TOPOLOGIES AND RANK FUNCTIONS FOR GALOIS TYPES

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An Abstract Elementary Class (AEC) is a nonempty class of structures in a given signature, closed under isomorphism, equipped with a strong submodel relation, $\prec_{\mathcal{K}}$, that satisfies (among other axioms):

- $\prec_{\mathcal{K}}$ is a partial order.
- Closure under unions of $\prec_{\mathcal{K}}$-increasing chains (etc.).
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

A $\mathcal{K}$-embedding $f : M \to N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

Example 1: Let $\mathcal{K}$ be the class of models of a first order theory T, and $\prec_{\mathcal{K}}$ the elementary submodel relation. The class $\mathcal{K}$ is an AEC with $\mathrm{LS}(\mathcal{K}) = \aleph_0 + |L(T)|$.

One would not go far wrong in thinking of AECs as the category-theoretic hulls of ECs—abandoning syntax, but retaining certain basic properties of the elementary submodel relation.

Example 2: Let ϕ be a sentence of $L_{\omega_1,\omega}$ (L countable), L^* a countable fragment containing ϕ. $\mathcal{K} = \mathrm{Mod}(\phi)$, with $\prec_{\mathcal{K}}$ elementary embedding with respect to L^*, is an AEC ($\mathrm{LS}(\mathcal{K}) = \aleph_0$). With suitable $\prec_{\mathcal{K}}$, can do the same with models of sentences in $L(Q)$, $L_{\omega_1,\omega}(Q)$, etc., though not, most likely, with $\mathrm{LS}(\mathcal{K}) = \aleph_0$.

Having discarded syntax, we need a new notion of type. In AECs with sufficient amalgamation (and hence a monster model $\mathfrak{C}$), there is a particularly nice notion: Galois types. In particular:

Definition

Assume $\mathcal{K}$ has amalgamation property, contains monster model $\mathfrak{C}$. For $a \in \mathfrak{C}$, $M \in \mathcal{K}$, the Galois type of a over M is defined to be the orbit of a under automorphisms of $\mathfrak{C}$ that fix M. The set of all types over M is denoted by ga-S(M). For $N \prec_{\mathcal{K}} M$, the restriction of $q =$ ga-tp(a/M) to N, denoted $q \restriction N$, is ga-tp(a/N).

Why nice: consistent with category-theoretic nature of AECs, saturation w.r.t. Galois types corresponds to homogeneity.

Example: If $\mathcal{K}$ is an elementary class and $\prec_{\mathcal{K}}$ the elementary submodel relation, the Galois types over $M \in \mathcal{K}$ are precisely the complete types over M:

$$\text{ga-tp}(a/M) = \text{ga-tp}(b/M) \quad \text{iff} \quad \text{tp}(a/M) = \text{tp}(b/M)$$

In general, though, the Galois types will not have a nice syntactic description.

Question

What can we say about the stability spectra of AECs? Can we use vaguely classical techniques to address the problem? Topology? Rank functions?

In this talk, we give:

- A way of topologizing the sets ga-S(M) in such a way that topological properties of the resulting spaces correspond to semantic properties of M and $\mathcal{K}$.
- A closely related notion of rank (actually, a family of ranks) which shows some promise in analyzing the stability spectra of reasonably well-behaved AECs.

Let $\mathcal{K}$ be an AEC with monster model $\mathfrak{C}$. Let $\lambda \geq \mathrm{LS}(\mathcal{K})$ and $M \in \mathcal{K}$.

Definition (X_M^λ)

For each $N \prec_{\mathcal{K}} M$ with $|N| \leq \lambda$ and type $p \in \text{ga-S}(N)$, let

$$U_{p,N} = \{q \in \text{ga-S}(M) \,:\, q \upharpoonright N = p\}$$

The sets $U_{p,N}$ form a basis for a topology on ga-S(M). We denote by X_M^λ the set ga-S(M) endowed with this topology.

