ACCESSIBLE CATEGORIES AND ABSTRACT ELEMENTARY CLASSES

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An outline of the content of this talk:

Model Theoretic Background:

- AECs
- Galois Types
- Partial Spectrum Results

Accessible Categories:

- Connections to AECs
- Implications for Stability
- A Structure Theorem for Categorical AECs

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  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$

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A strong embedding $f : M \hookrightarrow_{\mathcal{K}} N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

Example 1: Let $\mathcal{K}$ be the class of models of a first order theory T, and $\prec_{\mathcal{K}}$ the elementary submodel relation. Then $\mathcal{K}$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0 + |L(T)|$.

One can think of AECs as the category-theoretic hulls of elementary classes—abandoning syntax, but retaining certain basic properties of the elementary submodel relation.

Example 2: Let ϕ be a sentence of $L_{\infty,\omega}$, $\mathcal{A}$ a fragment containing ϕ. The class $\mathcal{K} = \mathrm{Mod}(\phi)$, with $\prec_{\mathcal{K}}$ elementary embedding with respect to $\mathcal{A}$, is an AEC ($\mathrm{LS}(\mathcal{K}) = |\mathcal{A}|$). With suitable $\prec_{\mathcal{K}}$, can do the same with models of sentences in $L(Q)$, $L_{\omega_1,\omega}(Q)$, etc.

We fix our context: $\mathcal{K}$ is an AEC, with Löwenheim-Skolem number $\mathrm{LS}(\mathcal{K})$. Unless otherwise indicated, $\mathcal{K}$ satisfies both the amalgamation and joint embedding properties, hence contains a monster model $\mathfrak{C}$.

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1. For any M, $a \in \mathfrak{C}$, and $N \prec_{\mathcal{K}} M$, the restriction of ga-tp(a/M) to N, denoted ga-tp$(a/M) \restriction N$, is the orbit of a under $\mathrm{Aut}_N(\mathfrak{C})$.

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2. $M \in \mathcal{K}$ is λ-Galois saturated if for every type p over $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$, p is realized in M, i.e. the orbit corresponding to p meets M.
3. $\mathcal{K}$ is λ-Galois stable if for every $M \in \mathcal{K}_\lambda$, $|\text{ga-S}(M)| = \lambda$.

One virtue of this setup: λ-Galois saturation corresponds to λ-model-homogeneity—for any $N \prec_{\mathcal{K}} M$ with $|N| < \lambda$, if $N' \succ_{\mathcal{K}} N$ and $|N'| < \lambda$, there is an embedding $f : N' \hookrightarrow_{\mathcal{K}} M$ that fixes N.

Final condition:

Definition

An AEC $\mathcal{K}$ is said to be χ-tame if for any $M \in \mathcal{K}$, if $q, q' \in \text{ga-S}(M)$ are distinct, then there is submodel $N \prec_{\mathcal{K}} M$ with $|N| \leq \chi$ such that $q \upharpoonright N \neq q' \upharpoonright N$.

Intuition: if we regard types over small models as formulas, tameness means that types are determined entirely by their constituent formulas.

Categoricity: Story fragmentary, results only for tame classes.

Theorem (Grossberg-VanDieren)

If $\mathcal{K}$ is categorical in λ and λ^+, it is categorical in λ^{++}.

Theorem (G-V)

If $\mathcal{K}$ is categorical in $\lambda^+ > H(\mathcal{K})$, it is categorical in all $\mu > H(\mathcal{K})$.

These results are obtained by resorting to syntax, EM-models, etc.

Question: Can a purely category-theoretic perspective reveal anything new about the structure of categorical AECs?

Stability: if anything, the picture is even less clear. For tame AECs, we have

Theorem (G-V)

If $\mathcal{K}$ is λ-stable, it is κ-stable for all κ such that $\kappa^\lambda = \kappa$.

Theorem (L)

Let $\mathcal{K}$ be λ-stable, where $\lambda^{\aleph_0} > \lambda$. If $cf(\kappa) > \lambda$ and $|ga\text{-}S(M)| \leq \kappa$ for every $M \in \mathcal{K}_{<\kappa}$, then $\mathcal{K}$ is κ-stable.

