Toward a categorical model theory

Michael Lieberman[1] Jiří Rosický[1]

[1]Department of Mathematics and Statistics
Masaryk University

Joint Mathematics Meetings, San Antonio
January 11, 2015

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a strong substructure relation, $\prec_{\mathcal{K}}$, that satisfies:

- $\prec_{\mathcal{K}}$ is a partial order.
- Unions of chains: if $(M_i \mid i < \delta)$ is a $\prec_{\mathcal{K}}$-increasing chain,
  1. $\bigcup_{i<\delta} M_i \in \mathcal{K}$
  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

A strong embedding $f : M \hookrightarrow_{\mathcal{K}} N$ is an isomorphism from M to a strong submodel of N, $f : M \cong M' \prec_{\mathcal{K}} N$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\prec_{\mathcal{K}}$ is a partial order.
- Unions of chains: if $(M_i \mid i < \delta)$ is a $\prec_{\mathcal{K}}$-increasing chain,
  1. $\bigcup_{i<\delta} M_i \in \mathcal{K}$
  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\mathcal{K}$, with the maps in $\mathcal{M}$, forms a subcategory of $\mathbf{Str}(L(\mathcal{K}))$
- Unions of chains: if $(M_i \mid i < \delta)$ is a $\prec_{\mathcal{K}}$-increasing chain,
  1. $\bigcup_{i<\delta} M_i \in \mathcal{K}$
  2. for each $j < \delta$, $M_j \prec_{\mathcal{K}} \bigcup_{i<\delta} M_i$
  3. if each $M_j \prec_{\mathcal{K}} M \in \mathcal{K}$, $\bigcup_{i<\delta} M_i \prec_{\mathcal{K}} M$
- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\mathcal{K}$, with the maps in $\mathcal{M}$, forms a subcategory of $\mathbf{Str}(L(\mathcal{K}))$
- which is closed under directed colimits (i.e. direct limits).

- Coherence: If $M_0 \prec_{\mathcal{K}} M_2$, $M_0 \subseteq M_1 \prec_{\mathcal{K}} M_2$, then $M_0 \prec_{\mathcal{K}} M_1$
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\mathcal{K}$, with the maps in $\mathcal{M}$, forms a subcategory of $\mathbf{Str}(L(\mathcal{K}))$
- which is closed under directed colimits (i.e. direct limits).

- Coherence: Given any $L(\mathcal{K})$-structure embedding $f : M \to N$ and any map $g : N \to N'$ in $\mathcal{M}$, if $gf \in \mathcal{M}$, then $f \in \mathcal{M}$.
- Löwenheim-Skolem: Exists cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, subset $A \subseteq M$, there is an $M_0 \in \mathcal{K}$ with $A \subseteq M_0 \prec_{\mathcal{K}} M$ and $|M_0| \leq |A| + \mathrm{LS}(\mathcal{K})$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\mathcal{K}$, with the maps in $\mathcal{M}$, forms a subcategory of $\mathbf{Str}(L(\mathcal{K}))$
- which is closed under directed colimits (i.e. direct limits).

- Coherence: Given any $L(\mathcal{K})$-structure embedding $f : M \to N$ and any map $g : N \to N'$ in $\mathcal{M}$, if $gf \in \mathcal{M}$, then $f \in \mathcal{M}$.
- Löwenheim-Skolem: There is a cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, M is a directed colimit of structures of size $\mathrm{LS}(\mathcal{K})$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\mathcal{K}$, with the maps in $\mathcal{M}$, forms a subcategory of $\mathbf{Str}(L(\mathcal{K}))$
- which is closed under directed colimits (i.e. direct limits).
- Coherence: Given any $L(\mathcal{K})$-structure embedding $f : M \to N$ and any map $g : N \to N'$ in $\mathcal{M}$, if $gf \in \mathcal{M}$, then $f \in \mathcal{M}$.
- Löwenheim-Skolem: There is a cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, M is a directed colimit of structures of size $\mathrm{LS}(\mathcal{K})$.

An *Abstract Elementary Class (AEC)* is a nonempty class $\mathcal{K}$ of structures in a given signature, closed under isomorphism, equipped with a family of embeddings $\mathcal{M}$, that satisfies:

- $\mathcal{K}$, with the maps in $\mathcal{M}$, forms a subcategory of $\mathbf{Str}(L(\mathcal{K}))$
- which is closed under directed colimits (i.e. direct limits).

- Coherence: Given any $L(\mathcal{K})$-structure embedding $f : M \to N$ and any map $g : N \to N'$ in $\mathcal{M}$, if $gf \in \mathcal{M}$, then $f \in \mathcal{M}$.

- Löwenheim-Skolem: There is a cardinal $\mathrm{LS}(\mathcal{K})$ such that for any $M \in \mathcal{K}$, M is a directed colimit of structures of size $\mathrm{LS}(\mathcal{K})$.