Note

The $U_{p,N}$ are, in fact, clopen. Types over small submodels play a role analogous to formulas in topologizing spaces of syntactic types.

Remark

The assignment $(M, \lambda) \mapsto X_M^\lambda$ is functorial in both arguments:

- *For any $\mu > \lambda$, the set-theoretic identity map $Id_{\mu,\lambda} : X_M^\mu \to X_M^\lambda$ is continuous.*
- *For $M, M' \in \mathcal{K}$ and $f : M \to M'$ a $\mathcal{K}$-embedding, the induced map from ga-$S(M')$ to ga-$S(M)$ is a continuous surjection from $X_{M'}^\lambda$ to X_M^λ.*

So for each $M \in \mathcal{K}$, we obtain a well-behaved spectrum of spaces, with topological properties passing up and down the line.

Properties and peculiarities:

- Intersection of λ many open sets in X_M^λ is open.
- The X_M^λ are uniform spaces, hence completely regular, and one can speak of Cauchy nets, completeness, etc.
- The X_M^λ are noncompact.
- For $a \in M$, ga-tp(a/M) an isolated point of X_M^λ.

Fact

A model $M \in \mathcal{K}$ is λ-saturated iff the types ga-tp(a/M) with $a \in M$ are dense in X_M^μ for all $\mu < \lambda$. Also M is λ-saturated only if isolated points are dense in X_M^μ for all $\mu < \lambda$.

Tameness has a particularly nice topological characterization. Recall:

Definition

An AEC $\mathcal{K}$ is said to be χ-tame if for any $M \in \mathcal{K}$, if $q, q' \in$ ga-$S(M)$ are distinct, then there is submodel $N \prec_{\mathcal{K}} M$ with $|N| \leq \chi$ such that $q \upharpoonright N \neq q' \upharpoonright N$.

Theorem (Tameness As Separation Principle)

The AEC $\mathcal{K}$ is χ-tame iff for all $M \in \mathcal{K}$, X_M^χ is Hausdorff (in fact, totally disconnected).

Important consequence: If $\mathcal{K}$ is χ-tame, and $\lambda \geq \chi$, a type $q \in X_M^\lambda$ is an accumulation point of $S \subseteq X_M^\lambda$ only if every neighborhood of q contains more than λ elements of S.

In light of this fact, we define a related, slightly Morley-like rank:

Definition (RM^λ)

Assume $\mathcal{K}$ is χ-tame. For $\lambda \geq \chi$, we define RM^λ by the following induction: for any $q \in \text{ga-S}(M)$ with $|M| \leq \lambda$,

- $\mathrm{RM}^\lambda[q] \geq 0$.
- $\mathrm{RM}^\lambda[q] \geq \alpha$ for limit α if $\mathrm{RM}^\lambda[q] \geq \beta$ for all $\beta < \alpha$.
- $\mathrm{RM}^\lambda[q] \geq \alpha + 1$ if there exists a structure $M' \succ_{\mathcal{K}} M$ such that q has strictly more than λ many extensions to types q' over M' with $\mathrm{RM}^\lambda[q'] \geq \alpha$.

For types q over M of arbitrary size, we define

$$\mathrm{RM}^\lambda[q] = \min\{\mathrm{RM}^\lambda[q \upharpoonright N] : N \prec_{\mathcal{K}} M, |N| \leq \lambda\}.$$

The ranks RM^λ are: monotonic, invariant under automorphisms of $\mathfrak{C}$, decreasing in λ. Bound above (typewise) Cantor-Bendixson ranks in spaces X_M^λ.

Definition (λ-t.t.)

We say that $\mathcal{K}$ (or perhaps $\mathfrak{C}$) is λ-totally transcendental if for every $M \in \mathcal{K}$ and $q \in \text{ga-S}(M)$, $\mathrm{RM}^\lambda[q]$ is an ordinal.

