Abstract tameness from large cardinals, via accessible categories

(Joint with Jiří Rosický)

Michael Lieberman

Masaryk University, Department of Mathematics and Statistics
http://www.math.muni.cz/~lieberman

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We consider the interplay between three areas. Template:

Theorem (Boney/Unger, '16)

The following are equivalent:

1. *There is a proper class of almost strongly compact cardinals.*
2. *The powerful image of any accessible functor is accessible.*
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Extensive use of ultraproducts.

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Simple argument involving relevant categories of diagrams.

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Refined by latter from strongly compact to almost strongly compact.

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Hart-Shelah style construction.

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We will focus on the machinery provided by (2), and its consequences in abstract model theory.

In particular, we will consider the phenomenon of tameness, first in AECs, but then much, much more broadly.

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For a regular cardinal λ, we say a category **C** is λ-*accessible* if

- it has at most a set of λ-presentable objects.
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A general λ-accessible category need not be closed under arbitrary directed colimits...

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"Accessible" means "λ-accessible for some λ."

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Assume V=L. For every successor κ, there is a nonfree abelian group A of size κall of whose subgroups of size less than κ are free.

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Corollary

Assuming $V = L$, $\mathcal{F}$ is not accessible.

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Notes

- *The free abelian group functor* $F : \mathbf{Sets} \to \mathbf{Ab}$ *is accessible, and* $\mathcal{F}$ *is its image.*

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- *The free abelian group functor $F : \mathbf{Sets} \to \mathbf{Ab}$ is accessible, and $\mathcal{F}$ is its image.*
- *$\mathcal{F}$ is closed under subobjects, hence the* powerful image *of F.*

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If image is accessible, completability is determined entirely on the small models...

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Definition

Let $\mathcal{K}$ be, say, an AEC.

- Version 1: Given a monster $\mathfrak{C}$ in $\mathcal{K}$, the type of $a \in \mathfrak{C}$ over $M \in \mathcal{K}$ is the orbit of a in $\mathcal{K}$ under automorphisms fixing M.

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- Version 2: For $M \prec_{\mathcal{K}} N_i$ and $a_i \in UN_i$, $i = 0, 1$, the triples (M, a_0, N_0) and (M, a_1, N_1) have the same Galois type over M if there are $f_i : N_i \to N$ such that $U(f_0)(a_0) = U(f_1)(a_1)$ and f_0 and f_1 agree on M.

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This notion behaves as you would like. Version 2 has unexpected benefits.

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One route: ensure, via tameness, that Galois types over arbitrary models are determined by their restrictions to submodels of a uniform small size...

Definition

We say that an AEC $\mathcal{K}$ is χ-*tame* if for any $M \in \mathcal{K}$ and types p and q over M, when $p \upharpoonright K = q \upharpoonright K$ for all $K \prec_{\mathcal{K}} M$ with $|UK| \leq \chi$, then $p = q$.

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Question: But is every AEC tame?

Answer: Not absolutely. Under V=L, Baldwin/Shelah, '08, produce nontame class of short exact sequences of groups. Uses Shelah's construction of non-Whitehead group of size $\aleph_1$.

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We can also argue via the diagrams themselves, using the accessibility of powerful images...

Metric abstract elementary classes (mAECs) are a recent development (due to Hirvonen/Hyttinen) in the project to develop a model theory relevant to structures arising in analysis, e.g. Banach spaces.

Slogan

Metric AECs represent an amalgam of AECs and the program of continuous logic.

Roughly, an mAEC is an AEC whose structures are built on complete metric spaces, rather than discrete sets.

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Here tameness looks different: if p, q over M should be close together whenever their restrictions to small submodels are sufficiently close together.

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An mAEC $\mathcal{K}$ is *χ-d-tame* if for every $\epsilon > 0$, there is $\delta > 0$ such that for any $M \in \mathcal{K}$ and p, q over M, if $d(p \upharpoonright K, q \upharpoonright K) < \delta$ for all $K \prec_{\mathcal{K}} M$ of size $\leq \chi$, then $d(p, q) < \epsilon$.

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Here again, the proof involves ultraproducts, but now metric ultraproducts. Everything much more delicate.

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3. By counting and κ-directedness, there is cofinal diagram of $K \prec_{\mathcal{K}} M$ giving the same distance $\delta < \epsilon$.
4. After some fiddling, pass to colimit, obtaining $d(p, q) < \epsilon$.

- Quantale-valued structures, Ω-structures, sheaves (?). Joint with Rosický and Zambrano.
- Less obvious contexts? Abstract model theory over combinatorial geometries/matroids?
- Suggestions?