Note

Here we describe $\mathcal{K}$ in terms of properties of the inclusion functor $\mathcal{K} \to \mathbf{Str}(L(\mathcal{K}))$.

Terminological Note:

"Direct limit" and "directed colimit" are essentially interchangeable.

The latter term will be preferred here, as it identifies the construction as a colimit, but both are built from a system of maps indexed by a directed poset, and share the same universal diagram.

In any category of structured sets, then, the directed colimit can be identified with the familiar quotient of the disjoint union or, in the case of an increasing chain, with the union itself.

There is another reason to prefer "directed colimit"—it admits an important generalization:

Definition

For λ regular, a poset I is λ-directed if any subset $J \subseteq I$, $|J| <$ lambda, has an upper bound in I. A colimit is said to be λ-directed if the indexing poset is λ-directed.

This is an important distinction: **Ban** (as a concrete category) is not closed under direct limits, but is closed under ω_1-directed colimits. This prevents it from being an AEC, but it is an accessible category...

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. Basic terminology:

Definition

An object N in a category $\mathbf{C}$ is finitely presentable (ω-presentable) if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves directed colimits.

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. Basic terminology:

Definition

An object N in a category $\mathbf{C}$ is finitely presentable (ω-presentable) if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves directed colimits.

Example: In **Grp**, the category of groups, an object G is finitely presentable iff G is finitely presented. Same for any finitary algebraic variety.

Roughly speaking, an accessible category is one that is generated by colimits of a set of small objects. Basic terminology:

Definition

An object N in a category $\mathbf{C}$ is finitely presentable (ω-presentable) if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves directed colimits.

Example: In **Grp**, the category of groups, an object G is finitely presentable iff G is finitely presented. Same for any finitary algebraic variety.

Definition

A category $\mathbf{C}$ is finitely accessible (ω-accessible) if

- it has at most a set of finitely presentable objects,
- it is closed under directed colimits, and
- every object is a directed colimit of finitely presentable objects.

For general regular cardinal λ:

Definition

An object N in a category $\mathbf{C}$ is λ-presentable if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves λ-directed colimits.

For general regular cardinal λ:

Definition

An object N in a category $\mathbf{C}$ is λ-presentable if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves λ-directed colimits.

Definition

A category $\mathbf{C}$ is λ-accessible if

- it has at most a set of λ-presentables
- it is closed under λ-directed colimits
- every object is a λ-directed colimit of λ-presentables

For general regular cardinal λ:

Definition

An object N in a category $\mathbf{C}$ is λ-presentable if the functor $\mathrm{Hom}_{\mathbf{C}}(N, -)$ preserves λ-directed colimits.

Definition

A category $\mathbf{C}$ is λ-accessible if

- it has at most a set of λ-presentables
- it is closed under λ-directed colimits
- every object is a λ-directed colimit of λ-presentables

Example: As noted, **Ban** lacks directed colimits, so is not finitely accessible. It is, however, $\aleph_1$-accessible.

Definition

For any object M in an accessible category $\mathcal{K}$, we define its **presentability rank**, $\pi(M)$, *to be the least λ such that M is λ-presentable.*

Fact

In any accessible category with arbitrary directed colimits, $\pi(M)$ is always a successor, say $\pi(M) = \lambda^+$. In this case we say λ is the **size** *of M.*

Theorem

If $\mathcal{K}$ is an AEC (resp. mAEC), $M \in \mathcal{K}$ is of size λ iff $|M| = \lambda$ (resp. $dc(M) = \lambda$). By DLS, it follows that any AEC (resp. mAEC [?]) $\mathcal{K}$ is LS($\mathcal{K}$)$^+$-accessible.

In a general accessible category, objects are not structured sets. To define Galois types, though, we do need to introduce sets and elements into the picture.

We do this via a functor $U : \mathcal{K} \to \textbf{Sets}$, which assigns

- to each $M \in \mathcal{K}$ a set $U(M)$, and
- to each $\mathcal{K}$-map $f : M \to N$ a set map $U(f) : U(M) \to U(N)$

To ensure good behavior, we insist that this functor

- Is faithful: If $f \neq g$ in $\mathcal{K}$, then $U(f) \neq U(g)$.
- Preserves directed colimits: the image of any colimit in $\mathcal{K}$ is the colimit of the corresponding diagram of sets.

We say $(\mathcal{K}, U)$ is an accessible category with *concrete directed colimits*.

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits. A Galois type is an equivalence class of pairs (f, a), where $f : M \to N$ and $a \in U(N)$.