As a particularly nice special case:

Corollary

If $\mathcal{K}$ is $\aleph_0$-stable, and κ is of uncountable cofinality, then if $\mathcal{K}$ is stable in every cardinality below κ, it is κ-stable as well.

If $\mathcal{K}$ is only weakly χ-tame (the defining condition of tameness holds only for Galois saturated models), less is known.

Theorem (Baldwin-Kueker-VanDieren)

If $\mathcal{K}$ is λ-stable, it is stable in λ^{+n} for all $n < \omega$.

Theorem (L)

If $\mathcal{K}$ is λ-t.t., and κ is such that cf(κ) $> \lambda$ and each M of size κ has a saturated extension also of size κ, then $\mathcal{K}$ is κ-stable.

Can we guarantee the existence of saturated extensions without making the standard model-theoretic assumption: $|\text{ga-S}(M)| < \kappa$ for all $M \in \mathcal{K}_{<\kappa}$?

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Yes: weak κ-stability, a purely category-theoretic (and weaker) notion, will suffice.

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- (Rosický, 1997) Accessible categories with directed colimits, considers exceedingly model-theoretic notions.
- (Beke/Rosický; L) Accessible categories and AECs.

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Equivalently, N is finitely presentable if for any directed diagram $D : (I, \leq) \to \mathbf{C}$ with colimit cocone $(\phi_i : D(i) \to M)_{i \in I})$, any map $f : N \to M$ factors through one of the cocone maps: $f = \phi_i \circ g$ for some $i \in I$ and $g : N \to D(i)$, as below.

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$$\begin{tikzcd}
 & & M & \\
 & D(i) \arrow[ur, "\phi_i"] \arrow[rr, "D(i \to j)"'] & & D(j) \arrow[ul, "\phi_j"'] \\
N \arrow[uurr, bend left, "f"] \arrow[ur, dashed, "g"'] & & &
\end{tikzcd}$$

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Example: **Grp**, or any other finitary algebraic variety, is finitely accessible.

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Example: **Hilb**, the category of Hilbert spaces with linear contractions, lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.

The Downward Löwenheim-Skolem Property ensures that models in an AEC are generated as directed unions of their submodels of size LS($\mathcal{K}$). As it happens (no AP or JEP needed),

Theorem

As a category, an AEC $\mathcal{K}$ is μ-accessible for all regular $\mu \geq LS(\mathcal{K})^+$, and the μ-presentable objects are precisely the models of size less than μ. Moreover, $\mathcal{K}$ is closed under directed colimits.

Rosický/Beke consider categories of this form, in which context the former has defined a number of category-theoretic analogues of notions from model theory. Most notably: weak κ-stability.

Definition

An object N is λ-saturated if for any λ-presentable objects M, M' and morphisms $f : M \to N$ and $g : M \to M'$, there is a morphism $h : M' \to N$ such that $h \circ g = f$.

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In an AEC $\mathcal{K}$ (with AP, JEP now),

$$\begin{aligned} N \text{ is } \lambda\text{-saturated} \quad &\Leftrightarrow \quad N \text{ is } \lambda\text{-model-homogeneous} \\ &\Leftrightarrow \quad N \text{ is } \lambda\text{-Galois-saturated} \end{aligned}$$

Definition

A morphism $f : M \to N$ in a category $\mathbf{C}$ is said to be λ-pure (λ regular) if for any commutative square

$$\begin{array}{ccc} M & \xrightarrow{f} & N \\ u \uparrow & & \uparrow v \\ C & \xrightarrow{g} & D \end{array}$$

in which C and D are λ-presentable, there is a morphism $h : D \to M$ such that $h \circ g = u$.

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In an AEC $\mathcal{K}$, $M \hookrightarrow_{\mathcal{K}} N$ is λ-pure iff M is λ-Galois-saturated relative to N. An inclusion $M \hookrightarrow_{\mathcal{K}} \mathfrak{C}$ is λ-pure iff M is λ-Galois-saturated.

