Metric AECs as accessible categories

(Joint with Jiří Rosický)

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Big picture:

We give a uniform treatment of several doctrines of abstract model theory (today, just AECs and mAECs) as pairs

$$\mathcal{K}, U : \mathcal{K} \to \mathbf{Sets}$$

with $\mathcal{K}$ an accessible category with all directed colimits and all morphisms monomorphisms, and U a functor whose properties can be tuned to the desired model theoretic frequency.

Analysis of $\mathcal{K}$ as an abstract category allows uniform treatment of

- Presentation theorems
- Ehrenfeucht-Mostowski functors, $E : \mathbf{Lin} \to \mathcal{K}$

Adjusting U allows us to capture subtle differences in concreteness/discreteness...

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. In particular, *directed colimits*.

Definition

For λ a regular cardinal, a poset I is λ-*directed* if any subset $J \subseteq I$, $|J| < \lambda$, has an upper bound in I. A colimit is said to be λ-*directed* if the indexing poset is λ-directed. When $\lambda = \aleph_0$, we simply say the colimit is *directed*.

(Compare: directed union, "direct limit.")

Note: Any λ-directed colimit is, in particular, directed. The converse is far from true.

For general regular cardinal λ:

Definition

An object N in a category $\mathbf{C}$ is λ-*presentable* if the functor $\mathrm{Hom}_{\mathbf{C}}(N,-)$ preserves λ-directed colimits.

Definition

A category $\mathbf{C}$ is λ-*accessible* if

- it has at most a set of λ-presentables
- it is closed under λ-directed colimits
- every object is a λ-directed colimit of λ-presentables

Note

A general λ-accessible category need not be closed under arbitrary directed colimits...

Question: When is a λ-accessible category $\mathcal{K}$ μ-accessible for $\mu > \lambda$?

Answer: Just in case $\mu \triangleright \lambda\ldots$

This is a delicate condition, *sharp inequality*, but appears again and again.

Examples:

- $\mu \triangleright \omega$ for all uncountable μ.
- $\mu^+ \triangleright \mu$.
- If $\mu > \lambda$ and $\mu^{<\lambda} = \mu$, $\mu \triangleright \lambda$.

Theorem (Beke/Rosický)

If $\mathcal{K}$ is λ-accessible with all directed colimits, it is μ-accessible for all $\mu \geq \lambda$.

Definition

For any object M in an accessible category $\mathcal{K}$, we define its *presentability rank*, $\pi(M)$, to be the least λ such that M is λ-presentable.

Fact (Beke/Rosický)

In any accessible category $\mathcal{K}$ that is closed under directed colimits, $\pi(M)$ is always a successor, say $\pi(M) = \lambda^+$.
In this case we say λ is the internal size of M in $\mathcal{K}$, and write $|M|_{\mathcal{K}} = \lambda$.

This ends up meaning precisely what we'd like it to in our special cases. . .

Theorem (L/Rosický)

If $\mathcal{K}$ is λ-accessible with directed colimits and all morphisms mono, there is a finitely presentable category $\mathcal{L}$ and a functor

$$\mathcal{L} \xrightarrow{F} \mathcal{K}$$

that is faithful, surjective on objects, and preserves directed colimits.

Notes

- $\mathcal{L} = \mathit{Ind}(\mathcal{A})$, *where $\mathcal{A}$ is the subcategory of λ-presentables.*
- *Requires essentially no AEC-related structure, especially coherence...*

Theorem (L/Rosický)

If $\mathcal{K}$ is a large accessible category with directed colimits and all morphisms mono, it admits an EM-functor

$$E : \mathbf{Lin} \to \mathcal{K}$$

that is faithful and preserves directed colimits.

Because E preserves directed colimits, it eventually preserves (internal) sizes: for sufficiently large I in **Lin**,

$$|EI|_{\mathcal{K}} = |I|$$

This, and simple functoriality of E, are surprisingly powerful.

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$$f \neq g \text{ in } \mathcal{K} \Longrightarrow U(f) \neq U(g) \text{ in } \mathbf{Sets}$$

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  Whenever the following diagram commutes

$$\begin{tikzcd} UL \arrow[rr, "U(h)"] \arrow[dr, "f"'] & & UN \\ & UM \arrow[ur, "U(g)"'] & \end{tikzcd}$$

  then $f = U(\bar{f})$ *for some* $\bar{f} : L \to M$.

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$$U(f) \text{ is injective for all } f \in \mathcal{K}$$

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$$(M_i \to M \mid i \in I) \text{ a directed colimit in } \mathcal{K} \implies$$

$$(UM_i \to UM \mid i \in I) \text{ a directed colimit in } \mathbf{Sets}.$$

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  Form the signature Σ_U consisting of all implicitly defined finitary relations in $\mathcal{K}$; that is, directed colimit-preserving subfunctors of U^n. $\mathcal{K}$ embeds in $\mathbf{Str}(\Sigma_U)$: we say $(\mathcal{K}, U)$ is iso-full if this embedding is iso-full.