No guarantee that types have unique extensions of same RM^λ rank, but:

Proposition (Quasi-unique Extension)

Let $M \prec_{\mathcal{K}} M'$, $q \in$ ga-S(M), and say that $RM^\lambda[q] = \alpha$. Given any rank α extension q' of q to a type over M', there is an intermediate structure M'', $M \prec_{\mathcal{K}} M'' \prec_{\mathcal{K}} M'$, $|M''| \leq |M| + \lambda$, and a rank α extension $p \in$ ga-S(M'') of q with $q' \in$ ga-S(M') as its unique rank α extension.

Connections with Galois stability, in case $\mathcal{K}$ is tame:

Theorem
If $\mathcal{K}$ is λ-stable where λ satisfies $\lambda^{\aleph_0} > \lambda$, then $\mathcal{K}$ is λ-t.t.

From the previous proposition,

Theorem
If $\mathcal{K}$ is λ-t.t., $\mathcal{K}$ is κ-stable for every κ such that $\kappa^\lambda = \kappa$.

Together they yield a (relatively weak) stability transfer result.
Better:

Theorem
If $\mathcal{K}$ is μ-totally transcendental, and $M \in \mathcal{K}$ with cf($|M|$) $> \mu$, then $|$ga-S(M)$| \leq |M| \cdot \sup\{|\text{ga-}S(N)| \mid N \prec_{\mathcal{K}} M, |N| < |M|\}$.

Theorem

If $\mathcal{K}$ is μ-stable with $\mu^{\aleph_0} > \mu$, and κ is such that cf(κ) $> \mu$, if there is an interval $[\lambda, \kappa)$ such that every M $\in \mathcal{K}_{[\lambda,\kappa)}$ satisfies $|$ga-S(M)$| \leq \kappa$, then $\mathcal{K}$ is κ-stable.

This generalizes (and improves upon) a previously known result (Baldwin-Kueker-Van Dieren, 2004), which has $\mu = \aleph_0$ and requires full stability below κ. We can also say something in case $\mathcal{K}$ is only weakly χ-tame (the defining condition of tameness holds only for saturated models):

Theorem

If $\mathcal{K}$ is μ-t.t., and κ is regular with $\kappa > \mu$, if there is an interval $[\lambda, \kappa)$ on which $\mathcal{K}$ is stable, then $\mathcal{K}$ is κ-stable.

It is worth noting that the stronger stability assumption in the last result serves only to guarantee the existence of enough saturated models that the weakening of tameness is not an issue. There is another, more category-theoretic condition that does as well. First, notice

Theorem

Every AEC $\mathcal{K}$ is a LS($\mathcal{K}$)$^+$-accessible category with directed colimits.

Such categories have been considered by Rosický (1997), in which context he proposes a notion of weak κ-stability. As it happens, this is enough:

Theorem

If $\mathcal{K}$ is weakly χ-tame and μ-t.t., and κ is such that cf(κ) $> \mu$, if $\mathcal{K}$ is weakly κ-stable, then $\mathcal{K}$ is κ-stable.

Curiously, weak stability occurs in any category of this form. This means, in a particularly simple case (and taking into account earlier results),

Proposition

If $LS(\mathcal{K}) = \aleph_0$, then for any $\mu > |\mathcal{K}^{mor}_{\aleph_0}|$ with $\aleph_1 \trianglelefteq \mu$, $\mathcal{K}$ is weakly $\mu^{<\mu}$-stable.

The notion of sharp inequality, $\trianglelefteq$, is well treated in texts on accessible categories. Suffice to say, there are many (and arbitrarily large) cardinals μ with $\aleph_1 \trianglelefteq \mu$. This lends savor to:

Corollary

If $LS(\mathcal{K}) = \aleph_0$, $\mathcal{K}$ is $\aleph_0$-t.t. and weakly $\aleph_0$-tame, then for any $\mu > |\mathcal{K}^{mor}_{\aleph_0}|$ with $\aleph_1 \trianglelefteq \mu$, $\mathcal{K}$ is $\mu^{<\mu}$-stable.