Definition

A category $\mathbf{C}$ *is weakly* κ*-stable if for every* κ^+*-presentable* M *and morphism* $f : M \to N$, f *factors as*

$$M \longrightarrow M' \longrightarrow N$$

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If an AEC $\mathcal{K}$ is weakly κ-stable, then for any $M \in \mathcal{K}_\kappa$, the inclusion $M \hookrightarrow_{\mathcal{K}} \mathfrak{C}$ factors through a κ^+-presentable object M' (i.e. a model $M' \in \mathcal{K}_\kappa$) such that $M' \hookrightarrow_{\mathcal{K}} \mathfrak{C}$ is λ-pure, whence M' is saturated.

That is, every $M \in \mathcal{K}_\kappa$ has a saturated extension $M' \in \mathcal{K}_\kappa$.

The partial spectrum result for weakly tame AECs becomes:

Proposition

If $\mathcal{K}$ is λ-t.t., and weakly κ-stable with cf(κ) $> \lambda$, $\mathcal{K}$ is κ-stable.

As it happens, any accessible category—hence any AEC—is weakly stable in many cardinalities:

Theorem (R)

Let $\mathbf{C}$ be a λ-accessible category, and μ a regular cardinal such that $\lambda \trianglelefteq \mu$ and $|\mathbf{Pres}_\lambda(\mathbf{C})^{mor}| < \mu$. Then $\mathbf{C}$ is weakly $\mu^{<\mu}$-stable.

Taken together, these yield new spectrum results for weakly tame AECs.

For example,

Proposition

If $LS(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-t.t. and weakly $\aleph_0$-tame, then for any $\mu > |\mathcal{K}_{\aleph_0}^{mor}|$ with $\mu \trianglerighteq \aleph_1$, $\mathcal{K}$ is $\mu^{<\mu}$-stable.

Suffice to say, there are many (and arbitrarily large) cardinals μ with $\aleph_1 \trianglelefteq \mu$.

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Suffice to say, there are many (and arbitrarily large) cardinals μ with $\aleph_1 \trianglelefteq \mu$.

For example, if $\mathcal{K}$ is $\aleph_0$-categorical, $|\mathcal{K}_{\aleph_0}^{\text{mor}}| \leq 2^{\aleph_0}$. Then $\mathcal{K}$ is stable in

$$[(2^{\aleph_k})^{+(n+1)}]^{(2^{\aleph_k})^{+n}} \text{ for } n < \omega \text{ and } 1 \leq k < \omega,$$

and so on.

Unassume AP, JEP. Suppose $\mathcal{K}$ is λ-categorical, C is the unique structure of size λ, and M is its monoid of endomorphisms.

Theorem (R,L)

*If $\mathcal{K}$ is λ-categorical, the sub-AEC $\mathcal{K}_{\geq\lambda}$ consisting of models of size at least λ is equivalent to (M^{op}, λ^+)-**Set**, the full subcategory of M^{op}-**Set** consisting of λ^+-directed colimits of M.*

The equivalence is induced by the composition

$$\mathcal{K}_{\geq\lambda} \xrightarrow{y} \mathbf{Set}^{(\mathcal{K}_{\geq\lambda})^{op}} \xrightarrow{r} \mathbf{Set}^{M^{op}} \longrightarrow M^{op}\text{-}\mathbf{Set}$$

where y is the Yoneda embedding, the second map is restriction, and the final map is the obvious equivalence $\mathbf{Set}^{M^{op}} \xrightarrow{\sim} M^{op}$-**Set**.

The assignment is:

$$N \in \mathcal{K}_{\geq\lambda} \quad \mapsto \quad \mathrm{Hom}_{\mathcal{K}}(C, N)$$

where $M = \mathrm{Hom}_{\mathcal{K}}(C, C)$ acts by precomposition.

That this gives the desired equivalence is an exercise in definitions.

This amounts to an astonishing transformation of a very abstract entity—an AEC—into a category of relatively simple algebraic objects.

Further Reading

Accessible Categories and AECs:

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Further Reading II

AEC Context:

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