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Metric abstract elementary classes (mAECs) are a recent development (due to Hirvonen/Hyttinen) in the project to develop a model theory relevant to structures arising in analysis, e.g. Banach spaces.

Slogan

Metric AECs represent an amalgam of AECs and the program of continuous logic.

Roughly, an mAEC is an AEC whose structures are built on complete metric spaces, rather than discrete sets.

A few crucial changes to the axioms for an mAEC $\mathcal{K}$:
(1) In the Löwenheim-Skolem axiom, cardinality is replaced by *density character*,

$$\mathrm{dc}(M) = \min\{|X| \mid X \text{ is a dense subset of } M\}$$

Upshot: the crucial notion of size in an mAEC is density character, not cardinality:

- $\mathcal{K}$ is λ-*d-categorical* if it contains one model of density character λ up to iso.
- $\mathcal{K}$ is λ-*d-stable* if any model of density character λ has Galois type space of density character at most λ.

(2) While the union of an increasing chain may not belong to the $\mathcal{K}$, the completion of the union must. Upshot:

- $\mathcal{K}$ is closed under colimits of chains, hence under arbitrary directed colimits.
- These limits need not be concrete: if $U : \mathcal{K} \to \mathbf{Sets}$ is the forgetful functor, in general we may have

$$U(\mathrm{colim}_{i \in I} M_i) \subsetneq \mathrm{colim}_{i \in I} UM_i$$

  That is, U will not preserve directed colimits...

Fact

$\aleph_1$*-directed colimits are concrete!*

Theorem (L/Rosický)

Let $\mathcal{K}$ be an mAEC, with $U : \mathcal{K} \to$ **Sets** *the forgetful functor. Then*

- *$\mathcal{K}$ is accessible,* **has all directed colimits***, and all morphisms are monomorphisms.*
- *U is faithful, coherent, and preserves monomorphisms and* **directed colimits***.*
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Note

Good news: for any $M \in \mathcal{K}$, $|M|_{\mathcal{K}} = \mathrm{dc}(M)$. So, working internally in $\mathcal{K}$, we get the correct notion of size by default.

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What, if anything, goes wrong?

Because $U : \mathcal{K} \to \mathbf{Sets}$ does not preserve directed colimits, it need not preserve all sufficiently large sizes. Still, it preserves $\aleph_1$-directed colimits, so:

Fact (L/Rosický)

If $|M|_{\mathcal{K}} = \lambda$ *for sufficiently large* $\lambda \triangleright \aleph_1$,

$$dc(M) = |M|_{\mathcal{K}} = |UM|$$

Note that this is, a priori, weaker than the condition $\lambda^{<\aleph_1} = \lambda$, i.e.

$$\lambda^{\aleph_0} = \lambda,$$

which would be a more obvious way of ensuring $\mathrm{dc}(M) = |UM|$.

A vaguely classical argument using this fact and the existence of a robust EM-functor gives lots of stability in categorical mAECs:

Theorem (L/Rosický)

Let $\mathcal{K}$ be an mAEC. If $\mathcal{K}$ is ν-categorical (i.e. ν-d-categorical), then it is λ-d-stable for all sufficiently large λ with $\lambda^+ \leq \nu$ and $\lambda^+ \triangleright \aleph_1$.

Granted, this is a very technical result, but an improvement on existing ones—it gives, a priori, stability in more cardinals—and is an easy consequence of the general theory.

Definition

A μ-concrete AEC, or μ-CAEC, consists of a pair $(\mathcal{K}, U)$, where $U : \mathcal{K} \to$ **Sets** and

- $\mathcal{K}$ is accessible, has all directed colimits, and all morphisms are monomorphisms.
- U is faithful, coherent, and preserves monomorphisms and μ-**directed colimits**.
- $\mathcal{K}$ is iso-full.

This is one of many possible generalized frameworks for abstract model theory that have popped up recently. See also μ-AECs (Boney/Grossberg/Vasey).

Coda

We've done something slightly funny in our analysis of mAECs: an extra act of forgetting.

$$\mathcal{K} \to \textbf{Met} \to \textbf{Sets}$$

This "discretization" loses us structure, clearly, and the ability to analyze, e.g. μ-d-tameness.
Perhaps we could (should?) have stuck with

$$\mathcal{K} \stackrel{U}{\to} \textbf{Met}$$

Question: How much meaningful theory can we develop in this way?

Coda

A bigger question: Let $\mathcal{K}$ be accessible with directed colimits, monomorphisms.

- AECs: abstract model theory in sense of **Sets**,

$$\mathcal{K} \stackrel{U}{\to} \textbf{Sets}$$

- mAECs: abstract model theory in sense of **Met**,

$$\mathcal{K} \stackrel{U}{\to} \textbf{Met}$$

- Abstract model theory in sense of a general accessible category with directed colimits, $\mathcal{A}$,

$$\mathcal{K} \stackrel{U}{\to} \mathcal{A}?$$