. *$\mathcal{K}$ is an accessible category (because an $\aleph_1$-AEC...),*
$$|M|_{\mathcal{K}} = |M|,$$
and there are λ^+ models in every infinite cardinality λ.

Example

Let $\mathcal{K}^*$ be the full subcategory of $\mathcal{K}$ on the $M \in \mathcal{K}$ isomorphic to $(L_\alpha, \in)$ such that for all $\beta < \alpha$, $[L_\beta]^{\leq\aleph_0} \cap L \subseteq L_\alpha$.

Fact

$\mathcal{K}^*$ is also accessible (still an $\aleph_1$-AEC).

Lemma

For λ an infinite cardinal we have:

$$I(\mathcal{K}^*, \lambda) = \begin{cases} 1 & \textit{if } |[L_\lambda]^{\leq\aleph_0} \cap L| > \lambda \\ \lambda^+ & \textit{if } (\forall\alpha < \lambda^+)(\exists\beta < \lambda^+)([L_\alpha]^{\leq\aleph_0} \cap L \subseteq L_\beta) \\ \mu & \textit{otherwise, for some } \mu \in [1, \lambda^+) \end{cases}$$

Cardinalities!

Theorem

Assume $V = L$ and let λ be an infinite cardinal. Then:

$$I(\mathcal{K}^*, \lambda) = \begin{cases} 1 & \text{if } cf(\lambda) = \aleph_0 \\ \lambda^+ & \text{if } cf(\lambda) > \aleph_0 \end{cases}$$

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Or not: one discovers that objects of cardinality λ_0^+ with $cf(\lambda_0) = \aleph_0$ can have internal size λ_0 or λ_0^+, and enough drop down to destroy our hopes...

Although AECs are very general, they are, for some purposes, not general enough: many classes of interest lack (concrete) directed colimits (e.g. mAECs), only have models in certain cardinalities (e.g. $\mathrm{Sat}_\lambda(\mathcal{K}) \subseteq \mathcal{K}$), or call for infinitary operations (e.g. μ-complete BAs). This led to:

Definition (Boney/Grossberg/L/Rosický/Vasey)

An abstract class of structures $\mathcal{K}$ in a μ-ary signature is a μ-AEC if it satisfies the AEC axioms, but with the following modifications:

- $\mathcal{K}$ is only assumed to have μ-directed colimits, and
- there is λ with $\lambda^{<\mu} = \lambda$ such that for any $A \subseteq M \in \mathcal{K}$, there is $A \subseteq N \prec_{\mathcal{K}} M$ and $|N| \leq |A|^{<\mu} + \lambda$. Define $LST(\mathcal{K})$ to be least such λ.

While this characterization isn't terribly easy to motivate, there is another:

Theorem (BGLRV)

1. *Any μ-AEC $\mathcal{K}$ is a LST$(\mathcal{K})^+$-accessible category with all morphisms mono.*
2. *Any μ-accessible category with all morphisms mono is (equivalent to) a μ-AEC.*

Good news in many ways: allows extensive application of MT tools to accessible categories with monos, on the one hand, and allows clean, uncluttered CT arguments involving μ-AECs, on the other.

As already seen, we have a tension here: $|-|_{\mathcal{K}}$ versus $|-|$.

We consider only the case GCH*. Given a μ-AEC $\mathcal{K}$, $|-|_{\mathcal{K}}$ and $|-|$ mostly agree, except—as in Shelah's example—when it comes to successors of cardinals of cofinality less than μ.

Theorem (L/Rosický/Vasey)

Let $\mathcal{K}$ be a μ-AEC, $M \in \mathcal{K}$, and $\lambda = |M|$.

$$|M|_{\mathcal{K}} = \begin{cases} \lambda \textit{ or } \lambda_0 & \textit{if } \lambda = \lambda_0^+,\ \mathit{cf}(\lambda_0) < \mu \\ \lambda & \textit{else} \end{cases}$$

Here you can clearly see the smoothing that comes with passing to internal sizes.

In a μ-AEC, it is easy to create, say, gaps in cardinalities λ with $\mathit{cf}(\lambda) < \mu$, but this is precisely where internal sizes drop back to fill the holes.

There are two natural ways of generalizing Shelah's conjecture to μ-AECs:

Conjecture (Eventual categoricity in power)

If a μ-AEC is categorical in a large enough cardinal λ with $\lambda = \lambda^{<\mu}$, it is categorical in all sufficiently large κ such that $\kappa = \kappa^{<\mu}$.

Note: we write off all cardinals of cofinality less than μ.

Conjecture (Eventual IS-categoricity)

If a μ-AEC is λ-IS-categorical for some sufficiently large λ, then it is κ-IS-categorical in sufficiently large κ.

In AECs, these are the same. In mAECs, research is (secretly) focused on the second. But what to pursue in a general μ-AEC?

Example

Let **Hilb** *denote the category of (complex) Hilbert spaces and linear isometries.*

Note

1. **Hilb** *is* $\aleph_1$*-accessible, hence an* $\aleph_1$*-AEC.*
2. *For any* $V \in$ **Hilb**, $|V|_{\mathbf{Hilb}}$ *is the size of an orthonormal basis of* V.
3. *Clearly,* V *is* λ*-IS-categorical for every* λ*.*

So what about categoricity in power?

Fact (Bartoszyński/Džamonja/Halbeisen/Murtinová/Plichko)

Any $V \in$ **Hilb** *has cardinality* $\lambda^{\aleph_0}$ *for some* λ*.*

Assume GCH. For any infinite cardinal λ,

$$I(\mathbf{Hilb}, \lambda) = \begin{cases} 0 & \text{if } cf(\lambda) = \aleph_0 \\ 2 & \text{if } \lambda = \lambda_0^+,\ cf(\lambda_0) = \aleph_0 \\ 1 & \text{else} \end{cases}$$

Here we have a failure of eventual categoricity in power, even in the limited sense of $\aleph_1$-AECs.

In fact, if for all α, $\aleph_\alpha^{\aleph_0} = \aleph_{\alpha+\beta}$ for some β, there are $|\beta| + 1$ objects of cardinality $\aleph_{\alpha+\beta}$, encompassing internal sizes

$$\aleph_\alpha, \aleph_{\alpha+1}, \ldots, \aleph_{\alpha+\beta}$$

Possible moral: categoricity in power is wildly extrinsic and dependent on background set theory. IS-categoricity, though...

All occurrences of "GCH*" should be interpreted as follows: "GCH or even $\mathrm{SCH}_{\mu,\geq\theta}$ for suitable θ," where:

Definition

Let $\mu \leq \lambda$.

1. We say that λ is *almost μ-closed* if $\theta^{<\mu} \leq \lambda$ for all $\theta < \lambda$.
2. For S a class of infinite cardinals greater than or equal to μ, we write $\mathrm{SCH}_{\mu,S}$ for the statement "every $\lambda \in S$ is almost μ-closed." $\mathrm{SCH}_{\mu,\geq\theta}$ has the obvious meaning.

Note

If μ is strongly compact, $SCH_{\mu,\geq\mu}$ holds, so the above results hold not just under V=L/GCH, but also in the newly favored context of ZFC+(strongly compacts).