Pairs (f_0, a_0) and (f_1, a_1) are equivalent if there are morphisms $h_0 : N_0 \to N$ and $h_1 : N_1 \to N$ such that $h_0 f_0 = h_1 f_1$ and $U(h_0)(a_0) = U(h_1)(a_1)$.

If $\mathcal{K}$ has the amalgamation property (which, of course, is purely diagrammatic), this is an equivalence relation.

No surprises: this is a straightforward generalization of the definition for AECs.

In an AEC, Galois types are said to be tame if they are determined by restriction to small submodels of their domains. The situation here is the same:

Definition

Let $(\mathcal{K}, U)$ be an accessible category with concrete directed colimits and κ regular. We say that $\mathcal{K}$ is κ-tame if for two non-equivalent types (f, a) and (g, b) there is a morphism $h : X \to M$ with X of size κ such that the types (fh, a) and (gh, b) are not equivalent.

$\mathcal{K}$ is called tame if it is κ-tame for some regular cardinal κ.

By the way: you would not lose much in thinking of this U as a *forgetful functor* (or *underlying object functor*), in the usual sense. There are peculiarities, however.

Note

The size of an object M in $\mathcal{K}$ need not correspond to $|U(M)|$. In principle, they could disagree for arbitrarily large M.

This poses little problem for the theory, but one might ask how it can be avoided.

Fact

If U reflects split epimorphisms, it preserves sufficiently large sizes.

We can achieve the same through a stronger, but more familiar condition on U:

Definition

We say $U : \mathcal{K} \to$ **Sets** *is* **coherent** *if, given any set map* $f : U(M) \to U(N)$ *and* $\mathcal{K}$*-map* $g : N \to N'$*, if* $U(g) \circ f = Uh$ *for some* $h : M \to N'$*, then there is* $\bar{f} : M \to N$ *with* $U(\bar{f}) = f$*.*

Definition

We say that an accessible category with concrete directed colimits, $(\mathcal{K}, U)$*, is coherent if* U *is coherent.*

This guarantees sizes behave well, and seems to be indispensable in the element-by-element construction of morphisms...

Big Picture

Accessible categories

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

-

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

-

Accessible categories with concrete directed colimits

-

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

-

Accessible categories with concrete directed colimits

-

Coherent accessible categories with concrete directed colimits

-

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

-

Accessible categories with concrete directed colimits

-

Coherent accessible categories with concrete directed colimits

-

. . . **AECs!**

In AECs, the following looms large:

Theorem (Shelah's Presentation Theorem)

For any AEC $\mathcal{K}$ in signature L, there is a signature $L' \supseteq L$, a first order theory T' in L', and a set of T'-types Γ such that if $\mathcal{K}'$ is the class of L'-structures

$$\{M' \mid M' \models T', M' \textit{ omits } \Gamma\}$$

then $\mathcal{K}' \restriction L = \mathcal{K}$. Moreover, if $M' \subseteq_{L'} N'$ in $\mathcal{K}'$, $M' \restriction_L \prec_{\mathcal{K}} N' \restriction_L$.

That is, the reduct $\restriction_L\colon \mathcal{K}' \to \mathcal{K}$ is functorial. More: $\restriction_L$ is faithful, surjective on objects, and preserves directed colimits.

We can package Shelah's Presentation Theorem as asserting the existence of such a well-behaved covering of any AEC $\mathcal{K}$:

$$\mathcal{K}' \xrightarrow{\restriction_L} \mathcal{K}$$

There are several things to note:

- This result is essential for the computation of Hanf numbers, used in the construction of the EM-functor for AECs.
- The proof makes essential use of coherence.
- The expansion L' and set Γ are a little *ad hoc*.

Theorem (Alternative Presentation Theorem, L/Rosický)

If $\mathcal{K}$ is an accessible category with directed colimits, there is a finitely accessible category $\mathcal{K}'$ and a functor $F : \mathcal{K}' \to \mathcal{K}$ that is faithful and surjective on objects, and preserves directed colimits:

$$\mathcal{K}' \xrightarrow{F} \mathcal{K}$$

Note:

- No coherence is required.
- This $\mathcal{K}'$ is actually nicer: $L^*_{\kappa\omega}$, not $L_{\kappa\omega}$.

But is it useful in the same way as Shelah's Presentation Theorem?

Theorem (Makkai/Paré)

If $\mathcal{K}'$ is finitely accessible, there is a faithful, directed colimit-preserving functor $E : \mathbf{Lin} \to \mathcal{K}'$.

So we get a composition $\mathbf{Lin} \to \mathcal{K}' \to \mathcal{K}\ldots$

Corollary

If $\mathcal{K}$ is a large accessible category with directed colimits, there is a faithful functor $EM : \mathbf{Lin} \to \mathcal{K}$ that preserves directed colimits.

This is a genuine *EM*-functor—partial results in [L/Rosický], more to come.

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

-

Accessible categories with concrete directed colimits

-

Coherent accessible categories with concrete directed colimits

-

. . . **AECs!**

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

- Presentation theorem, EM-functor.

Accessible categories with concrete directed colimits

-

Coherent accessible categories with concrete directed colimits

-

. . . **AECs!**

Theorem

Assuming a proper class of strongly compact cardinals, henceforth (C), any AEC is tame.

Using an immensely powerful result of Makkai and Paré, this generalizes:

Theorem (L/Rosický)

Assuming (C), any accessible category with concrete directed colimits is tame.

Proof (Idea):

Consider the following categories of configurations:

- $\mathcal{L}_2 : (f_0, f_1, a_0, a_1)$, with $f_i : M \to N_i$, $a_i \in U(N_i)$.
- $\mathcal{L}_1 : (f_0, f_1, a_0, a_1, h_0, h_1)$, with the h_i witnessing equivalence.

Let $G : \mathcal{L}_1 \to \mathcal{L}_2$ be the forgetful functor. Both categories are accessible, as is G.

It is a matter of some subtlety to prove (following Makkai/Pare, 5.5.1) that, assuming (C), the full image of G in $\mathcal{L}_2$ is κ-accessible for a compact cardinal κ. Proving closure under κ-directed colimits involves a delicate compactness argument—$L_{\kappa,\kappa}$ must be large enough to capture the relevant atomic diagrams, among other things.

If you believe that $G(\mathcal{L}_1)$, the subcategory consisting of equivalent pairs, is κ-accessible, the rest is easy:

Consider (f_0, f_1, a_0, a_1), where $(f_0 u, a_0)$ and $(f_1 u, a_1)$ are equivalent for all $u : X \to M$, X κ-presentable. Then $(f_0 u, f_1 u, a_0, a_1)$ belongs to $G(\mathcal{L}_1)$ for all such u, and since (f_0, f_1, a_0, a_1) is their κ-directed colimit, it belongs to $G(\mathcal{L}_1)$ as well. That is, (f_0, a_0) and (f_1, a_1) are equivalent.

Thus $\mathcal{K}$ is κ-tame. $\square$

This proof makes no use of coherence—the apparent connection between tameness and large cardinals goes well beyond AECs.

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

- Presentation theorem, EM-functor.

Accessible categories with concrete directed colimits

-

Coherent accessible categories with concrete directed colimits

-

... **AECs!**

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

- Presentation theorem, EM-functor.

Accessible categories with concrete directed colimits

- Boney's Theorem

Coherent accessible categories with concrete directed colimits

-

. . . **AECs!**

One might ask where coherence is absolutely essential.

In trying to build maps element-by-element, we run into such situations—without it there is no reason to think the set maps being constructed actually arise from maps in the category.

In particular, we can't prove

Theorem

An object M is Galois-saturated iff it is model-homogeneous.

This, of course, is essential in the proof of the uniqueness of saturated models, and in the transfer of categoricity.

If we do assume coherence? The EM-functor works beautifully, much of classification theory for AECs seems to generalize...

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

- Presentation theorem, EM-functor.

Accessible categories with concrete directed colimits

- Boney's Theorem

Coherent accessible categories with concrete directed colimits

-

. . . **AECs!**

Big Picture

Accessible categories

- Foundations: Makkai/Paré, Adámek/Rosický

Accessible categories with directed colimits

- Presentation theorem, EM-functor.

Accessible categories with concrete directed colimits

- Boney's Theorem

Coherent accessible categories with concrete directed colimits

- Fragment of classification theory...

. . . **AECs!**

Bibliography

- J. Adámek and J. Rosický, *Locally Presentable and Accessible Categories*, Cambridge University Press 1994.
- J. Baldwin, *Categoricity*, AMS 2009.
- J. Baldwin, *Ehrenfeucht-Mostowski models in abstract elementary classes*. Logic and Its Applications (Y. Zhang and A. Blass, eds.), vol. 380, Contemporary Mathematics.
- T. Beke and J. Rosický, *Abstract elementary classes and accessible categories*, Annals Pure Appl. Logic 163 (2012), 2008-2017.
- R. Grossberg, Classification theory for abstract elementary classes, Logic and Algebra (Yi Zhang, ed.), vol. 302, AMS 2002.

- M. Lieberman, *Category theoretic aspects of abstract elementary classes*, Annals Pure Appl. Logic 162 (2011), 903-915.
- M. Lieberman and J. Rosický, *Classification theory for accessible categories*. To appear in J. Symb. Logic.
- M. Makkai and R. Paré, *Accessible Categories: The Foundations of Categorical Model Theory*, AMS 1989.
- J. Rosický, *Accessible categories, saturation and categoricity*, J. Symb. Logic 62 (1997), 891-901.